Spotting Trees with Few Leaves

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(speaker)

Lund University

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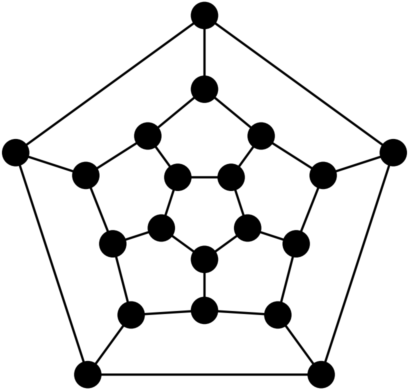
Satisfiability Lower Bounds and Tight Results for Parameterized and Exponential-Time Algorithms
Simons Institute, Berkeley
4th Nov 2015

Hamiltonian Cycle in undirected graphs

Problem

Input: **undirected** graph G.
Goal: Find a Hamiltonian cycle.



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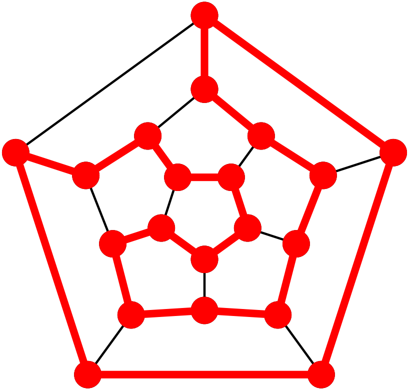

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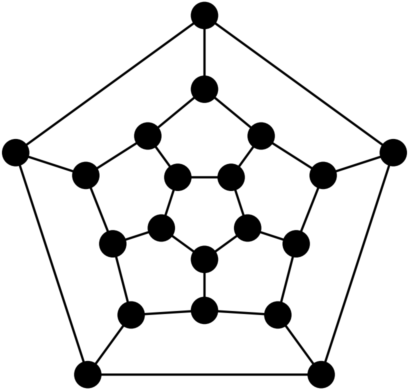

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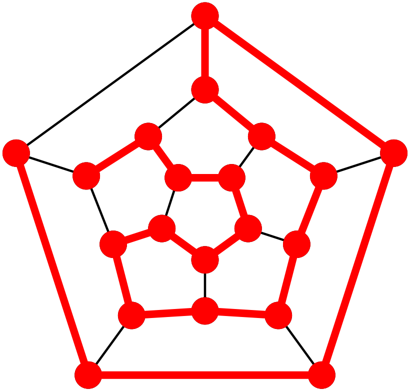

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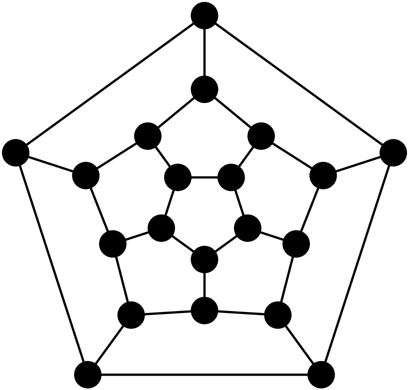
INPUT: **undirected** graph G, integer k.
GOAL: Find a k-vertex path (shortly: k-path).



k-PATH in undirected graphs

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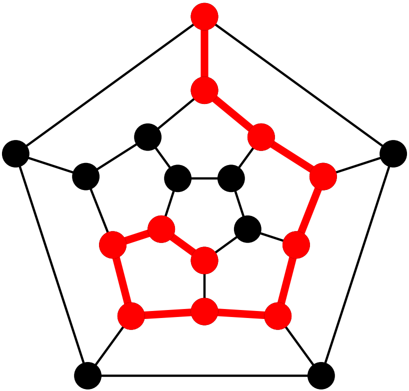

(k, ℓ)-TREE in undirected graphs

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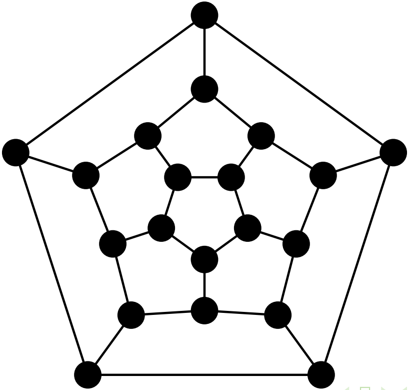
INPUT: **undirected** graph G, integers k, ℓ.
GOAL: Find a tree T with k vertices including exactly ℓ leaves, (shortly: (k, ℓ)-tree).



