Orthogonally Decomposable Tensors

Elina Robeva
UC Berkeley

November 13, 2014

Symmetric Tensors

T is an $\underbrace{n \times \ldots \times n}_{d \text{ times}}$ *symmetric tensor* with elements in a field $\mathbb{K}(= \mathbb{R}, \mathbb{C})$ if

$$T_{i_1 i_2 \ldots i_d} = T_{i_{\sigma_1} i_{\sigma_2} \ldots i_{\sigma_d}}$$

for all permutations σ of $\{1, 2, \ldots, d\}$. Notation: $T \in S^d(\mathbb{K}^n)$.

Example ($d = 2$)

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{12} & T_{22} & \cdots & T_{2n} \\ & & \vdots & \\ T_{1n} & T_{2n} & \cdots & T_{nn} \end{pmatrix}$$

Example ($n = 3, d = 3$)

$$T = \underbrace{\begin{pmatrix} T_{111} & T_{112} & T_{113} \\ T_{112} & T_{122} & T_{123} \\ T_{113} & T_{123} & T_{133} \end{pmatrix}}_{T_{1..}}, \underbrace{\begin{pmatrix} T_{112} & T_{122} & T_{123} \\ T_{122} & T_{222} & T_{223} \\ T_{123} & T_{223} & T_{233} \end{pmatrix}}_{T_{2..}}, \underbrace{\begin{pmatrix} T_{113} & T_{123} & T_{133} \\ T_{123} & T_{223} & T_{233} \\ T_{133} & T_{233} & T_{333} \end{pmatrix}}_{T_{3..}}.$$

Symmetric Tensors and Polynomials

An equivalent way of representing a symmetric tensor $T \in S^d(\mathbb{K}^n)$ is by a *homogeneous polynomial* $f_T \in \mathbb{K}[x_1, ..., x_n]$ of degree d.

Example ($d = 2$)

In the case of matrices,

$$\begin{aligned} f_T(x_1, ..., x_n) &= x^T T x \\ &= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{12} & T_{13} & \cdots & T_{2n} \\ & & \vdots & \\ T_{1n} & T_{2n} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \sum_{i,j} T_{ij} x_i x_j. \end{aligned}$$

Symmetric Tensors and Polynomials

For general $T \in S^d(\mathbb{K}^n)$,

$$
\begin{aligned}
f_T(x_1, ..., x_n) &= T \cdot x^d := \sum_{i_1,...,i_d=1}^{n} T_{i_1...i_d} x_{i_1}...x_{i_d} \\
&= \sum_{j_1+\cdots+j_n=d} \binom{d}{j_1, \ldots, j_n} T_{\underbrace{1\ldots1}_{j_1}\ldots\underbrace{n\ldots n}_{j_n}} x_1^{j_1} \ldots x_n^{j_n} \\
&= \sum_{j_1+\cdots+j_n=d} u_{j_1,...,j_n} x_1^{j_1} \ldots x_n^{j_n}.
\end{aligned}
$$

Example $(n = 3, d = 2)$

For 3×3 matrices,

$$
\begin{aligned}
f_T(x_1, x_2, x_3) &= \sum_{i_1,i_2=1}^{3} T_{i_1 i_2} x_{i_1} x_{i_2} \\
&= \underbrace{T_{11}}_{u_{2,0,0}} x_1^2 + \underbrace{2T_{12}}_{u_{1,1,0}} x_1 x_2 + \underbrace{2T_{13}}_{u_{1,0,1}} x_1 x_3 + \underbrace{T_{22}}_{u_{0,2,0}} x_2^2 + \underbrace{2T_{23}}_{u_{0,1,1}} x_2 x_3 + \underbrace{T_{33}}_{u_{0,0,2}} x_3^2.
\end{aligned}
$$

Symmetric Tensor Decomposition

A *decomposition* of a tensor $T \in S^d(\mathbb{K}^n)$ is

$$T = \sum_{i=1}^{r} \lambda_i v_i^{\otimes d}.$$

If $f_T \in \mathbb{K}[x_1, ..., x_n]$ is the corresponding polynomial, then

$$f_T(x_1, ..., x_n) = \sum_{i=1}^{r} \lambda_i (v_i \cdot x)^d = \sum_{i=1}^{r} \lambda_i (v_{i1}x_1 + v_{i2}x_2 + \cdots + v_{in}x_n)^d.$$

The smallest r for which such a decomposition exists is the *symmetric rank* of T.

