Characterization of cutoff for reversible Markov chains
Yuval Peres

Joint work with Riddhi Basu and Jonathan Hermon

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Transition matrix - $P$ (reversible).

Stationary dist. - $\pi$.

Reversibility: $\pi(x)P(x,y) = \pi(y)P(y,x), \forall x, y \in \Omega$.

Laziness $P(x,x) \geq 1/2, \forall x \in \Omega$. 
For any 2 dist. $\mu, \nu$ on $\Omega$, their **total-variation distance** is:

$$\|\mu - \nu\|_{TV} \overset{d}{=} \max_{A \subseteq \Omega} \mu(A) - \nu(A).$$

$$d(t, x) \overset{d}{=} \|P_t^x - \pi\|_{TV}, \quad d(t) \overset{d}{=} \max_{x \in \Omega} d(t, x).$$
TV distance

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\[ \|\mu - \nu\|_{TV} \overset{d}{=} \max_{A \subset \Omega} \mu(A) - \nu(A). \]

- The $\epsilon$-mixing-time $(0 < \epsilon < 1)$ is:

\[ t_{\text{mix}}(\epsilon) \overset{d}{=} \min \{ t : d(t) \leq \epsilon \} \]

\[ t_{\text{mix}} \overset{d}{=} t_{\text{mix}}(1/4). \]
Def: a sequence of MCs \((X_t^{(n)})\) exhibits \textit{cutoff} if

\[
t_{\text{mix}}^{(n)}(\epsilon) - t_{\text{mix}}^{(n)}(1 - \epsilon) = o(t_{\text{mix}}^{(n)}), \quad \forall 0 < \epsilon < 1/4.
\] (1)
Def: a sequence of MCs \((X_t^{(n)})\) exhibits **cutoff** if

\[
t^{(n)}(\epsilon) - t^{(n)}(1 - \epsilon) = o(t^{(n)}_{\text{mix}}), \quad \forall 0 < \epsilon < 1/4.
\]  

\((w_n)\) is called a **cutoff window** for \((X_t^{(n)})\) if: \(w_n = o\left(t^{(n)}_{\text{mix}}\right)\), and

\[
t^{(n)}(\epsilon) - t^{(n)}(1 - \epsilon) \leq c_{\epsilon} w_n, \quad \forall n \geq 1, \forall \epsilon \in (0, 1/4).
\]
Cutoff

Figure: cutoff
Cutoff was first identified for random transpositions Diaconis & Shashahani 81 and RW on the hypercube by Aldous 83.
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The name cutoff was coined by Aldous and Diaconis in their seminal 86 paper.

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Spectral gap & relaxation-time

Let $\lambda_2$ be the largest non-trivial e.v. of $P$.

Definition: $\text{gap} = 1 - \lambda_2$ - the **spectral gap**.

Def: $t_{\text{rel}} := \text{gap}^{-1}$ - the **relaxation-time**.
In a 2004 Aim workshop I proposed that **The product condition (Prod. Cond.)** -
\[ \text{gap}^{(n)} t_{\text{mix}}^{(n)} \to \infty \text{ (equivalently, } t_{\text{rel}}^{(n)} = o(t_{\text{mix}}^{(n)}) \text{) } \]
should imply cutoff for "nice" reversible chains.

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- It is not always sufficient - examples due to Aldous and Pak.

- Problem: Find families of MCs s.t. \textbf{Prod. Cond.} \( \implies \) cutoff.
Aldous’ example

- The mass is concentrated in a small neighborhood of \( y \).
- Last step away from \( z \) before \( T_y \) “determines” \( T_y \).

Figure: Fixed bias to the right conditioned on a non-lazy step.

Different laziness probabilities along the 2 paths.