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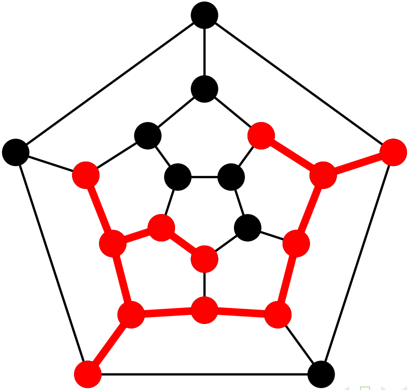
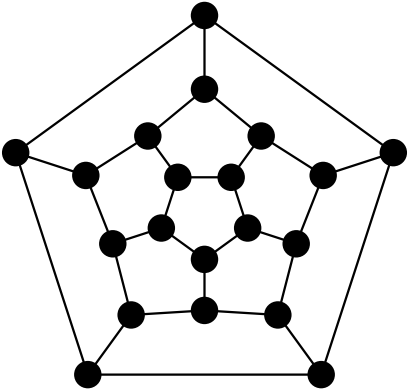


k-Internal Spanning Tree (k-IST)

Problem

Input: **undirected** graph G, integer k.
Goal: Find a **spanning** tree T with at least k **internal** vertices



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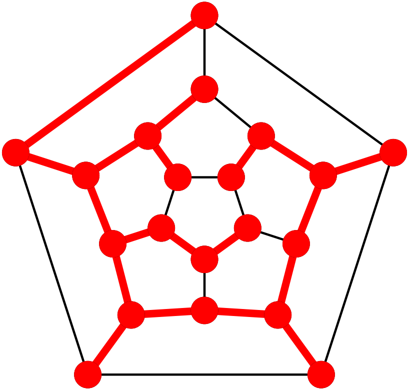

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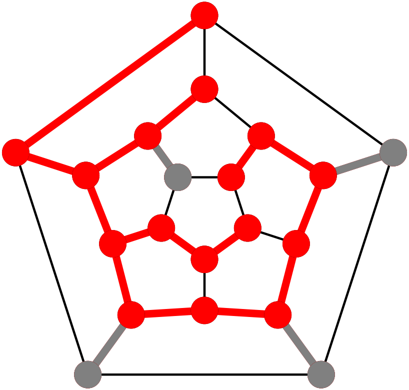

k-Internal Spanning Tree (k-IST)

Problem

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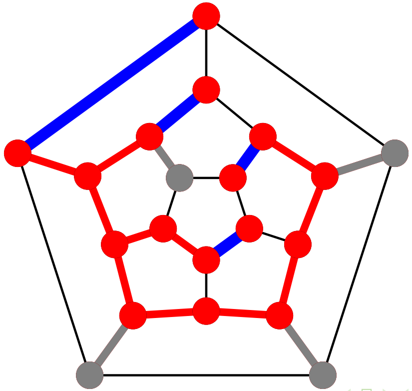
Goal: Find a tree T with at least k **internal** vertices s.t. edges from leaves to parents form a matching



k-Internal Spanning Tree (k-IST)

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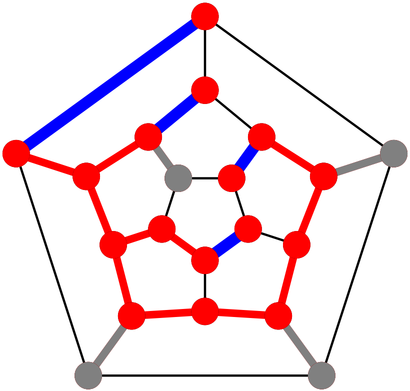
Input: **undirected** graph G, integer k.
Goal: Find a $(k + l, l)$-tree for some $l = 2, \ldots, k$



FPT and exponential-time algorithms

All these problems are NP-complete.

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In this talk we are interested in

- **Parameterized** (FPT) algorithms: running time of $O^*(c^k)$
- **Exponential-time** algorithms: running time of $O^*(c^n)$

and we want c to be small.