Orthogonal Tensor Decomposition

An *orthogonal decomposition* of a symmetric tensor $T \in S^d(\mathbb{R}^n)$ is a decomposition

$$T = \sum_{i=1}^{r} \lambda_i v_i^{\otimes d} \quad \text{with corresponding} \quad f_T = \sum_{i=1}^{r} \lambda_i (v_i \cdot x)^d$$

such that the vectors $v_1, \ldots, v_r \in \mathbb{R}^n$ are orthonormal. In particular, $r \leq n$.

Definition

A tensor $T \in S^d(\mathbb{R}^n)$ is *orthogonally decomposable*, or *odeco*, if it has an orthogonal decomposition.

Examples

1. All symmetric matrices are odeco: by the spectral theorem

$$T = V^T \Lambda V = \begin{bmatrix} | & \cdots & | \\ v_1 & \cdots & v_n \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i v_i v_i^T = \sum_{i=1}^{n} \lambda_i v_i^{\otimes 2},$$

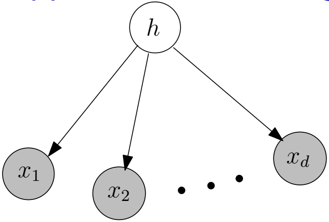
   where $v_1, \ldots, v_n$ is an orthonormal basis of eigenvectors.

2. The Fermat polynomial: If $v_i = e_i$, for $i = 1, \ldots, n$, then

$$f_T(x_1, \ldots, x_n) = x_1^d + x_2^d + \cdots + x_n^d,$$

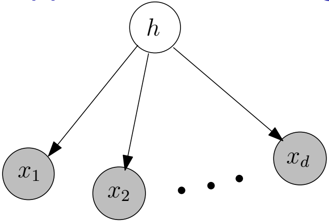
$$T = e_1^{\otimes d} + e_2^{\otimes d} + \cdots + e_n^{\otimes d}.$$

An Application: Exchangeable Single Topic Models



Pick a topic $h \in \{1, 2, ..., k\}$ with distribution $(w_1, ..., w_k) \in \Delta_{k-1}$. Given $h = j$, $x_1, ..., x_d$ are *i.i.d* random variables taking values in $\{1, 2, ..., n\}$ with distribution $\mu_j = (\mu_{j1}, ..., \mu_{jn}) \in \Delta_{n-1}$.

An Application: Exchangeable Single Topic Models

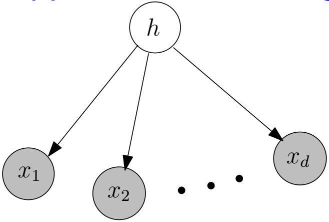


Pick a topic $h \in \{1, 2, ..., k\}$ with distribution $(w_1, ..., w_k) \in \Delta_{k-1}$. Given $h = j$, $x_1, ..., x_d$ are $i.i.d$ random variables taking values in $\{1, 2, ..., n\}$ with distribution $\mu_j = (\mu_{j1}, ..., \mu_{jn}) \in \Delta_{n-1}$.

Then, the joint distribution of $x_1, ..., x_d$ is an $\underbrace{n \times n \times \cdots \times n}_{d \text{ times}}$ symmetric tensor $T \in S^d(\mathbb{R}^n)$ whose entries sum to 1. Moreover,

$$T = \sum_{j=1}^{k} \mathbb{P}(h = j) \prod_{i=1}^{d} \mathbb{P}(x_i | h = j) = \sum_{j=1}^{k} w_j \mu_j^{\otimes d}.$$

An Application: Exchangeable Single Topic Models



Pick a topic $h \in \{1, 2, ..., k\}$ with distribution $(w_1, ..., w_k) \in \Delta_{k-1}$. Given $h = j$, $x_1, ..., x_d$ are $i.i.d$ random variables taking values in $\{1, 2, ..., n\}$ with distribution $\mu_j = (\mu_{j1}, ..., \mu_{jn}) \in \Delta_{n-1}$.