- \( t_{\text{rel}}^{(n)} = O(1) \).
- \( d_n(t) \sim P_x[T_y > t] \implies \epsilon \leq d_n(130n) \leq d_n(128n) \leq 1 - \epsilon \), for some \( \epsilon \).
Aldous’ example

Figure: Fixed bias to the right conditioned on a non-lazy step.

\[ d_n(t) \]
Def: The **hitting time** of a set $A \subset \Omega = T_A := \min\{t : X_t \in A\}$.
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Hitting times of “worst” sets are related to mixing - mid 80’s (Aldous).

Refined independently by Oliviera (2011) and Peres-Sousi (2011) (case $\alpha = 1/2$ due to Griffiths-Kang-Oliviera-Patel 2012): for any irreducible reversible lazy MC and $0 < \alpha \leq 1/2$:

$$t_H(\alpha) = \Theta_\alpha(t_{\text{mix}}), \text{ where } t_H(\alpha) := \max_{x, A: \pi(A) \geq \alpha} \mathbb{E}_x[T_A]. \quad (2)$$
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We relate $d(t)$ and $\max_{x,A: \pi(A) \geq \alpha} \mathbb{P}_x[T_A > t]$ and refine (2) by also allowing $1/2 < \alpha \leq 1 - \exp[-Ct_{\text{mix}}/t_{\text{rel}}]$ and improving $\Theta_{\alpha}$ to $\Theta$.

Remark: (2) may fail for $\alpha > 1/2$. 

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counter-example

Figure: $n$ is the index of the chain
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Diaconis & Saloff-Coste (06) (separation cutoff) and Ding-Lubetzky-Peres (10) (TV cutoff):

A seq. of BD chains exhibits cutoff iff the Prod. Cond. holds.
Concentration of hitting times of “worst” sets is related to cutoff in birth and death (BD) chains.

Diaconis & Saloff-Coste (06) (separation cutoff) and Ding-Lubetzky-Peres (10) (TV cutoff):

A seq. of BD chains exhibits cutoff iff the Prod. Cond. holds.

We extend their results to weighted nearest-neighbor RWs on trees.
Cutoff for trees

**Theorem**

Let \((V, P, \pi)\) be a lazy Markov chain on a tree \(T = (V, E)\) with \(|V| \geq 3\). Then

\[
t_{\text{mix}}(\epsilon) - t_{\text{mix}}(1 - \epsilon) \leq C \sqrt{\log \epsilon} t_{\text{rel}} t_{\text{mix}}, \text{ for any } 0 < \epsilon \leq 1/4.
\]

In particular, the Prod. Cond. implies cutoff with a cutoff window \(w_n = \sqrt{t_{\text{rel}}^{(n)} t_{\text{mix}}^{(n)}}\) and \(c_\epsilon = C \sqrt{\log \epsilon}\). 

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- Ding Lubetzky Peres (10) - For BD chains \(t_{\text{mix}}(\epsilon) - t_{\text{mix}}(1 - \epsilon) \leq O(\epsilon^{-1} \sqrt{t_{\text{rel}}t_{\text{mix}}})\) and in some cases \(w_n = \Omega \left( \sqrt{t_{\text{rel}}(n) t_{\text{mix}}(n)} \right)\).
To mix - escape and then relax

Definition: $\text{hit}_\alpha := \text{hit}_\alpha(1/4)$, where

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\text{hit}_{\alpha,x}(\epsilon) := \min\{t : P_x[T_A > t] \leq \epsilon : \text{for all } A \subset \Omega \text{ s.t. } \pi(A) \geq \alpha\},
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- **Easy direction:** to mix, the chain must first escape from small sets = “first stage of mixing”. 

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- Easy direction: to mix, the chain must first escape from small sets = “first stage of mixing”.