Previous results

HAMILTONIAN CYCLE / PATH

- Held, Karp 1962, $O^*(2^n)$ time, $O(2^n)$ space.
- Kohn et al 1969 $O^*(2^n)$ time, poly space.
- Björklund 2010 $O^*(1.66^n)$ time, poly space

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k-PATH

- Alon, Yuster, Zwick 1994: $O^*((2e)^k)$, poly space
- Kneis, Mölle, Richter, Rossmanith 2006: $O^*(4^k)$, poly space
- Chen, Lu, She, Zhang 2007: $O^*(4^k)$, poly space
- Koutis 2008: $O^*(2^{3/2k})$, poly space
- Williams 2009: $O^*(2^k)$, poly space
- Björklund, Husfeldt, Kaski, Koivisto 2010: $O^*(1.66^k)$, poly space

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(k, ℓ)-TREE

- Cohen et al. 2010: $O^*(6.14^k)$, poly space
- Dauligault 2011, Zehavi 2013: $O^*(2^k)$, poly space

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k-INTERNAL SPANING TREE: FPT algorithms

- Prieto, Sloper 2005: $O^*(2^{k \log k})$, poly space
- Cohen et al. 2010: $O^*(49.4^k)$, poly space
- Fomin et al. 2012: $O^*(16^k)$, poly space
- Fomin et al. 2013: $O^*(8^k)$, poly space
- Daligault 2011; Zehavi 2013; Li et al. 2014: $O^*(4^k)$, poly space
- Zehavi 2015: $O^*(3.62^k)$, exp space

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k-INTERNAL SPANING TREE: exponential time algorithms

- Raible et al. 2008: $O^*(2^n)$, exp space
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Open question (Raible et al): Can the approach of Björklund (2010) be extended from finding paths to finding trees?

Our results (1): from paths to trees

k-PATH
BHKK 2010
$O^*(1.66^k)$

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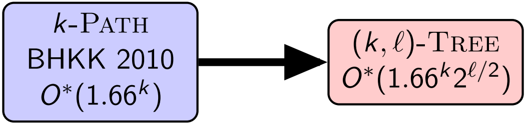

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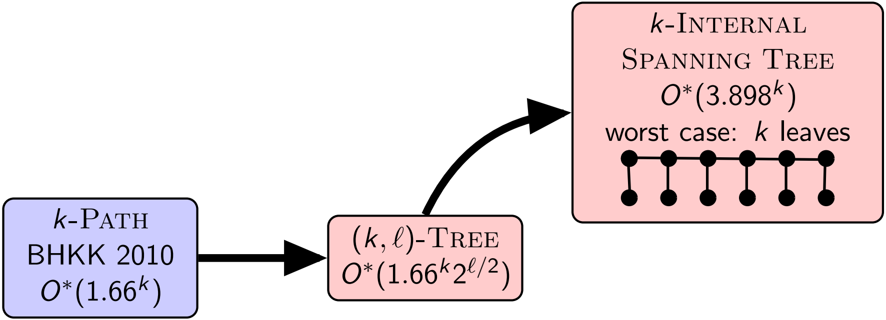

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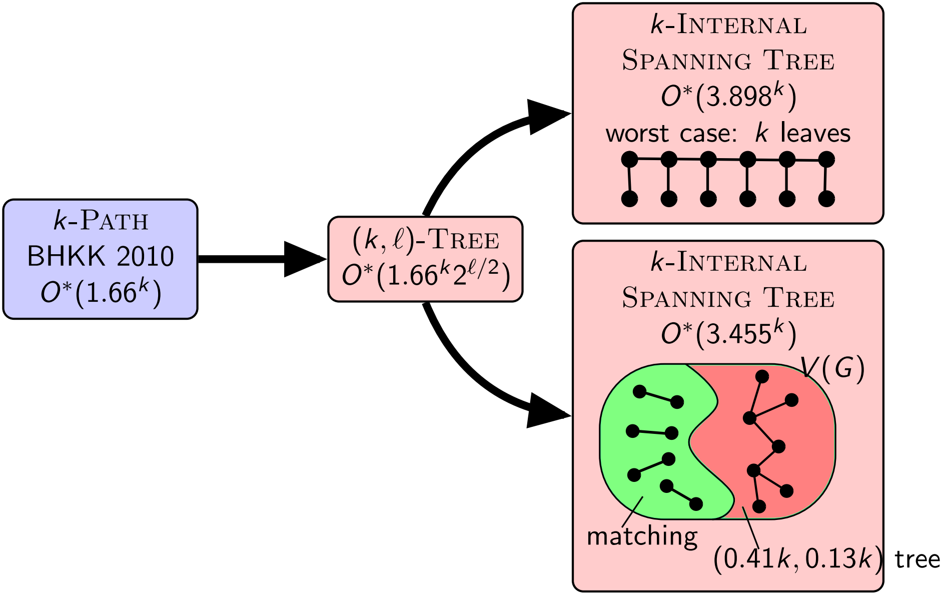