Then, the joint distribution of $x_1, ..., x_d$ is an $\underbrace{n \times n \times \cdots \times n}_{d \text{ times}}$ symmetric tensor $T \in S^d(\mathbb{R}^n)$ whose entries sum to 1. Moreover,

$$T = \sum_{j=1}^{k} \mathbb{P}(h = j) \prod_{i=1}^{d} \mathbb{P}(x_i | h = j) = \sum_{j=1}^{k} w_j \mu_j^{\otimes d}.$$

Given T, to recover the parameters w, μ, use a transformation $T \mapsto T_{od}$ and decompose T_{od} [Anandkumar et al.].

Eigenvectors of Tensors

Consider a symmetric tensor $T \in S^d(\mathbb{K}^n)$.

Example ($d = 2$)

T is an $n \times n$ matrix and $w \in \mathbb{K}^n$ is an eigenvector if

$$Tw = \begin{pmatrix} \vdots \\ \sum_{j=1}^n T_{i,j} w_j \\ \vdots \end{pmatrix} = \lambda w.$$

Example ($d = 3$)

T is an $n \times n \times n$ tensor and $w \in \mathbb{K}^n$ is an eigenvector if

$$Tw^2 := \begin{pmatrix} \vdots \\ \sum_{j,k=1}^n T_{i,j,k} w_j w_k \\ \vdots \end{pmatrix} = \lambda w.$$

Eigenvectors of Symmetric Tensors

Definition

- Given a symmetric tensor $T \in S^d(\mathbb{K}^n)$, an *eigenvector* of T with *eigenvalue* λ is a vector $w \in \mathbb{C}^n$ such that

$$Tw^{d-1} := \begin{pmatrix} \vdots \\ \sum_{i_2,\ldots,i_d=1}^{n} T_{i,i_2,\ldots,i_d} w_{i_2} \ldots w_{i_d} \\ \vdots \end{pmatrix} = \lambda w.$$

  Two eigenvector-eigenvalue pairs (w, λ) and (w', λ') are equivalent if there exists $t \in \mathbb{K} \setminus \{0\}$ such that $t^{d-2}\lambda = \lambda'$ and $tw = w'$.

- For the corresponding $f \in \mathbb{K}[x_1, \ldots, x_n]$, $w \in \mathbb{C}^n$ is an *eigenvector* with *eigenvalue* λ if

$$\nabla f(w) = d\lambda w.$$

Eigenvectors of Symmetric Tensors

Definition

- Given a symmetric tensor $T \in S^d(\mathbb{K}^n)$, an *eigenvector* of T with *eigenvalue* λ is a vector $w \in \mathbb{C}^n$ such that

$$Tw^{d-1} := \begin{pmatrix} \vdots \\ \sum_{i_2,\dots,i_d=1}^{n} T_{i,i_2,\dots,i_d} w_{i_2} \dots w_{i_d} \\ \vdots \end{pmatrix} = \lambda w.$$

  Two eigenvector-eigenvalue pairs (w, λ) and (w', λ') are equivalent if there exists $t \in \mathbb{K} \setminus \{0\}$ such that $t^{d-2}\lambda = \lambda'$ and $tw = w'$.

- For the corresponding $f \in \mathbb{K}[x_1, \dots, x_n]$, $w \in \mathbb{C}^n$ is an *eigenvector* with *eigenvalue* λ if

$$\nabla f(w) = d\lambda w.$$

Therefore, the eigenvectors of f are given by the vanishing of the

$$2 \times 2 \text{ minors of the matrix } [\nabla f(x)|x].$$

Eigenvectors of Symmetric Tensors

Example

Let

$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} \text{ and } f(x,y,z) = x^3 + y^3 + z^3.$$

