Quantum Hamiltonian Complexity Reunion Workshop
Berkeley, CA  5/4/2015

Quantum information theory with overlapping qubits

\[ \mathcal{H} = (\mathbb{C}^2)^{\otimes n} \]
Problem:
How to “pull apart” qubits that are almost in tensor product...?

qubits: $X, Y, Z \quad X, Y, Z \quad X, Y, Z
\quad X, Y, Z \quad X, Y, Z$

Motivation:

Goal: Establish substantial entanglement (with structure and control) between two untrusted devices—by giving them black-box tests.
**Approach:** Repeated CHSH games

- If devices pass the tests (w.h.p.), then they must share lots of entanglement, which they measure in a very particular way.

**Main Theorem:** [R., Unger, Vazirani '18]

\[
\begin{array}{ccccccc}
\times & \times & \times & \times & \times & \times & \times \\
n \text{games} & n \text{ games} & n \text{ games} & \ldots & n \text{ games}
\end{array}
\]

- If: \( \Pr[\text{win} \geq w - \epsilon \text{ of games}] \geq 1-\epsilon \)

- Then: At the beginning of a random block of \( n \) games,

  Alice & Bob's strategy \( \approx \) Ideal strategy \( \begin{pmatrix} 100 \end{pmatrix}, \begin{pmatrix} 111 \end{pmatrix} \) for those games

  pair \( j \) for game \( j \)

**Proof Idea:**
Start with arbitrary shared state, arbitrary strategy.

- Find subsequence of sequential games so
$y_j, \quad P\{\text{win game } j \} = w^*$
regardless of games 1 to $j-1$

(2) Find near-EPR pairs,
move them into tensor product,
glu together history dependence,
replace with perfect EPR pairs,

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**Big picture:**

(Operational, approximate)

Indepedence assumption  \[ \rightarrow \]  EPR pairs in tensor product

\[
\begin{align*}
\text{unstructured } & \mathcal{H} \\
\rightarrow & \quad \emptyset \otimes \emptyset \otimes \emptyset \otimes \emptyset \otimes \emptyset
\end{align*}
\]

This gives lots of entanglement, in a very nice form
but inefficiently: $N = \text{poly}(n)$,
final error $= \varepsilon^r$.

**Open question:**
• Make the test more efficient:
  Can we determine a tensor-product structure even starting with a constant noise rate?

Outline:
State-independent separation
State-dependent separation

State independent:
\[ X, Y, Z \]
\[ X, Y, Z \]
\[ X, Y, Z \]

State dependent:
\[ add \ 147 \]

(State-independent)
Separation of overlapping qubit operators
Given: Pauli operators
\[ \sigma_x', \sigma_z', \ldots, \sigma_x^n, \sigma_z^n \]
almost commuting:
\[ \| [\sigma_x^i, \sigma_z^j] \| < \varepsilon \]

Goal: Find nearby operators (possibly on an extended space) so
\[ [\tilde{\sigma}_x^i, \tilde{\sigma}_z^j] = 0, \]
i.e., operators for qubits in tensor product.

Claim: \(\Theta(n \varepsilon)\) movement is sufficient & necessary.

Nearby = \(\Theta(n \varepsilon)\)

Remark: Related work (more sophisticated)

Lin '95: almost-commuting Hermitian pairs close to commuting Hermitians

if \(\| [A, B] \| < \varepsilon\), \exists \( A' \approx A, B' \approx B \) Hermitian, Hermitian, \([A', B'] = 0\)

Hastings '10: \( \delta = \Theta(\varepsilon^{1/6}) \) dimension-independent!
Hastings 10: $d = O(\epsilon^{-\epsilon})$. dimension-independent!

- false for almost-commuting unitary pairs
- false for almost-commuting Hermitian triples
  (no dimension-independent bound)
- but Paulis have much more structure
  Hermitian & unitary $\Rightarrow$ e-values $\pm 1$

Upper bound: $O(n\epsilon)$ is enough.

Proof: * 2 qubits

\[ \sigma_2^{\pm} = \text{SWAP}_{0,1}, \sigma_2^{\pm} \text{SWAP}_{1,2} \approx \sigma_2^{\pm} \]

* $n$ qubits
  - $\text{SWAP}_{0,1}$ to fix qubits 2 to $n$
  - $\text{SWAP}_{0,2}$ to fix qubits 3 to $n$
  - $\text{SWAP}_{0,3}$ for qubits 4 to $n$
  \[ \ldots \]
  \[ \sigma_2^{\pm} = \text{SWAP}_{0,1} \sigma_2^{\pm} \text{SWAP}_{1,2} \ldots \text{SWAP}_{n-1,n} \approx \sigma_2^{\pm} \]
\[ \sigma_2^z = \text{SWAP}_0, \sigma_2^z \text{ SWAP}_0, \text{ \checkmark \ commutator} \Rightarrow \sigma_3 \]

\[ \Rightarrow \sigma^n \text{ moves } n \text{E total } \checkmark \]

