Addition is exponentially harder than counting for shallow monotone circuits

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Joint work with **Xi Chen** (Columbia) and **Rocco Servedio** (Columbia)

What is this talk about?

1. Exponential weights in bounded-depth monotone majority circuits.

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1. **Exponential weights in bounded-depth monotone majority circuits.**

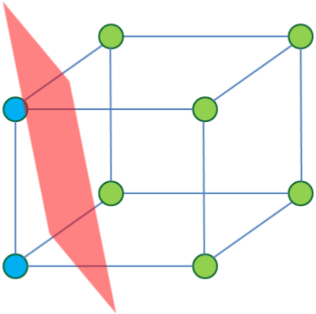
2. **The power of negation gates in bounded-depth AND/OR/NOT circuits.**

Part 1. Monotone majority circuits.

Weighted threshold functions

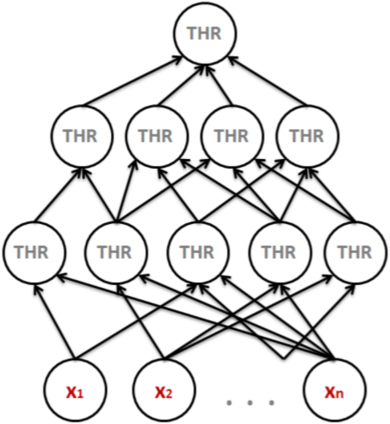
Def. $f\colon \{0,1\}^m \to \{0,1\}$ is a ***weighted threshold function*** if there are integers (**"weights"**) $w_1, \ldots, w_m$ and t such that

$$f(x) = 1 \quad \Leftrightarrow \quad \sum_{i=1}^{m} w_i x_i \geq t.$$



Threshold circuits: Definition

- Each internal gate computes a weighted threshold function.



- This circuit has **depth** 3 (# layers) and **size** 10 (# gates).

Threshold circuits: The frontier

Simple computational model whose power remains mysterious.

Open Problem. Can we solve **s-t-connectivity** using constant-depth polynomial size threshold circuits?

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However, relative success in understanding the role of large weights in the gates of the circuit:

"Exponential weights vs. polynomial weights".

Threshold Circuits vs. Majority Circuits

◦ **Majority circuits:** “We care about the weights.”

Example: $3x_1 - 4x_3 + 2x_7 - x_2 \overset{?}{\geq} 5.$

The weight of this gate is $3 + 4 + 2 + 1 = 10$**.**

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Size of Majority Circuit: Total weight in the circuit.

Polynomial weight is sufficient

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Simplification/better parameters:
[Hofmeister, 1996] and **[Amano and Maruoka, 2005]**.

[Goldmann and Karpinski, 1993]

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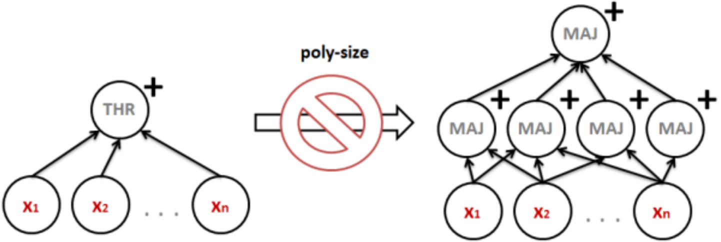
[GK'93] Is there a monotone transformation?

(Question recently reiterated by J. Hastad, 2010 & 2014)

Previous Work [Hofmeister, 1992]



No efficient monotone simulation in depth **2**:
Total weight must be $2^{\Omega(\sqrt{n})}$.

Our first result.

Solution to question posed by Goldmann and Karpinski:

No efficient monotone simulation in any fixed depth $d \in \mathbb{N}$.

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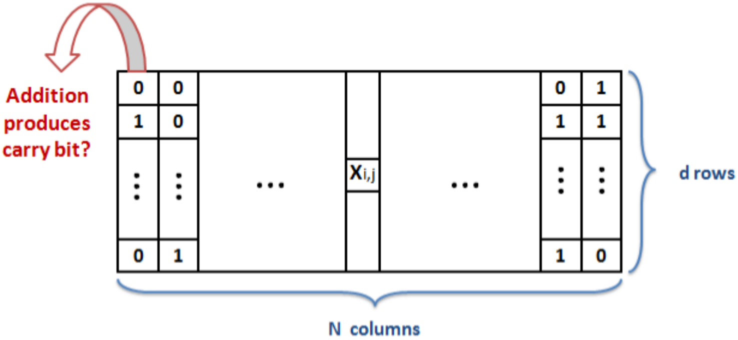
Our hard monotone threshold gate: $U_{d,N}$

Checks if the addition of d natural numbers (each with N bits) is at least 2^N.