Then, $(x,y,z)^T$ is an eigenvector of f if and only if the 2×2 minors of the matrix $\begin{bmatrix} & x \\ \nabla f & y \\ & z \end{bmatrix} = \begin{bmatrix} 3x^2 & x \\ 3y^2 & y \\ 3z^2 & z \end{bmatrix}$ vanish. Therefore,

$$x^2y - xy^2 = x^2z - xz^2 = y^2z - yz^2 = 0.$$

This is equivalent to

$$xy(x-y) = xz(x-z) = yz(y-z) = 0.$$

The solutions are (up to scaling):

$$\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}.$$

Eigenvectors of Odeco Tensors

If $T = \sum_{i=1}^{n} \lambda_i v_i^{\otimes d}$ is an odeco tensor, i.e. $v_1, ..., v_n$ are orthonormal, then the vectors v_k, $k = 1, ..., n$ are eigenvectors of T with corresponding eigenvalues λ_k, $k = 1, ..., n$:

$$Tv_k^{d-1} = \sum_{i=1}^{n} \lambda_i v_i (v_k \cdot v_i)^{d-1} = \lambda_k v_k.$$

Eigenvectors of Odeco Tensors

If $T = \sum_{i=1}^{n} \lambda_i v_i^{\otimes d}$ is an odeco tensor, i.e. $v_1, ..., v_n$ are orthonormal, then the vectors v_k, $k = 1, ..., n$ are eigenvectors of T with corresponding eigenvalues λ_k, $k = 1, ..., n$:

$$Tv_k^{d-1} = \sum_{i=1}^{n} \lambda_i v_i (v_k \cdot v_i)^{d-1} = \lambda_k v_k.$$

- Is there an easy way of finding these vectors, i.e. finding the orthogonal decomposition of an odeco tensor?

Eigenvectors of Odeco Tensors

If $T = \sum_{i=1}^{n} \lambda_i v_i^{\otimes d}$ is an odeco tensor, i.e. $v_1, \ldots, v_n$ are orthonormal, then the vectors v_k, $k = 1, \ldots, n$ are eigenvectors of T with corresponding eigenvalues λ_k, $k = 1, \ldots, n$:

$$Tv_k^{d-1} = \sum_{i=1}^{n} \lambda_i v_i (v_k \cdot v_i)^{d-1} = \lambda_k v_k.$$

- Is there an easy way of finding these vectors, i.e. finding the orthogonal decomposition of an odeco tensor?
- Are these all of the eigenvectors of an odeco tensor?

Robust Eigenvectors

Definition

A unit vector $u \in \mathbb{R}^n$ is a *robust eigenvector* of a tensor $T \in S^d(\mathbb{R}^n)$ if there exists $\epsilon > 0$ such that for all $\theta \in \mathcal{B}_\epsilon(u) = \{u' : ||u - u'|| < \epsilon\}$, repeated iteration of the map

$$\theta \mapsto \frac{T\theta^{d-1}}{||T\theta^{d-1}||}, \qquad (1)$$

starting from θ converges to u.

Theorem (Anandkumar et al.)

Let T have an orthogonal decomposition $T = \sum_{i=1}^n \lambda_i v_i^{\otimes d}$ as in the definition.

1. *The set of $\theta \in \mathbb{R}^n$ which do not converge to some v_i under repeated iteration of (1) has measure 0.*
2. *The set of robust eigenvectors of T is equal to $\{v_1, v_2, ..., v_n\}$.*

The Tensor Power Method

The tensor power method consists of repeated iteration of the map

$$u \mapsto \frac{Tu^{d-1}}{||Tu^{d-1}||},$$

or equivalently,

$$u \mapsto \frac{\nabla f(u)}{||\nabla f(u)||}.$$

Algorithm

Input: An odeco tensor T.
Output: An orthogonal representation of T.

Repeat
  Find $v_i \leftarrow$ power method output starting from a random $u \in \mathbb{R}^n$.
  Recover $\lambda_i = T \cdot v_i^d$.
  $T \leftarrow T - \lambda_i v_i^{\otimes d}$.
Return $v_1, ..., v_n$ and $\lambda_1, ..., \lambda_n$.

The Number of Eigenvectors of a Tensor

Recall: Given a tensor $T \in S^d(\mathbb{C}^n)$ with corresponding polynomial f, the eigenvectors $x \in \mathbb{C}^n$ are the solutions to the equations given by the 2×2 minors of the matrix

$$\left[\nabla f(x) | x\right].$$

Theorem (Sturmfels and Cartwright)

If a tensor $T \in S^d(\mathbb{C}^n)$ has finitely many eigenvectors, then their number is $\frac{(d-1)^n - 1}{d-2}$.