- Loosely speaking - we show that in the 2nd “stage of mixing”, the chain mixes at the fastest possible rate (governed by its relaxation-time).
Fact: Let $A \subset \Omega$ be such that $\pi(A) \geq 1/2$. Then (under reversibility)

$$P_\pi[T_A > 2st_{rel}] \leq \frac{e^{-s}}{2}, \text{ for all } s \geq 0.$$
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By a coupling argument,

$$P_x[T_A > t + 2s_{rel}] \leq d(t) + P_\pi[T_A > 2s_{rel}].$$
Hitting of worst sets

For any reversible irreducible finite lazy chain and any $0 < \epsilon \leq 1/4$, 

$$\text{hit}_{1/2}(3\epsilon) - |t_{\text{rel}} \log(2\epsilon)| \leq t_{\text{mix}}(2\epsilon) \leq \text{hit}_{1/2}(\epsilon) + |t_{\text{rel}} \log(4\epsilon)|$$

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- Terms involving $t_{\text{rel}}$ are negligible under the Prod. Cond.

- A similar two sided inequality holds for $t_{\text{mix}}(1 - 2\epsilon)$. 
Main abstract result

Definition: A sequence has $\text{hit}_\alpha$-cutoff if

$$\text{hit}^{(n)}(\epsilon) - \text{hit}^{(n)}(1 - \epsilon) = o(\text{hit}^{(n)}) \text{ for all } 0 < \epsilon < 1/4.$$
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Theorem

Let $(\Omega_n, P_n, \pi_n)$ be a seq. of finite reversible lazy MCs. Then TFAE:

- The seq. exhibits cutoff.
- The seq. exhibits a $\text{hit}_\alpha$-cutoff for some $\alpha \in (0, 1/2)$.
- The seq. exhibits a $\text{hit}_\alpha$-cutoff for some $\alpha \in [1/2, 1)$ and the Prod. Cond. holds.

Joint work with Riddhi Basu and Jonathan Hermon
Main abstract result

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\text{hit}_\alpha(n)(\epsilon) - \text{hit}_\alpha(n)(1 - \epsilon) = o(\text{hit}_\alpha(n)) \quad \text{for all} \quad 0 < \epsilon < 1/4.
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The equivalence of cutoff to \( \text{hit}_{1/2} \)-cutoff under the Prod. Cond. follows from the ineq. from the prev. slide together with the fact that \( \text{hit}_{1/2}(n) = \Theta(t_{\text{mix}}(n)) \).
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For general $\alpha$ we show under the Prod. Cond. (using the tail decay of $T_A/t_{\text{rel}}$ when $X_0 \sim \pi$):

$$\text{hit}_\alpha$$-cutoff for some $\alpha \in (0, 1) \implies \text{hit}_\beta$$-cutoff for all $\beta \in (0, 1)$. 

Joint work with Riddhi Basu and Jonathan Hermon
Def: For $f \in \mathbb{R}^\Omega$, $t \geq 0$, define $P^t f \in \mathbb{R}^\Omega$ by

$$P^t f(x) := \mathbb{E}_x[f(X_t)] = \sum_y P^t(x, y)f(y).$$

The following is well-known and follows from elementary linear-algebra.

**Lemma (Contraction Lemma)**

Let $(\Omega, P, \pi)$ be a finite rev. irr. lazy MC. Let $A \subset \Omega$. Let $t \geq 0$. Then

$$\text{Var}_\pi P^t 1_A \leq e^{-\frac{2t}{t_{\text{rel}}}}.$$ (3)

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Tools

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Maximal Inequality

The main ingredient in our approach is Starr’s maximal-inequality (66) (refines Stein’s max-inequality (61))

Theorem (Maximal inequality)

Let $(\Omega, P, \pi)$ be a lazy irreducible reversible Markov chain. Let $f \in \mathbb{R}^{\Omega}$. Define the corresponding maximal function $f^* \in \mathbb{R}^{\Omega}$ as

$$f^*(x) := \sup_{0 \leq k < \infty} |P^k(f)(x)| = \sup_{0 \leq k < \infty} |E_x[f(X_k)]|.$$