**Lower bound:** \( \Omega(nE) \) is sometimes required

**Idea:**

\[ e \text{ overlaps } \eta/2 \quad \eta/2 \]

**Proposal 1:** \( \ln (\mathbb{C}^2)^{\otimes \eta} \)

\[ \bullet \times \bullet \times \bullet \text{ usual qubits} \]

\[ U \bullet \times \bullet \times \bullet \text{ } U^\dagger \text{ same, conjugated by a random unitary} \]

**Intuition:** Overlap \[ \| [\sigma^z_2, \sigma^z_\beta] \| \approx \frac{1}{\sqrt{\eta/2}} \] ??

Wrong! **Fact:** \( \ln (\mathbb{C}^2)^{\otimes \eta} \), for random unitary \( U \),

\[ \| [\sigma^z_2, U \sigma^z_\beta U^\dagger] \| \approx 2 \text{ (maximal)} \]

**Moral:** Qubit overlap is not "monogamous."
Moral: Qubit overlap is not monogamous.

huge overlaps ⇒ need more structure

Proposal 2: \( \ln \left( \mathbb{C}^2 \right)^{\otimes n} \)

perturb, eg., by \( e^{-i \epsilon H}, \quad H = \frac{\sqrt{2}}{j=1} X_j \)

\( \Rightarrow \tilde{X}_j = X_j, \quad \tilde{Z}_j \approx Z_j + \epsilon \sum_{\ell=1}^{n/2} X_\ell \)

Track total commutator

\[
C = \sum_{i \neq j} \| [\tilde{Z}_i, \sigma_\ell] \|_F
\]

\( \approx (\frac{n}{2}) \times \epsilon \) initially

Claim: Move qubit by \( \delta \) (spectral norm)

\( \Rightarrow C \) changes by \( O(\delta) \).

\( \Rightarrow \Omega(n^2 \epsilon) \) total change needed to make \( C = 0 \).

Proof:

\[
\sum_{i=1}^{n/2} \left\| \left[ Z_k + \epsilon \sum_{\ell=1}^{n/2} X_\ell + \Delta, Z_i \right] \right\|_F
= \sum_{i \in \mathcal{M}_k} \left\| \left[ \epsilon \sum_{\ell=1}^{n/2} X_\ell, Z_i \right] + [\Delta, Z_i] \right\|_F
\]
(State-dependent)

Separation of overlapping qubits

Goal: Understand error accumulation in analysis

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\[ \Delta = \frac{-f}{n} \sum_{i \leq n/2} X_i \]
\[ \approx n(\varepsilon - \frac{f}{n}) = n\varepsilon - f \]
Possible theorem statements

Must we lose a factor of $n$?

Does fixing one qubit help/hurt the others?

**General problem:**

Given: $\text{Tr}_A 14 \times 41 = \rho^*$
$\text{Tr}_B 14 \times 41 = \rho^*$
$\text{Tr}_C 14 \times 41 = \rho^*$

Goal: Understand effect of "fixing" $147$ so (changing state, not qubits)
Example: \( \rho^* = \text{EPR pair } |10\rangle + |11\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \)

Toy problem: Depolarizing qubits

\[ \rho^* = \frac{1}{2} |0\rangle \langle 0| \] maximally mixed state

"Fixing" a qubit in \( |1\rangle \) \( \rightarrow \) depolarize it

Question:
Does depolarizing one qubit help or hurt the others?

Intuition?: Adding noise on one qubit can only make overlapping qubits more noisy.

Claim:
\[ |1\rangle \] \( \rightarrow \) \( \frac{1}{2} |0\rangle \langle 0| \) initially

Depolarize \( |1\rangle \) qubits

\[ |1\rangle = 10\rangle \] finally
Moral: Adding noise to some qubits can purify others.

Proof:

\[ |\Psi\rangle = |00\rangle + |11\rangle, \quad \bigcirc = \text{first qubit} \]

1) Depolarize \( \bigcirc \)

\[
\begin{align*}
Z_L &= 1 \\
X_L &= \frac{1}{2} \\
= & \frac{1}{2} \langle 00| + \frac{1}{2} \langle 11|
\end{align*}
\]

\[
\begin{align*}
ev_{\langle 00|} &= \frac{1}{2} \\
ev_{\langle 11|} &= \frac{1}{2}
\end{align*}
\]

\[
|\Psi\rangle = \frac{1}{2} |00\rangle \otimes |00\rangle + \frac{1}{2} |11\rangle \otimes |11\rangle
\]

2) (Generalized) Zeno effect:

\[
\begin{align*}
\text{Any state } &\frac{1}{2}, \frac{1}{2}, 0, 0 \\
\rightarrow & \text{slowly rotating depolarizations}
\end{align*}
\]

\[
\begin{align*}
\text{Any other } &\frac{1}{2}, \frac{1}{2}, 0, 0 \\
\rightarrow & \text{pure measurements dephasing}
\end{align*}
\]

More problems:

- Extend analysis to more states
- Connect to operational assumptions, e.g., parallel CHSH games with constant noise