A Robust Form of Kruskal's Identifiability Theorem

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Joint work with

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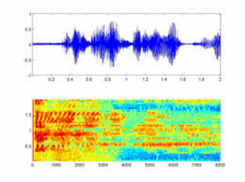
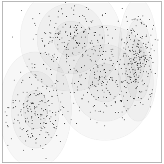
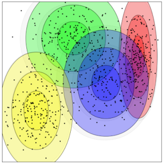
Background: “understanding” data

Data from various sources:

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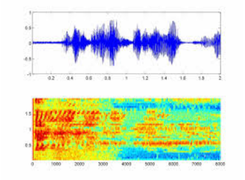
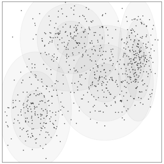
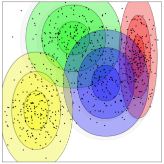

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Very successful; e.g. topic models for documents, hidden markov models for speech, . . .

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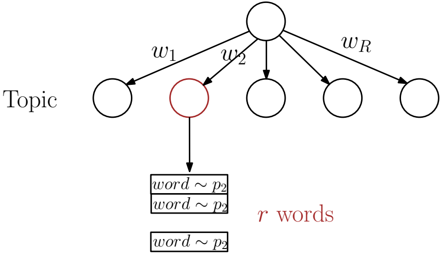
Model parameters:

- probability that document is on topic i : w_i (sum to 1)
- word probability vector for topic i : $p_i \in \mathbb{R}^n$

Topic model for documents

Generating a document: say r-word document



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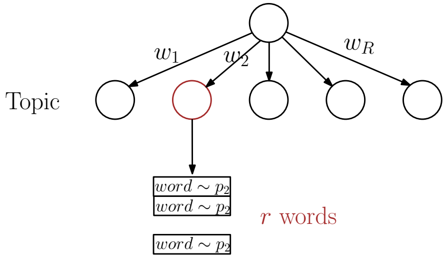

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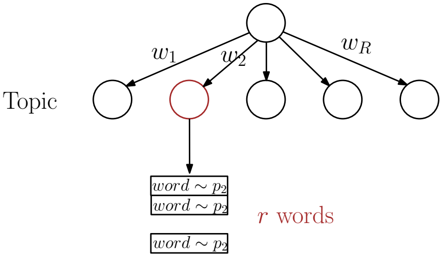

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Goal: find the $\{w_r, p_r\}$

What about tensors?

Experiment: Pick three random words from random document

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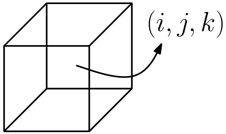

$$= \quad \sum_{r=1}^{R} w_r (p_r \otimes p_r \otimes p_r).$$

$\therefore$ Finding parameters $\equiv$ tensor decomposition!

Recipe for tensor methods in mixture models

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Many applications: mixtures of gaussians, hidden markov models, communities, crabs, ...

Are we done?

Two caveats

Efficiency

We need polynomial time algorithms for decomposition!

Given $T = \sum_{r=1}^{R} p_r \otimes p_r \otimes p_r$, can we find $\{p_r\}_{r=1}^{R}$?

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GOAL: Given, target accuracy ε, and

$$T = \sum_{r=1}^{R} p_r \otimes p_r \otimes p_r + \mathcal{N}, \text{ with } \|\mathcal{N}\| < \varepsilon/\text{poly}(n),$$

recover $\{p_r\}$ up to error ε.

A success story: the full rank case

$$\text{Given } T = \sum_{r=1}^{R} p_r \otimes p_r \otimes p_r + \mathcal{N}, \quad \|\mathcal{N}\| < \varepsilon/\text{poly}(n)$$

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Discovered many times: [Harshman 72], [Leurgans, et al. 93], [Chang 96], [Anandkumar, et al. 10], [Goyal et al. 13], ...