Then for $1 < p < \infty$,

$$\|f^*\|_p \leq q\|f\|_p \quad \frac{1}{p} + \frac{1}{q} = 1$$

(4)
Combining the Max-in. with the Contraction Lemma

Goal: want for every $A \subset \Omega$ to have $G = G_s(A) \subset \Omega$ s.t. $T_G \leq t$ serves as a certificate of “being $\epsilon$-mixed w.r.t. $A$” and to control its $\pi$ measure from below.
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- Let $\sigma_s := e^{-s/t_{\text{rel}}} \geq \sqrt{\text{Var}_\pi P^s 1_A}$ (contraction lemma).
- Consider

$$G = G_s(A) := \left\{ g : \forall \tilde{s} \geq s, |P^\tilde{s}_g(A) - \pi(A)| \leq 4\sigma_s \right\}.$$
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- Want precision $4\sigma_s = \epsilon \implies s := t_{rel} \times \log(4/\epsilon)$.
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Claim

$$\pi(G') \geq 1/2. \quad (5)$$
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Claim

$$\pi(G) \geq 1/2. \quad (5)$$

**Proof:** Set $f_s := P^s (1_A - \pi(A))$. Then

$$G^c \subset \{ f_s^* > 4\|f_s\|_2 \}.$$  

Apply Starr’s inequality.
Main idea

Claim:

\[ t_{\text{mix}}(2\epsilon) \leq \text{hit}_{1/2}(\epsilon) + t_{\text{rel}} \times \log(4/\epsilon). \]

Proof: Recall

\[ G := G_s(A, m) := \left\{ g : \forall \tilde{s} \geq s, |P_{\tilde{s}}^g(A) - \pi(A)| \leq \epsilon \right\}, \ s := t_{\text{rel}} \times \log(4/\epsilon) \]

Set \( t := \text{hit}_{1/2}(\epsilon) \). By prev. claim \( \pi(G) \geq 1/2 \implies P_x[T_G > t] \leq \epsilon \) (by def. of \( t \)).
Main idea

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- For any \( x, A \):
  \[ |P_x^{t+s}(A) - \pi(A)| \leq P_x[T_G > t] + \max_{g \in G, \tilde{s} \geq s} |P_{\tilde{s}}g(A) - \pi(A)| \leq 2\epsilon. \]
Let: $T := (V, E)$ be a finite tree.

$(V, P, \pi)$ a lazy MC corresponding to some (lazy) weighted nearest-neighbor walk on $T$ (i.e. $P(x, y) > 0$ iff $\{x, y\} \in E$ or $y = x$).

Fact: (Kolmogorov’s cycle condition) every MC on a tree is reversible.
Can the tree structure be used to determine the identity of the “worst” sets?
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Easier question: what set of $\pi$ measure $\geq 1/2$ is the “hardest” to hit in a birth & death chain with state space $[n] := \{1, 2, \ldots, n\}$?
Can the tree structure be used to determine the identity of the “worst” sets?

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Answer: take a state $m$ with $\pi([m]) \geq 1/2$ and $\pi([m - 1]) < 1/2$. Then the set worst set would be either $[m]$ or $[n] \setminus [m - 1]$. 
How to generalize this to trees?
Central vertex

Figure: A vertex $o \in V$ is called a **central-vertex** if each connected component of $T \setminus \{o\}$ has stationary probability at most $1/2$. 
There is always a central-vertex (and at most 2). We fix one, denote it by \( o \) and call it the root.
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It follows from our analysis that for trees the Prod. Cond. holds iff $T_o$ is concentrated (from worst leaf).

