Parameterized Inapproximability of Max k-Subset Intersection under ETH

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ETH can be used to refute the existence of exponential time approximation algorithms.
Max-$k$-Subset-Intersection

**Input:** A collection $\mathcal{F} = \{S_1, S_2, \cdots, S_n\}$ of subsets over $[n]$.  

**Solution:** $k$ distinct subsets $S_{j_1}, S_{j_2}, \cdots, S_{j_k}$ from $\mathcal{F}$.  

**Cost:** $|S_{j_1} \cap \cdots \cap S_{j_k}|$.  

**Goal:** max.
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Another formulation: given a bipartite graph \( G = (A \cup B, E) \), find a \( k \)-vertex set \( V \in \binom{A}{k} \) with maximum number of common neighbors.
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**Remark**

1. **Max-k-Subset-Intersection** is **NP-hard**
2. **Max-k-Subset-Intersection** can be solved in time $n^{O(k)}$. 
### Max-k-Subset-Intersection

**Input:** A collection \( \mathcal{F} = \{S_1, S_2, \ldots, S_n\} \) of subsets over \([n]\).

**Solution:** \( k \) distinct subsets \( S_{j_1}, S_{j_2}, \ldots, S_{j_k} \) from \( \mathcal{F} \).

**Cost:** \( |S_{j_1} \cap \cdots \cap S_{j_k}| \).

**Goal:** max.

Let \( \text{OPT}_{kmsi}(\mathcal{F}) \) be the maximum \( k \)-subset intersection size of \( \mathcal{F} \).

**Question**

*Is there an \( f(k) \cdot n^{O(1)}\)-time algorithm that, given \( \mathcal{F} \), finds \( k \) distinct subsets from \( \mathcal{F} \) with intersection size at least \( \frac{1}{r} \cdot \text{OPT}_{kmsi} \)?*
Results of **Polynomial-time** inapproximability:

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It does not rule out approximate algorithms in $f(k) \cdot n^{O(1)}$-time.
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*It does not rule out approximate algorithms in $f(k) \cdot n^{O(1)}$-time.*
1. Most proofs of the classical inapproximability rely on the PCP theorem.

2. Reductions based on the PCP theorem produce instances with optimal solutions of relatively large size, e.g. $k = n^{\Theta(1)}$.

3. In parameterized complexity, we assume the value of $k$ is small, hence $k$ should not depend on $n$. 
A gap-producing reduction

**Theorem (main)**

We can construct a bipartite graph $H = (A \cup B, E)$ in polynomial time on input an $n$-vertex graph $G$ and $k \in \mathbb{N}$ with $(k + 1)! < n^{\Theta(1/k)}$ s.t.:

1. if $K_k \subseteq G$, then there are $s$ vertices in $A$ with at least $n^{\Theta(1/k)}$ common neighbors in $B$;

2. if $K_k \not\subseteq G$, every $s$ vertices in $A$ have at most $(k + 1)!$ common neighbors in $B$,

where $s = \binom{k}{2}$. 

**Remark**

- This reduction does not use the PCP theorem. It is based on some extremal combinatorics construction.
- It applies in case with small value of $k$. 

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- This reduction does not use the PCP theorem. It is based on some extremal combinatorics construction.
- It applies in case with small value of $k$. 

Consequence under ETH

Theorem (Chen et. al 04)
Assuming ETH, k-Clique cannot be solved in $f(k) \cdot n^{o(k)}$-time for any computable function $f$.

Corollary
Assuming ETH, Max-k-Subset-Intersection does not admit $f(k) \cdot n^{o(\sqrt{k})}$-time approximation algorithm with ratio $n^{1/\sqrt{k}}$. 
A variant of main theorem

Fix $\Delta \in \mathbb{N}^+$. 

**Theorem**

We can construct a bipartite graph $H = (A \cup B, E)$ in polynomial time on input an $n$-vertex graph $G$ and $k \in \mathbb{N}$ with $(k + 1)! < n^{\Theta(1/k)}$ s.t.:

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Theorem

Assuming ETH, Max-\textit{k-Subset-Intersection} does not admit $f(k) \cdot n^{o(\sqrt{k}/\Delta)}$-time approximation algorithm with ratio $n^{\sqrt{\Delta}/\sqrt{k}}$. 
A variant of main theorem

Let $\Delta = 2^k/s$.