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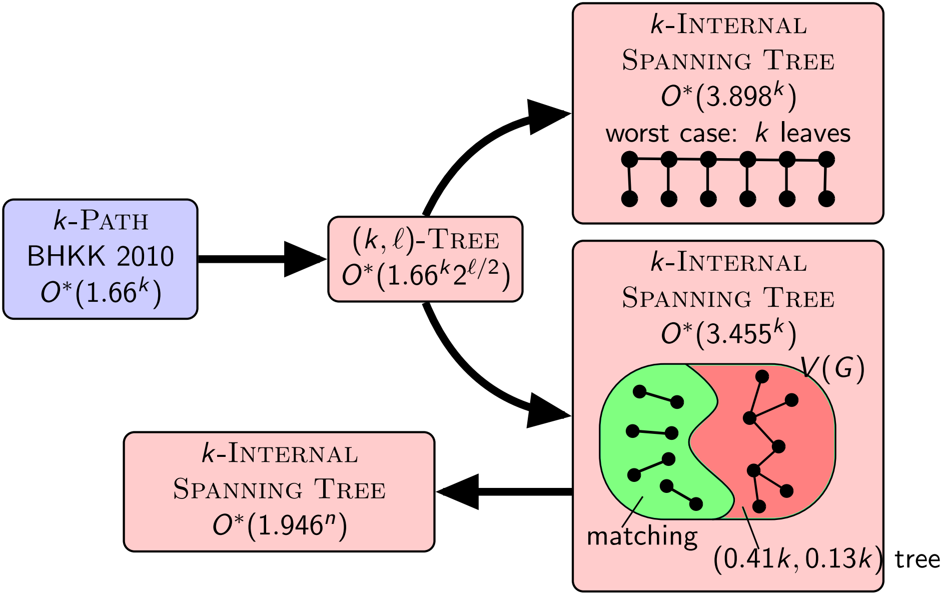

Note: All our algorithms

- use polynomial space,
- are randomized Monte-Carlo with one sided error, i.e.,
  - No solution $\rightarrow$ correct answer
  - Solution Exists $\rightarrow$ correct answer with probability 99%.

Previous results: restricted graph classes

HAMILTONIAN CYCLE / PATH in cubic graphs

- Eppstein WADS 2003 $O^*(1.26^n)$, poly space.
- Iwama and Nakashima 2007 $O^*(1.251^n)$, poly space.
- Cygan et al. 2011 $O^*(1.201^n)$, exp space

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HAMILTONIAN CYCLE / PATH in max degree 4 graphs

- Eppstein 2003 $O^*(1.89^n)$, poly space.
- Gebauer 2008 $O^*(1.733^n)$, exp space.
- Xiao, Namagochi 2012 $O^*(1.716^n)$, poly space.
- Björklund 2010 $O^*(1.66^n)$, poly space
- Cygan et al. 2011 $O^*(1.588^n)$, exp space

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k-Path in bipartite graphs

- Björklund, Husfeldt, Kaski, Koivisto 2010: $O^*(1.41^k)$, poly space

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Question: Is there anything interesting going on between bipartite and general graphs?

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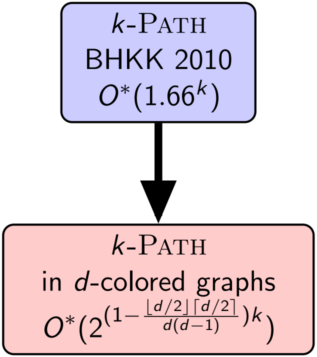
Question: Is there anything interesting going on between bipartite and general graphs?

What about d-colorable graphs?