A counterintuitive result $\iff \exists$ such unweighed trees (Peres-Sousi (13)).
Let $A$ be s.t. $\Pi(A) \geq 1/2$. Partition $V$ to $B$ and $D = V \setminus B$ s.t. $B$ is connected, $o$ is in $B$ and $\Pi(A') \geq 1/4$, where $A' := (D \cup \{o\}) \cap A$.

\[
P_o[T_A > s] \leq P_o[T_{A'} > s] \leq P_{\Pi_B}[T_{A'} > s] \leq 2P_{\Pi}[T_{A'} > s],
\]
where $\Pi_B$ is $\Pi$ conditioned on $B$.

Take $s := C_{rel}|\log(\varepsilon)|$.

\[
\Rightarrow P_o[T_A > s] \leq \varepsilon.
\]

\[
\Rightarrow \text{hit}_{1/2}(a + \varepsilon) \leq \min\{t : P_x[T_o > t] \leq a, \text{ for all } x\} + s.
\]

trivially: $\min\{t : P_x[T_o > t] \leq a, \text{ for all } x\} \leq \text{hit}_{1/2}(a)$

Figure: Hitting the worst set is roughly like hitting $o$. 

Joint work with Riddhi Basu and Jonathan Hermon

Characterization of cutoff for reversible Markov chains
Cutoff would follow if we show that $T_o$ is concentrated (under the Prod. Cond.).

More precisely, we need to show that $\mathbb{E}_x[T_o] = \Omega(t_{\text{mix}}) \implies T_{y\beta}(x)$ is concentrated if $X_0 = x$. 
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More precisely, we need to show that $\mathbb{E}_x[T_0] = \Omega(t_{\text{mix}}) \implies T_{y\beta}(x)$ is concentrated if $X_0 = x$. 

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Figure: Let \( v_0 = x, v_1, \ldots, v_k = o \) be the vertices along the path from \( x \) to \( o \).
Trees

Proof of Concentration: \( \text{Var}_x[T_o] \leq C t_{\text{rel}} t_{\text{mix}} \):

- It suffices to show that \( \text{Var}_x[T_o] \leq 4 t_{\text{rel}} \mathbb{E}_x[T_o] \).
Proof of Concentration: $\text{Var}_x[T_o] \leq C t_{\text{rel}} t_{\text{mix}}$

- It suffices to show that $\text{Var}_x[T_o] \leq 4 t_{\text{rel}} \mathbb{E}_x[T_o]$.

- If $X_0 = x$ then $T_o$ is the sum of crossing times of the edges along the path between $x$: $\tau_i := T_{v_i} - T_{v_{i-1}} \overset{d}{=} T_{v_i} \text{ under } X_0 = v_{i-1}$
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- $\tau_1, \ldots, \tau_k$ are independent $\implies$ it suffices to bound the sum of their 2nd moments

$$\text{Var}_x[T_o] = \sum \text{Var}_x[\tau_i] = \sum \text{Var}_{v_{i-1}}[T_{v_i}] \leq \sum \mathbb{E}_{v_{i-1}}[T_{v_i}^2].$$
Trees

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- $\tau_1, \ldots, \tau_k$ are independent $\implies$ it suffices to bound the sum of their 2nd moments $\text{Var}_x[T_o] = \sum \text{Var}_x[\tau_i] = \sum \text{Var}_{v_{i-1}}[T_{v_i}] \leq \sum \mathbb{E}_{v_{i-1}}[T_{v_i}^2]$.

- Denote the subtree rooted at $v$ (the set of vertices whose path to $o$ goes through $v$) by $W_v$. For $A \subset \Omega$ let $\pi_A$ be $\pi$ conditioned on $A$.

- Kac formula implies that for any $A$, there exists a dist. $\mu$ on the external vertex boundary of $A$ s.t. $E_{\mu}[T_A^2] \leq 2 E_{\mu}[T_A] E_{\pi_A \mid c}[T_A] \implies$

- By the tree structure $E_{v_{i-1}}[T_{v_i}^2] \leq 2 E_{v_{i-1}}[T_{v_i}] E_{\pi_{W_{v_{i-1}}}}[T_{v_i}]$.