Eigenvectors of Odeco Tensors

Odeco tensors are nice because we can characterize all of their eigenvectors.

Theorem

Let $f \in S^d(\mathbb{C}^n)$ be an odeco tensor with $f(x_1, ..., x_n) = \sum_{i=1}^n \lambda_i (v_i \cdot x)^d$, where V is the orthogonal matrix with columns $v_1, ..., v_n$. Then, f has $\frac{(d-1)^n - 1}{d-2}$ eigenvectors. Explicitly, they are

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = V \begin{bmatrix} \lambda_1^{-\frac{1}{d-2}} \\ \eta_2 \lambda_2^{-\frac{1}{d-2}} \\ \vdots \\ \eta_k \lambda_k^{-\frac{1}{d-2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $k = 1, ..., n$ and $\eta_2, ..., \eta_k$ are $(d-2)$-nd roots of unity, up to permutation.

Eigenvectors of Odeco Tensors

Example ($d = 3, n = 3$)

Let

$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3}.$$

Then, $V = I$, the identity matrix and the eigenvectors of T are:

$k = 1$ $(1 : 0 : 0)^T, (0 : 1 : 0)^T, (0 : 0 : 1)^T$

$k = 2$ $(1 : 1 : 0)^T, (1 : 0 : 1)^T, (0 : 1 : 1)^T$

$k = 3$ $(1 : 1 : 1)^T.$

The Set of Odeco Tensors

- Parametric representation:
  The set of orthogonally decomposable tensors can be parametrized by $\mathbb{R}^n \times O_n(\mathbb{R})$:
  $$\lambda, V \mapsto \sum_{i=1}^{n} \lambda_i v_i^{\otimes d}.$$
- Implicit representation:
  The set of orthogonally decomposable tensors can also be represented as the solutions to a set of equations.

Definition

The *odeco variety* is the Zariski closure of the set of all odeco tensors in $S^d(\mathbb{C}^n)$.

Goal: find equations defining this variety.

The Odeco Variety

Let $T \in S^d(\mathbb{C}^n)$. Let $\mathcal{F}$ be the set of the following equations:

$$\mathcal{F} = \langle \sum_{s=1}^{n} T_{i_1,\ldots,i_{d-1},s} T_{j_1,\ldots,j_{d-1},s} - T_{k_1,\ldots,k_{d-1},s} T_{l_1,\ldots,l_{d-1},s} \rangle,$$

where $i_1, \ldots, i_{d-1},\ j_1, \ldots, j_{d-1},\ k_1, \ldots, k_{d-1},\ l_1, \ldots, l_{d-1} \in \{1, 2, \ldots, n\}$ are such that $\{i_1, \ldots, i_{d-1}, j_1, \ldots, j_{d-1}\} = \{k_1, \ldots, k_{d-1}, l_1, \ldots, l_{d-1}\}$.

Conjecture

The odeco variety is given by $\mathcal{V}(\mathcal{F})$ for general n.

Lemma

The equations $\mathcal{F}$ vanish on the set of orthogonally decomposable tensors.

Proposition

The odeco variety is equal to $\mathcal{V}(\mathcal{F})$ for $n = 2$, i.e. in the case of $2 \times 2 \times \cdots \times 2$ tensors.

Nonsymmetric Tensor Decomposition

Let $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n = (\mathbb{R}^n)^{\otimes d}$. A *decomposition* of T is an expression of the form

$$T = \sum_{i=1}^{r} \lambda_i a_i \otimes b_i \otimes c_i \otimes \cdots .$$

A tensor $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ is *orthogonally decomposable*, or *odeco*, if we can decompose it as

$$T = \sum_{i=1}^{n} \lambda_i a_i \otimes b_i \otimes c_i \otimes \cdots ,$$

so that $a_1, \ldots, a_n \in \mathbb{R}^n$ are orthonormal, $b_1, \ldots, b_n \in \mathbb{R}^n$ are orthonormal, $c_1, \ldots, c_n \in \mathbb{R}^n$ are orthonormal, etc.

