Derandomization via Robust Algebraic Circuit Lower Bounds

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based on various work with Amir Shpilka

(I am on the job market this year)

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Question

When do lower bounds for a circuit class $C$ yield efficient deterministic algorithms for deciding properties of circuits from $C$?

Motivations

- Derandomize algorithms
- $C$ may be too weak for derandomization via hardness versus randomness paradigm
- Better understanding
Question (Polynomial Identity Testing (PIT))

Given a polynomial \( f \in \mathbb{Q}[x_1, \ldots, x_n] \), is \( f \not\equiv 0 \)?

Polynomials are given by a small computational device, e.g. 
\( x^2 - y^2 = (x + y)(x - y) \) can be given by

\[
\begin{array}{c}
\times \\
+ \\
x \\
+ \\
-1 \\
x \\
y \\
x \\
y
\end{array}
\]
Polynomial Identity Testing (ii)

Lemma (Schwartz-Zippel)

\[ f \not\equiv 0 \text{ iff } f(\vec{\alpha}) \not= 0 \text{ for random } \vec{\alpha}. \]

- Gives a randomized algorithm for testing if \( f \not\equiv 0 \), only uses \( f \) as a black-box. Deterministic algorithms?
- **hitting set** (\( \equiv \) **black-box PIT**): set of points \( \mathcal{H} \subseteq \mathbb{Q}^n \) such that
  
  \[ f \not\equiv 0 \text{ iff } f|_\mathcal{H} \not\equiv 0, \text{ for computationally simple } f. \]
- Non-constructively \( |\mathcal{H}| = \text{small} \), constructively?
Algebraic formulas typically use addition ($\sum$) and multiplication ($\prod$), but we can also use addition ($\sum$) and \textbf{powering} ($\bigwedge$)

\[
xy = \frac{1}{4} \left( (x + y)^2 - (x - y)^2 \right),
\]

Have equivalence for arbitrary formulas, but not for low-depth.

A \textbf{depth-3 powering formula} ($\sum \bigwedge \sum$) is a sum of powers of linear forms

\[
f(x_1, \ldots, x_n) = \sum_{i=1}^{s} (\alpha_{i,0} + \alpha_{i,1}x_1 + \cdots + \alpha_{i,n}x_n)^{d_i}, \quad \alpha_{i,j} \in \mathbb{Q}.
\]

$\sum \bigwedge \sum$ are a moral analogue of CNFs/DNFs from boolean complexity.
Theorem (NisanWigderson96,Kayal08,Sylvester1851)

The monomial $x_1 \cdots x_n$ requires $2^{\Omega(n)}$ size to be computed as a $\sum \land \sum$ formula.

Theorem (F-Shpilka13)

If $f(x_1, \ldots, x_n) = \sum_{\overrightarrow{a}} \alpha_{\overrightarrow{a}} x_1^{a_1} \cdots x_n^{a_n}$ is computed by a size-$s$ $\sum \land \sum$ formula, then $f$ is determined by its restriction to monomials involving $O(\log s)$ variables. This implies a $\text{poly}(n, s)^{O(\log s)}$ size hitting set.

This is proven by making the above lower bound robust.
The lower bound follows from a complexity measure argument. For $f \in \mathbb{Q}[x_1, \ldots, x_n]$ define

$$\mu(f) := \dim \text{span} \left\{ \frac{\partial}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}} f \right\}_{a_1, \ldots, a_n \geq 0}$$

**facts:**

- $\mu(f + g) \leq \mu(f) + \mu(g)$.
- $\mu\left((\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n)^d\right) \leq d + 1$.
- $\mu(x_1 \cdots x_n) = 2^n$.

$\implies$ $x_1 \cdots x_n$ needs size $2^{\Omega(n)}$ to be computed as

$$x_1 \cdots x_n = \sum_{i=1}^{s} (\alpha_{i,0} + \alpha_{i,1} x_1 + \cdots + \alpha_{i,n} x_n)^{d_i},$$

if $d_i \leq \text{poly}(n)$. 


Lower Bounds for Depth-3 Powering Formulas (iii)

Lower bound: small $\sum \land \sum$ formula cannot exactly equal large monomials. Approximately?

$$(x_1 + \cdots + x_n)^n = x_1 \cdots x_n + \cdots + x_1^n + \cdots + x_n^n + \cdots.$$ 

Express $f \neq 0$ as

$$f = \alpha x_1^{a_1} \cdots x_n^{a_n} + \text{lower order terms},$$

where monomials are ordered lexicographically with $x_1 \succ \cdots \succ x_n$

**fact:** $\mu(f) \geq \mu(x_1^{a_1} \cdots x_n^{a_n})$ — the measure $\mu$ is robust

$\implies$ the leading monomial of a small $\sum \land \sum$ formula involves few variables [F-Shpilka13]

$\implies$ quasipolynomial time deterministic blackbox PIT for $\sum \land \sum$
Conclusions

Robust Lower Bound:

\[ \mu(\text{extrema}(f) + \text{lower terms}) \geq \mu(\text{extrema}(f)) \].

Other PIT via Robust Lower Bounds:

- [SV09]: Read-Once Formula
- [FS12]: (commutative) read-once algebraic branching programs
- [MRS14, F15]: sums of powers of low-degree polynomials
- [GKST15, F15]: sparse polynomials

Open questions:

- polynomial-size hitting set for \( \sum \land \sum \) formula? best known is
  \( s^{O(\lg \lg s)} \) for size \( s \) [FSS14]