- Not hard to show $E_{\pi_{W_{v_{i-1}}}}[T_{v_i}] \leq 2 t_{\text{rel}}$ (generally $\pi(A^c) E_{\pi_A \mid c}[T_A] \leq t_{\text{rel}}$) so $\sum E_{v_{i-1}}[T_{v_i}^2] \leq \sum 4 t_{\text{rel}} E_{v_{i-1}}[T_{v_i}] = 4 t_{\text{rel}} \mathbb{E}_x[T_o]$. \qed
The tree assumption can be relaxed. In particular, we can treat jumps to vertices of bounded distance on a tree (i.e. the length of the path from $u$ to $v$ in the tree (which is now just an auxiliary structure) is $> r \implies P(u, v) = 0$) under some mild necessary assumption.
The tree assumption can be relaxed. In particular, we can treat jumps to vertices of bounded distance on a tree (i.e. the length of the path from \( u \) to \( v \) in the tree (which is now just an auxiliary structure) is \( > r \) \( \Rightarrow P(u, v) = 0 \)) under some mild necessary assumption.

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Beyond trees

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- Previously the BD assumption could not be relaxed mainly due to it being exploited via a representation of hitting times result for BD chains.

- In particular, if $P(u, v) \geq \delta > 0$ for all $u, v$ s.t. $d_T(u, v) \leq r$ (and otherwise $P(u, v) = 0$), then

  $\text{cutoff} \iff \text{the Prod. Cond. holds.}$
Previously “good expansion of small sets can improve mixing”.

Now know - considering expansion only of small sets and $t_{rel}$ suffices to bound $t_{mix}$!

$$t_{mix}(\epsilon) \leq \text{hit}_{1-\epsilon/4}(3\epsilon/4) + \frac{3t_{rel}}{2} \log \left(\frac{4}{\epsilon}\right).$$

From which it follows that

$$t_{mix} \leq 5 \max_{x, A: \pi(A) \geq 1 - \epsilon/4} \mathbb{E}_x[T_A] + \frac{3t_{rel}}{2} \log \left(\frac{4}{\epsilon}\right).$$

For any $x$ and $A$ with $\pi(A) \geq 1 - \epsilon/4$ we can bound $\mathbb{E}_x[T_A]$ using the expansion profile of sets only of $\pi$ measure at most $\epsilon/4$ (by an integral of the form used to bound the mixing time via the expansion profile).
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For any $x$ and $A$ with $\pi(A) \geq 1 - \varepsilon/4$ we can bound $\mathbb{E}_x[T_A]$ using the expansion profile of sets only of $\pi$ measure at most $\varepsilon/4$ (by an integral of the form used to bound the mixing time via the expansion profile).

In practice, we can take $\varepsilon = \exp[-ct_{mix}/t_{rel}]$ to determine $t_{mix}$ up to a constant.
What can be said about the geometry of the “worst” sets in some interesting particular cases (say, transitivity or monotonicity)?

When can the worst sets be described as \( \{ |f| \leq C \} \)? (would imply several new cutoff results if true in certain cases)

When can one relate escaping time from balls of \( \pi \)-measure \( \epsilon \) to escaping time from sets of \( \pi \)-measure \( \epsilon \)?

When can monotonicity w.r.t. a partial order (preserved by the chain) be used to describe the “worst” sets and their hitting time distributions?
Open problems

- What can be said about the geometry of the “worst” sets in some interesting particular cases (say, transitivity or monotonicity)?

- When can the worst sets be described as $\{|f_2| \leq C\} (P f_2 = \lambda_2 f_2)$? (would imply several new cutoff results if true in certain cases)

- When can one relate escaping time from balls of $\pi$-measure $\epsilon$ to escaping time from sets of $\pi$-measure $\epsilon^{100}/100$?

- When can monotonicity w.r.t. a partial order (preserved by the chain) be used to describe the “worst” sets and their hitting time distributions?

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