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- Can be done if the components are *nearly orthogonal* [Anandkumar et al. 14]

Kruskal's theorem (1977)

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Theorem (Kruskal)

Suppose T is defined as above, and A, B, C are $n \times R$ matrices with columns a_r, b_r, c_r, respectively.
Then a sufficient condition for uniqueness of decomposition is

$$k(A) + k(B) + k(C) \geq 2R + 2$$

Our result

$$T = \sum_{r=1}^{R} a_i \otimes b_i \otimes c_i + \mathcal{N}$$

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For matrix $A(n \times R)$ and parameter $\tau > 0$, the *rank* is the largest integer $k_\tau(A)$ s.t. every $n \times k_\tau(A)$ submatrix has condition number $< \tau$.

Note: Recall that condition number of B is $\sigma_{\max}(B)/\sigma_{\min}(B)$.

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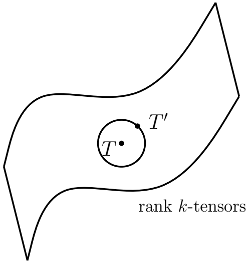
Theorem (Rough)

Let T be as above, and A, B, C be $n \times R$ matrices as before. Then the decomposition is robustly *unique if*

$$k_\tau(A) + k_\tau(B) + k_\tau(C) \geq 2R + 2$$

Our result



We show: For any $\varepsilon > 0$, there is an $\varepsilon' = \varepsilon/\text{poly}(n)$, such that $T =_{\varepsilon'} T'$ implies the decompositions are ε-close, up to a permutation.

Our result

$$T = \sum_{r=1}^{R} a_i \otimes b_i \otimes c_i \quad ; \quad T' = \sum_{r=1}^{R} a'_i \otimes b'_i \otimes c'_i$$

Theorem (Formal)

Suppose T, T' are defined as above, and A, B, C, A', B', C' are $n \times R$ matrices. Further, suppose for some $\tau > 0$, that

$$k_\tau(A) + k_\tau(B) + k_\tau(C) \geq 2R + 2.$$

Then for any $\varepsilon > 0$, there is an $\varepsilon' = \varepsilon/poly(n, \tau)$ such that if $\|T - T'\| < \varepsilon'$, then there exist diagonal matrices $\Gamma_A, \Gamma_B, \Gamma_C$, and a permutation Π such that $\Gamma_A \Gamma_B \Gamma_C = I$, and

$$A' =_\varepsilon \Gamma_A \Pi A, \quad B' =_\varepsilon \Gamma_B \Pi B, \quad C' =_\varepsilon \Gamma_C \Pi C.$$

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Main difficulty in proof: handling $1/\text{poly}(n)$ noise

(If $\varepsilon' = \varepsilon/\exp(n)$, much easier to show that decompositions are ε-close)

Proof overview

Suppose A, B, C, A', B', C' are $n \times R$, column lengths $\leq \rho$, and

$$k_\tau(A) = k_\tau(B) = k_\tau(C) = n \quad ; \quad R = 4n/3.$$

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Crux: first part – "permutation lemma"

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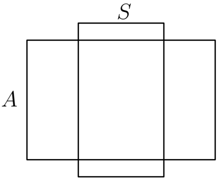
Definition: $\langle A_S \rangle$:= span of columns indexed by $S \subseteq [R]$



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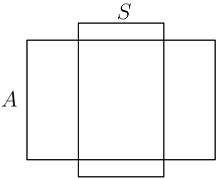

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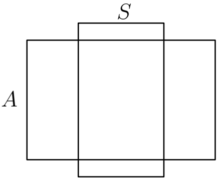

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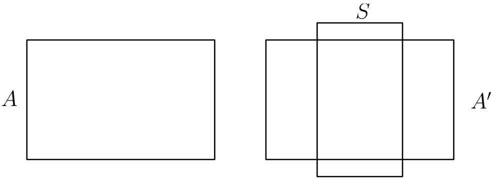
(If true for $|S| = 1$, then every column of A' is parallel to some column of A)