**Theorem**

We can construct a bipartite graph $H = (A \cup B, E)$ in fpt time on input an $n$-vertex graph $G$ and $k \in \mathbb{N}$ with $(k + 1)! < n^{\Theta(1/k)}$ s.t.:

1. if $K_k \subseteq G$, then there are $2^k = s \cdot \Delta$ vertices in $A$ with at least $n^{\Theta(1/k)}$ common neighbors in $B$;
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**Corollary**

Max-$k$-Subset-Intersection does not admit $f(k) \cdot n^{o(\log k)}$-time approximation algorithm to ratio $n^{1/\log k}$ under ETH.
What can we do with this gap?
Inapproximability of other natural parameterized problem

Question

Find gap-preserving fpt-reduction from Max-$k$-Subset-Intersection to

- $k$-Clique
- $k$-Dominating-Set
Question

Is there any fpt-algorithm $A$ such that on input a bipartite graph $H = (A \cup B, E)$, it constructs a graph $G$ satisfying:

- (1) if there exists $V \in \binom{A}{s}$ with $n^{\Theta(1/k)}$ common neighbors, then $G$ contains a $g(k)$ clique;
- (2) if every $V \in \binom{A}{s}$ has at most $(k + 1)!$ common neighbors, then $G$ contains no $\frac{g(k)}{2}$ clique.

Wrong: there might exist $s - 1$ vertices in $A$ with $n^{\Theta(1/k)}$ common neighbors, leading to a $(s - 1 + 2(k + 1)!)$-clique.
Question

Is there any fpt-algorithm \( A \) such that on input a bipartite graph \( H = (A \cup B, E) \), it construct a graph \( G \) satisfying:

- (1) if there exists \( V \in \binom{A}{s} \) with \( n^{\Theta(1/k)} \) common neighbors, then \( G \) contains a \( g(k) \)-clique;
- (2) if every \( V \in \binom{A}{s} \) has at most \((k + 1)!\) common neighbors, then \( G \) contains no \( \frac{g(k)}{2} \)-clique.

A naive idea: color \( A \) (resp. \( B \)) with \( s \) (resp. \( 2(k + 1)! \)) colors, add edges between vertices in \( A \) (resp. \( B \)) with different colors.

- in case (1), \( H \) has a \((s + 2(k + 1)!)\)-clique;
- in case (2), \( H \) has no clique with \( > (s + (k + 1)! \) vertices.
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**Wrong:** there might exist $s - 1$ vertices in $A$ with $n^{\Theta(1/k)}$ common neighbors, leading to a $(s - 1 + 2(k + 1)!)$-clique.
From Max-$k$-Subset-Intersection to $k$-Dominating-Set?

Let $\gamma(G)$ be the size of its minimum dominating set.

**Question**

Is there any fpt-algorithm $A$ such that on input a bipartite graph $H = (A \cup B, E)$, it construct a graph $G$ satisfying:

- (i) if there exists $V \in (A_s)$ with $n^{\Theta(1/k)}$ common neighbors, then $\gamma(G) < g(k)$;
- (ii) if every $V \in (A_s)$ has at most $(k + 1)!$ common neighbors, then $\gamma(G) > 2g(k)$.

where $s = \binom{k}{2}$. 
Constant inapproximability of dominating set

Theorem (Chen and Lin 15)

There is an algorithm \( \mathcal{A} \) such that on input a bipartite graph \( H = (A \cup B, E) \), it construct a graph \( G \) in \( f(k, d) \cdot |H|^{O(c)} \)-time satisfying:

- if there exists \( V \in \binom{A}{s} \) with \( d \) common neighbors, then \( \gamma(G) < (1 + \varepsilon)d^c \);
- if every \( V \in \binom{A}{s} \) has at most \( (k + 1)! \) common neighbors, then \( \gamma(G) > cd^c/3 \).

where \( s = \binom{k}{2} \), \( d = k^{O(k^3)} \).
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where $s = \binom{k}{2}$, $d = k^{O(k^3)}$.

Theorem

Assuming ETH, there is no $f(\gamma(G)) \cdot |G|^{O(1)}$-time algorithm which on every input graph $G$ outputs a dominating set of size at most $4 + \varepsilon \sqrt{\log(\gamma(G))} \cdot \gamma(G)$. 
Previous inapproximability results of dominating set

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Remark: Independent dominating set problem is not monotone.
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**Remark**

*Independent dominating set problem is not monotone.*
Proof of the gap-producing reduction
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**Notation:** \( \Gamma(X) \) is the set of common neighbors of all vertices in \( X \).

**Goal:** Given \( n \)-vertex graph \( G \) construct \( H = (A \dot{\cup} B, E) \) in FPT, such that:

1. If \( K_k \subseteq G \), then \( \exists V \in (\mathcal{A}_s) \), \( |\Gamma(V)| \geq h \); \( h = n \Theta(1/k) \)

2. If \( K_k \not\subseteq G \), then \( \forall V \in (\mathcal{A}_s) \), \( |\Gamma(V)| \leq \ell \); \( \ell = (k+1)! \)

where \( s = \binom{k}{2} \).