Our results (2): colored graphs

k-PATH
BHKK 2010
$O^*(1.66^k)$

Our results (2): colored graphs



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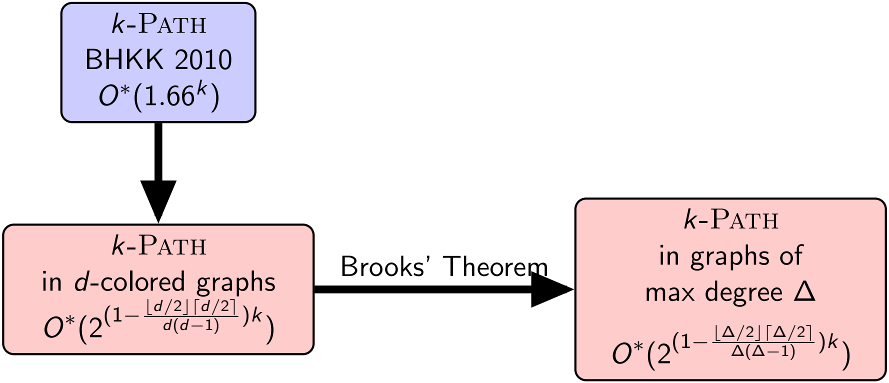

Our results (2): colored graphs

Δ	Running Time
$3, 4$	$O^*(2^{2k/3}) = O^*(1.5874^k)$
$5, 6$	$O^*(2^{7k/10}) = O^*(1.6245^k)$
$7, 8$	$O^*(2^{5k/7}) = O^*(1.6406^k)$
$9, 10$	$O^*(2^{13k/18}) = O^*(1.6497^k)$
$11, 12$	$O^*(2^{8k/11}) = O^*(1.6555^k)$

k-Path
BHKK 2010
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↓

k-Path
in d-colored graphs
$O^*(2^{(1-\frac{\lfloor d/2 \rfloor \lceil d/2 \rceil}{d(d-1)})k})$

→ Brooks' Theorem →

k-Path
in graphs of
max degree Δ
$O^*(2^{(1-\frac{\lfloor \Delta/2 \rfloor \lceil \Delta/2 \rceil}{\Delta(\Delta-1)})k})$

↑ (to the table)

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↑ (to the table above)

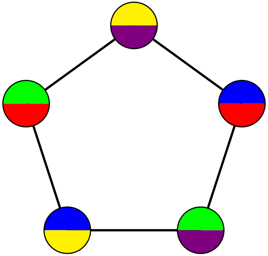
Note: These results generalize to (k, l)-TREE (for $l = O(1)$ we get the same bounds) and to *k*-IST.

Fractional coloring

$(a : b)$-coloring

- Assign b-element subsets of $\{1, \ldots, a\}$
- Adjacent vertices get disjoint sets.

Fractional chromatic number $\chi_f(G) = \inf\{\frac{a}{b} \mid G \text{ is } (a:b)\text{-colorable}\}$.
Example: C_5 is $(5:2)$-colorable, and $\chi_f(G) = \frac{5}{2}$.



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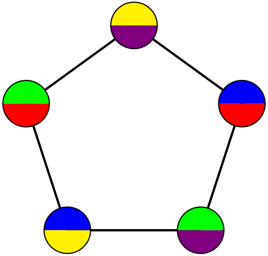

Note: $\chi_f(G) \leq \chi(G)$.

Our results (3): fractionally-colored graphs

Theorem

Given a proper $(a : b)$-coloring of the input graph, for any $t = 1, \ldots, b$ one can solve $(k, \ell)-\textsc{Tree}$ in time

$$2^{\left(1-\frac{\binom{a-b}{t}-\binom{a-2b}{t}}{\binom{a}{t}}\right)k+\left(1-\frac{\binom{a-b}{t}}{\binom{a}{t}}\right)l} n^{O(1)}.$$

Our results (3): fractionally-colored graphs

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Main consequences:

- $1.571^k 1.274^l n^{O(1)}$-time algorithm for (k,l)-$\textsc{Tree}$ in **triangle-free** graphs of maximum degree 3,

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

- $1.571^k n^{O(1)}$-time algorithm for k-$\textsc{Path}$ in **general** graphs of maximum degree 3

Our results: updated bounded degree table

Updated table:

Δ	Running Time
3	$O^*(1.571^k)$
4	$O^*(2^{2k/3}) = O^*(1.5874^k)$
$5, 6$	$O^*(2^{7k/10}) = O^*(1.6245^k)$
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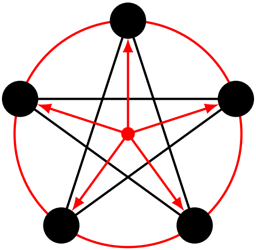
$11, 12$	$O^*(2^{8k/11}) = O^*(1.6555^k)$

Vector coloring

An r-**vector coloring** of a graph G is an assignment $c : V(G) \to \mathbb{R}^n$ s.t.