Downward induction: base case $|S| = (n-1)$



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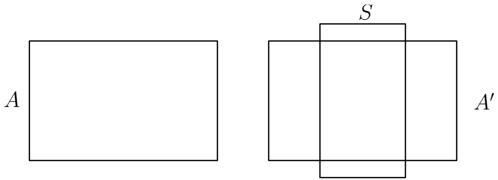

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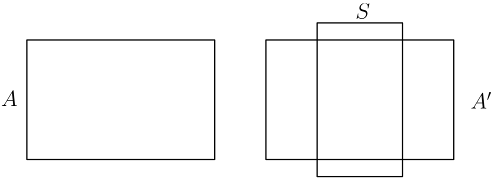

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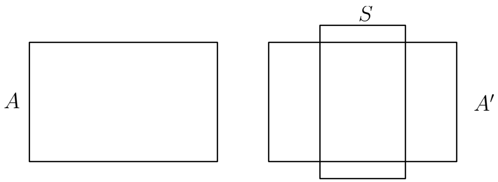

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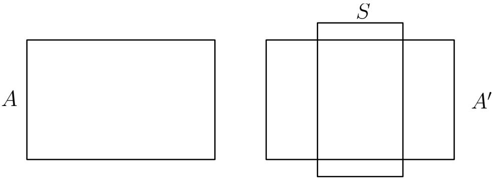

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1. Say we have: $a = \sum_{r \in S} \alpha_r a'_r + \alpha_i a'_i = \sum_{r \in S} \beta_r a'_r + \beta_j a'_j$

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1. Say we have: $a = \sum_{r \in S} \alpha_r a'_r + \alpha_i a'_i = \sum_{r \in S} \beta_r a'_r + \beta_j a'_j$
2. Since a'_i, a'_j are not parallel, this means $\alpha_i = \beta_j = 0$

Downward induction

Show for $|S| = (n-2), \ldots, 1$:

- start with some S of size $(n-2)$
- define $S_i = S \cup \{i\}$, for various $i \notin S$ — $(n/3) + 2$ such..
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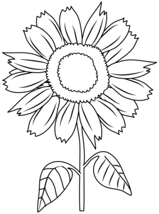

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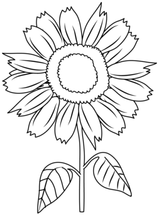

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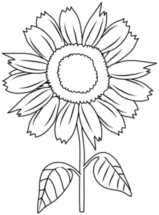

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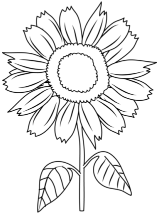

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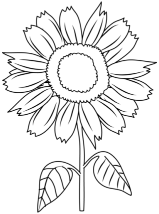

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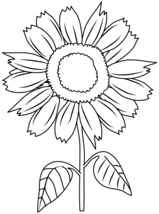

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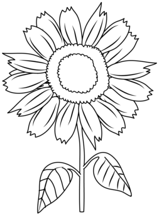

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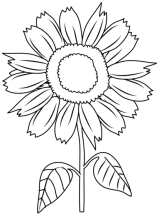

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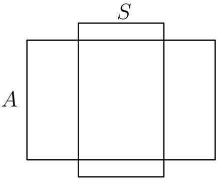
Key idea. Look at the spaces spanned by subsets of columns of A, A'.



Informal: Suppose for all S of size $(n-1)$, $\langle A'_S \rangle$ contains at least $|S|$ columns of A, up to error ε. Then the same is true for all S, with error ε'.

(If true for singletons, then every col of A' is ε'-close to a scaling of a col of A)

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Can we show "robust" uniqueness under more algebraic conditions?

Can we show that a generic $n \times n \times n$ tensor has a "stable" unique decomposition up to rank $n^2/4$?

Thank you!

Questions?