The lower bound



$$U_{d,N}: \quad \sum_{j=0}^{N-1} 2^j (x_{1,j} + \ldots + x_{d,j}) \overset{?}{\geq} 2^N$$

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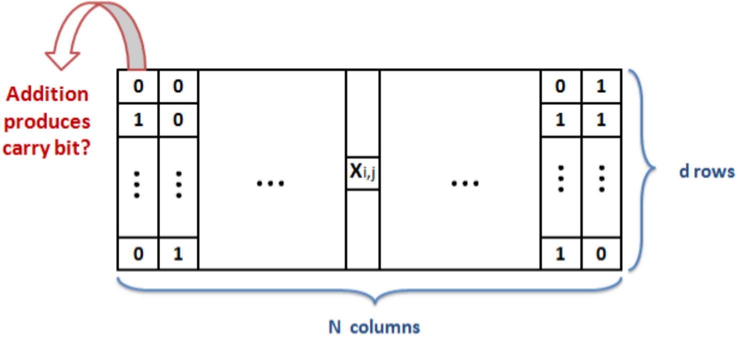

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Theorem 1. Any depth-d monotone MAJ circuit for $U_{d,N}$ has size $2^{\Omega(N^{1/d})}$.

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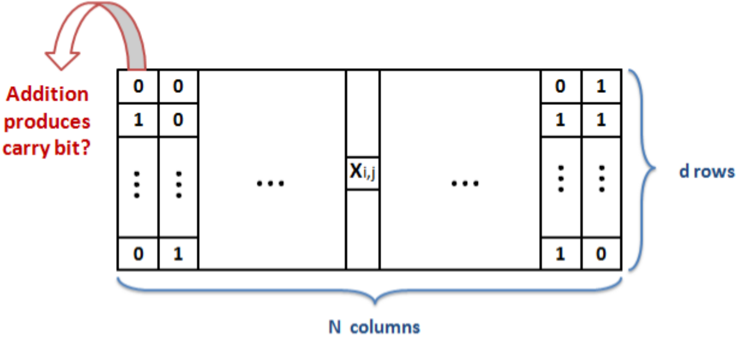

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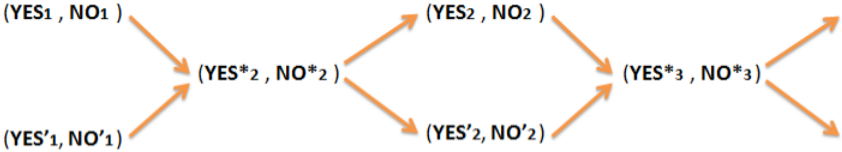
Theorem 1. Any depth-d monotone MAJ circuit for $U_{d,N}$ has size $2^{\Omega(N^{1/d})}$. Furthermore, there is a matching upper bound.

Our approach: pairs of pairs of distributions



Intuition: **YES*** distrib. supported over strings with sum $\geq 2^N$.
NO* distrib. supported over strings with sum $< 2^N$.

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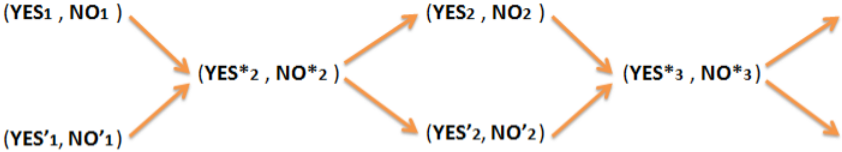

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Inductive Lemma. $\forall \ell \leq d$ any "small" depth-ℓ MAJ circuit $\mathcal{C}$ satisfies:

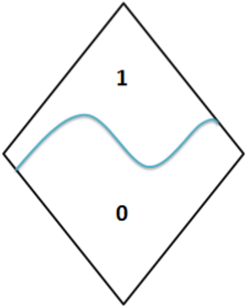
$$\Pr[\mathcal{C}(\mathrm{YES}^\star_\ell) = 1] + \Pr[\mathcal{C}(\mathrm{NO}^\star_\ell) = 0] \; < \; 1 + \frac{10^\ell}{10^d}.$$

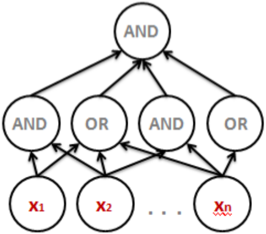
(Proof explores monotonicity and low weight in a crucial way.)

Part 2. Monotonicity and AC^0 circuits.

Monotone Complexity

Semantics vs. syntax:





Monotone Functions “ = ” Monotone Circuits

The Ajtai-Gurevich Theorem (1987)

◦ Motivated by question in Finite Model Theory.

There is **monotone** $g_n \colon \{0,1\}^n \to \{0,1\}$ such that:

- $g \in \mathrm{AC}^0$;
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Obs.: g_n computed by monotone AC^0 circuits of size $n^{O(\log n)}$.

Question.

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(Analogous question for **arbitrary** circuits answered positively **[Tardos, 1988]**.)

Our second result.

Theorem 2. There is a **monotone** $f_n \colon \{0,1\}^n \to \{0,1\}$ s.t.:

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Proof. Upper bound for our addition function $U_{k,N}$. □

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An interesting direction:

Formulation of a general theory to explain when non-monotone operations speed-up the computation of monotone functions (in bounded-depth complexity).

Thank you!