- $||c(v)|| = 1$ for every $v \in V(G)$ and
- for every edge uv, $c(u) \cdot c(v) \leq -1/(r-1)$.

The **vector chromatic number** $\chi_v(G)$ is the smallest such (real) r.



$c(u) \cdot c(v) = \cos \angle(u, v) = \cos(\frac{4\pi}{5}) = -\frac{1}{\sqrt{5}-1}$, so $\chi_v(C_5) \leq \sqrt{5}$.

Relaxation property and computational complexity

Let $\omega(G)$ be the size of a maximum clique in G.

Theorem (Folklore)

For any graph G

$$\omega(G) \leq \chi_v(G) \leq \chi_f(G) \leq \chi(G)$$

Relaxation property and computational complexity

Let $\omega(G)$ be the size of a maximum clique in G.

Theorem (Folklore)

For any graph G

$$\omega(G) \leq \chi_v(G) \leq \chi_f(G) \leq \chi(G)$$

Theorem (Karger, Motwani, Sudan)

The vector chromatic number can be $(1+\epsilon)$*-approximated in polynomial time.*

Our results (4): r-vector-colorable graphs

Theorem

For any $\epsilon > 0$, (k, ℓ)-TREE can be solved in time

$$O^*(2^{\left(\frac{k+\ell}{2}+\left(1-\frac{\arccos(-1/(r-1))}{\pi}\right)\frac{k-1}{2}+\epsilon\right)})$$

for r-vector colorable graphs (hence also for r-colorable or fractionally r-colorable).

Concrete running times for k-PATH:

$\chi_v(G)$	Running Time
3	$O^*(1.5875^k)$
4	$O^*(1.6199^k)$
5	$O^*(1.6356^k)$
6	$O^*(1.6448^k)$
7	$O^*(1.6510^k)$
8	$O^*(1.6554^k)$

A glimpse of eye at our approach for d-colored graphs

BHKK's k-Path algorithm as black box

partition $V = V_1 \cup V_2$ such that for a solution k-path P we have $|E(P) \cap V_1 \times V_2| \geq t$

↓

Some algebraic magic...

↓

$O^*(2^{k-t/2})$-time algorithm

partitions in colored graphs

d-colored graphs

- Idea: Pick $\lfloor d/2 \rfloor$ colors as V_1.

d-colored graphs

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partitions in colored graphs

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partitions in colored graphs

d-colored graphs

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$(a : b)$-colored graphs

- V_1 is a subset of $\binom{a}{t}$ colors $(t = 1, \ldots, \lfloor a/2 \rfloor)$.

partitions in colored graphs

d-colored graphs

- Idea: Pick $\lfloor d/2 \rfloor$ colors as V_1.
- Possibly, the saving $|E(P) \cap V_1 \times V_2|$ is small.
- Solution: test all the $\binom{d}{\lfloor d/2 \rfloor}$ choices!

$(a : b)$-colored graphs

- V_1 is a subset of $\binom{a}{t}$ colors $(t = 1, \ldots, \lfloor a/2 \rfloor)$.
- Test all choices

partitions in colored graphs

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- pick a random hyperplane through the origin: it divides the unit sphere into two half-spheres: green and red.

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- V_1 are the vertices mapped to the green half-space.
- The expected value of $|E(P) \cap V_1 \times V_2|$ is large.

Further work

- Some more interesting partitions (in other graph classes)?
- A better algorithm when many leaves?
- Lower bounds under SETH, Set Cover Conjecture, etc.?
- Max Leaf problem (is there a spanning tree with at least k leaves?).
  State of the art: Zehavi'15: $O^*(3.188^k)$-time branching algorithm
  Can we use the algebraic approach to get a faster algorithm? Ideally, $O^*(2^k)$-time algorithm?

Thank you!