**Example \((k = 3, s = 3)\):**

**Key idea:** construct a bipartite graph \( T = (\mathcal{V}(G) \dot{\cup} B, E(T)) \) satisfying:

1. For all \( V \in (\mathcal{V}(G))_{k+1} \), \( |\Gamma(V)| \leq \ell \);
2. For a random \( V \in (\mathcal{V}(G))_k \), with high probability \( |\Gamma(V)| \geq h \).
Proof of the gap-producing reduction

**Notation:** $\Gamma(X)$ is the set of common neighbors of all vertices in $X$. 
Proof of the gap-producing reduction

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- $H_1$ if $K_k \subseteq G$, then $\exists V \in \binom{A}{s}, |\Gamma(V)| \geq h; (h = n^{\Theta(1/k)})$
- $H_2$ if $K_k \nsubseteq G$, then $\forall V \in \binom{A}{s}, |\Gamma(V)| \leq \ell. (\ell = (k + 1)!)$

where $s = \binom{k}{2}$. 

Example ($k = 3$, $s = 3$)
**Proof of the gap-producing reduction**

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**Example** \((k = 3, s = 3)\)

---

**key idea**: construct a bipartite graph \( T = (V(G) \cup B, E(T)) \) satifying:

- **T1** \( \forall V \in \binom{V(G)}{k+1}, |\Gamma(V)| \leq \ell; \)
- **T2** for a random \( V \in \binom{V(G)}{k} \), with high probability \( |\Gamma(V)| \geq h; \)
**Proof of the gap-producing reduction**

**Notation:** $\Gamma(X)$ is the set of common neighbors of all vertices in $X$.

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**Example ($k = 3, s = 3$)**
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**Example** ($k = 3, s = 3$)
Probabilistic construction of $T$

**Bipartite Random Graph:** $T = (A \cup B, E)$
- $|A| = |B| = n$
- $u \in A$ and $v \in B$ is joined by an edge with probability $p = n^{-1/(k+1)}$
Probabilistic construction of $T$

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The expected number of common neighbors of a $(k+1)$-vertex subset of $A$ is

$$n \cdot p^{k+1} = O(1)$$
Bipartite Random Graph: \( T = (A \cup B, E) \)

- \( |A| = |B| = n \)
- \( u \in A \) and \( v \in B \) is joined by an edge with probability \( p = \frac{n-1}{k+1} \)

The expected number of common neighbors of a \((k+1)\)-vertex subset of \( A \) is

\[
n \cdot p^{k+1} = O(1)
\]

The expected number of common neighbors of a \( k \)-vertex subset of \( A \) is

\[
n \cdot p^k = n^{1/(k+1)}
\]
Define bipartite graph $T = (A \cup B, E) = ((V_1 \cup V_2 \cup \cdots \cup V_n) \cup B, E)$ satisfying:

**T1** every $k + 1$ vertices in $A$ has at most $\ell$ common neighbors;

**T2'** for every $k$ distinct indices $i_1, \cdots, i_k$, there exist $v_{i_1} \in V_{i_1}, \cdots, v_{i_k} \in V_{i_k}$ such that $v_1, \cdots, v_k$ have at least $h$ common neighbors.

Remark: The reduction can be adapted to $T$ satisfying $T_1$ and $T_2'$. 

Lemma: For $\ell = \binom{k+1}{2} < h = \Theta(\frac{1}{k})$, we can construct $T$ satisfying $T_1$ and $T_2'$ in polynomial time.
Derandomizing the reduction

Define bipartite graph $T = (A \cup B, E) = ((V_1 \cup V_2 \cup \cdots \cup V_n) \cup B, E)$ satisfying:

1. **T1** every $k + 1$ vertices in $A$ has at most $\ell$ common neighbors;
2. **T2’** for every $k$ distinct indices $i_1, \cdots, i_k$, there exist $v_{i_1} \in V_{i_1}, \cdots, v_{i_k} \in V_{i_k}$ such that $v_1, \cdots, v_k$ have at least $h$ common neighbors.

**Remark**

The reduction can be adapted to $T$ satisfying T1 and T2’.

**Lemma**

For $\ell = (k + 1)! < h = n^{\Theta(1/k)}$, we can construct $T$ satisfying T1 and T2’ in polynomial time.
Summary

- We give an fpt gap-producing reduction from $k$-Clique to Max-$k$-Subset-Intersection.
- Under ETH, we can rule out moderate exponential approximation algorithms for Max-$k$-Subset-Intersection.
- Inapproximability of other natural parameterized problem.
  - $k$-Dominating-Set: no constant fpt-approximation

Open questions

- Does $k$-Clique have constant fpt-approximation?
- Does $k$-Dominating-Set have fpt-approximation with ratio $\rho(k)$?
Summary

- We give an fpt gap-producing reduction from $k$-CLIQUE to MAX-$k$-SUBSET-INTERSECTION.
- Under ETH, we can rule out moderate exponential approximation algorithms for MAX-$k$-SUBSET-INTERSECTION.
- Inapproximability of other natural parameterized problem.
  - $k$-DOMINATING-SET: no constant fpt-approximation

Open questions
- Does $k$-CLIQUE have constant fpt-approximation?
- Does $k$-DOMINATING-SET have fpt-approximation with ratio $\rho(k)$?
Thank You!