Nonsymmetric Odeco Tensors

Example

1. If $T \in \mathbb{R}^n \otimes \mathbb{R}^n$ is a matrix, then T has singular value decomposition

$$T = U\Sigma V^T = \sum_{i=1}^{k} \sigma_i u_i v_i^T,$$

where $u_1, ..., u_k$ are orthonormal and $v_1, ..., v_k$ are orthonormal.

2. The tensor $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ given by

$$T = \sum_{i=1}^{n} \lambda_i e_i \otimes e_i \otimes \cdots \otimes e_i$$

is odeco.

Singular Vector Tuples

Example

Given a matrix $T \in \mathbb{R}^n \otimes \mathbb{R}^n$, (u, v) is a *singular vector tuple* of T if there exist λ_1 and λ_2 such that

$$Tu = \begin{bmatrix} \vdots \\ \sum_j T_{ij} u_j \\ \vdots \end{bmatrix} = \lambda_1 v \qquad \text{and} \qquad T^T v = \begin{bmatrix} \vdots \\ \sum_i T_{ij} v_i \\ \vdots \end{bmatrix} = \lambda_2 u.$$

Definition

Given a tensor $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ a *singular vector tuple* is a d-tuple $(x_{\cdot 1}, \cdots, x_{\cdot d}) \in \mathbb{C}^n \times \cdots \times \mathbb{C}^n$ such that for every $1 \leq k \leq d$,

$$\begin{bmatrix} \vdots \\ \sum_{i_1,\ldots,i_{k-1},\, i_{k+1},\ldots,i_d} T_{i_1 \ldots i_{k-1}\, i\, i_{k+1} \ldots i_d} x_{i_1 1} \cdots x_{i_{k-1}(k-1)}\, x_{i_{k+1}(k+1)} \cdots x_{i_d d} \\ \vdots \end{bmatrix} = \lambda_k x_{\cdot k},$$

for some $\lambda_k \in \mathbb{C}$.

Examples

1. If $T \in \mathbb{R}^n \otimes \mathbb{R}^n$ is a generic matrix with singular value decomposition

$$T = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T,$$

$(u_1, v_1), ..., (u_n, v_n)$ are all of the singular vector pairs of T.

2. Let $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ be given by $T = \sum_{i=1}^n \lambda_i e_i \otimes \cdots \otimes e_i$. Therefore, $(x_{\cdot 1}, \cdots, x_{\cdot d})$ is a singular vector tuple of T if and only if

$$\begin{bmatrix} \lambda_1 x_{11} \cdots x_{1(k-1)} x_{1(k+1)} \cdots x_{1d} \\ \vdots \\ \lambda_n x_{n1} \cdots x_{n(k-1)} x_{n(k+1)} \cdots x_{nd} \end{bmatrix} = \lambda_k x_{\cdot k},$$

i.e. the matrix

$$\begin{bmatrix} \lambda_1 x_{11} \cdots x_{1(k-1)} x_{1(k+1)} \cdots x_{1d} & x_{1k} \\ \vdots & \vdots \\ \lambda_n x_{n1} \cdots x_{n(k-1)} x_{n(k+1)} \cdots x_{nd} & x_{nk} \end{bmatrix}$$

has rank 1 for every $1 \leq k \leq d$.

The number of singular vector tuples of a tensor

Theorem (Friedland and Ottaviani)

Let $T \in \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ be a generic tensor. Then, T has finitely many singular vector tuples and they correspond to nonzero singular values. Their number is the coefficient of the monomial $\prod_{i=1}^{d} t_i^{n-1}$ in the polynomial

$$\prod_{i=1}^{d} \frac{(\sum_j t_j - t_i)^n - t_i^n}{\sum_j t_j - 2t_i}.$$

Singular Vectors of Odeco Tensors

Theorem

Let $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ be odeco with decomposition $T = \sum_{i=1}^n \lambda_i a_i \otimes b_i \otimes c_i \otimes \cdots$. Let $A = \left(\; a_1 \; \middle| \; a_2 \; \middle| \; \cdots | a_n \; \right)$, $B = \left(\; b_1 \; \middle| \; b_2 \; \middle| \; \cdots | b_n \; \right)$, etc., so that $A, B, C, \ldots$ are orthogonal matrices. Then, the singular vector tuples of T are given as follows:

Type I

$$A \begin{bmatrix} \lambda_1^{-\frac{1}{d-2}} \\ \chi_{12}\eta_2\lambda_2^{-\frac{1}{d-2}} \\ \vdots \\ \chi_{1k}\eta_k\lambda_k^{-\frac{1}{d-2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes B \begin{bmatrix} \lambda_1^{-\frac{1}{d-2}} \\ \chi_{22}\eta_2\lambda_2^{-\frac{1}{d-2}} \\ \vdots \\ \chi_{2k}\eta_k\lambda_k^{-\frac{1}{d-2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes C \begin{bmatrix} \lambda_1^{-\frac{1}{d-2}} \\ \chi_{32}\eta_2\lambda_2^{-\frac{1}{d-2}} \\ \vdots \\ \chi_{3k}\eta_k\lambda_k^{-\frac{1}{d-2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \cdots,$$

where $1 \leq k \leq n$, χ_{ij} is a 2-nd root of unity, η_i is a $(d-2)$-nd root of unity, up to permutation.

Type II

$$Ax_{\cdot 1} \otimes Bx_{\cdot 2} \otimes Cx_{\cdot 3} \otimes \cdots,$$

where the matrix $X = (x_{ij})_{ij}$ has at least two zeros in each row and no column is identically to 0.

Example

Let $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ be given by $T = \sum_{i=1}^n \lambda_i e_i \otimes \cdots \otimes e_i$. Therefore, $(x_{\cdot 1}, \cdots, x_{\cdot d})$ is a singular vector tuple of T if and only if

$$\begin{bmatrix} \lambda_1 x_{11} \cdots x_{1(k-1)} x_{1(k+1)} \cdots x_{1d} \\ \vdots \\ \lambda_n x_{n1} \cdots x_{n(k-1)} x_{n(k+1)} \cdots x_{nd} \end{bmatrix} = \lambda_k x_{\cdot k},$$

i.e. the matrix

$$\begin{bmatrix} \lambda_1 x_{11} \cdots x_{1(k-1)} x_{1(k+1)} \cdots x_{1d} & x_{1k} \\ \vdots & \vdots \\ \lambda_n x_{n1} \cdots x_{n(k-1)} x_{n(k+1)} \cdots x_{nd} & x_{nk} \end{bmatrix}$$

has rank 1 for every $1 \le k \le d$.

The Set of Odeco Tensors

Definition

The *odeco variety* is the Zariski closure of the set of all odeco tensors in $\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$.

Goal: find equations defining this variety.

Conjecture

The prime ideal of the set of odeco tensors is given by

$$I = \langle \sum_{s=1}^{n} T_{a_1 \ldots a_{r-1} s a_r \ldots a_d} T_{b_1 \ldots b_{r-1} s b_r \ldots b_d} - T_{c_1 \ldots c_{r-1} s c_r \ldots c_d} T_{d_1 \ldots d_{r-1} s d_r \ldots d_d} \rangle,$$

where $a, b, c, d \in \{1, \ldots, n\}^{d-1}$, $\{a_i, b_i\} = \{c_i, d_i\}$ and $r \in \{1, \ldots, n\}$.

Lemma

The ideal I vanishes on the set of odeco tensors.

References

- A. Anandkumar, R. Ge, D. Hsu, S. Kakade, and M. Telegarsky: *Tensor Decompositions for Learning Latent Variable Models*
- A. Anandkumar, D. Hsu, and S. Kakade: *A Method of Moments for Mixture Models and Hidden Markov Models*
- J. Brachat, P. Common, B. Mourrain, and E. Tsigaridas: *Symmetric Tensor Decomposition*
- P. Common, G. Golub, L. Lim, and B. Mourrain: *Symmetric Tensors and Symmetric Tensor Rank*
- D. Cartwright and B. Sturmfels: *The Number of Eigenvalues of a Tensor*
- D. Eisenbud and B. Sturmfels: *Binomial Ideals*
- S. Friedland and G. Ottaviani: *The number of singular vector tuples and uniqueness of best rank one approximation of tensors*
- L. Oeding and G. Ottaviani: *Eigenvectors of Tensors and Algorithms for Waring Decomposition*