Orthogonally Decomposable Tensors

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November 13, 2014
Symmetric Tensors

$T$ is an $n \times \ldots \times n$ symmetric tensor with elements in a field $\mathbb{K}(= \mathbb{R}, \mathbb{C})$ if

$$T_{i_1i_2\ldots i_d} = T_{i_{\sigma_1}i_{\sigma_2}\ldots i_{\sigma_d}}$$

for all permutations $\sigma$ of $\{1, 2, \ldots, d\}$. Notation: $T \in S^d(\mathbb{K}^n)$.

Example ($d = 2$)

$$T = \begin{pmatrix}
T_{11} & T_{12} & \cdots & T_{1n} \\
T_{12} & T_{22} & \cdots & T_{2n} \\
& & \ddots & \\
T_{1n} & T_{2n} & \cdots & T_{nn}
\end{pmatrix}$$

Example ($n = 3$, $d = 3$)

$$T = \begin{pmatrix}
T_{111} & T_{112} & T_{113} \\
T_{112} & T_{122} & T_{123} \\
T_{113} & T_{123} & T_{133}
\end{pmatrix}, \quad T = \begin{pmatrix}
T_{112} & T_{122} & T_{123} \\
T_{122} & T_{222} & T_{223} \\
T_{123} & T_{223} & T_{233}
\end{pmatrix}, \quad T = \begin{pmatrix}
T_{113} & T_{123} & T_{133} \\
T_{123} & T_{223} & T_{233} \\
T_{133} & T_{233} & T_{333}
\end{pmatrix}.$$
Symmetric Tensors and Polynomials

An equivalent way of representing a symmetric tensor $T \in S^d(\mathbb{K}^n)$ is by a \textit{homogeneous polynomial} $f_T \in \mathbb{K}[x_1, ..., x_n]$ of degree $d$.

\textbf{Example ($d = 2$)}

In the case of matrices,

$$f_T(x_1, ..., x_n) = x^T T x$$

$$= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{12} & T_{13} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{1n} & T_{2n} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \sum_{i,j} T_{ij} x_i x_j.$$
Symmetric Tensors and Polynomials

For general \( T \in S^d(\mathbb{K}^n) \),

\[
f_T(x_1, \ldots, x_n) = T \cdot x^d := \sum_{i_1, \ldots, i_d=1}^n T_{i_1 \ldots i_d} x_{i_1} \cdots x_{i_d}
\]

\[
= \sum_{j_1 + \cdots + j_n = d} \binom{d}{j_1, \ldots, j_n} T_{1 \ldots 1 \ldots n} x_1^{j_1} \cdots x_n^{j_n}
\]

\[
= \sum_{j_1 + \cdots + j_n = d} u_{j_1, \ldots, j_n} x_1^{j_1} \cdots x_n^{j_n}.
\]

Example \((n = 3, d = 2)\)

For \(3 \times 3\) matrices,

\[
f_T(x_1, x_2, x_3) = \sum_{i_1, i_2=1}^3 T_{i_1 i_2} x_{i_1} x_{i_2}
\]

\[
= T_{11} x_1^2 + 2 T_{12} x_1 x_2 + 2 T_{13} x_1 x_3 + T_{22} x_2^2 + 2 T_{23} x_2 x_3 + T_{33} x_3^2.
\]
Symmetric Tensor Decomposition

A decomposition of a tensor $T \in S^d(\mathbb{K}^n)$ is

$$T = \sum_{i=1}^{r} \lambda_i v_i \otimes d.$$ 

If $f_T \in \mathbb{K}[x_1, \ldots, x_n]$ is the corresponding polynomial, then

$$f_T(x_1, \ldots, x_n) = \sum_{i=1}^{r} \lambda_i (v_i \cdot x)^d = \sum_{i=1}^{r} \lambda_i (v_{i1}x_1 + v_{i2}x_2 + \cdots + v_{in}x_n)^d.$$ 

The smallest $r$ for which such a decomposition exists is the symmetric rank of $T$. 
Orthogonal Tensor Decomposition

An orthogonal decomposition of a symmetric tensor $T \in S^d(\mathbb{R}^n)$ is a decomposition

$$T = \sum_{i=1}^{r} \lambda_i v_i \otimes d$$

with corresponding

$$f_T = \sum_{i=1}^{r} \lambda_i (v_i \cdot x)^d$$

such that the vectors $v_1, ..., v_r \in \mathbb{R}^n$ are orthonormal. In particular, $r \leq n$.

Definition

A tensor $T \in S^d(\mathbb{R}^n)$ is orthogonally decomposable, or odeco, if it has an orthogonal decomposition.
Examples

1. All symmetric matrices are odeco: by the spectral theorem

\[ T = V^T \Lambda V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ & \ddots & \\ - & v_n & - \end{bmatrix} \]

\[ = \sum_{i=1}^{n} \lambda_i v_i v_i^T = \sum_{i=1}^{n} \lambda_i v_i \otimes^2, \]

where \( v_1, \ldots, v_n \) is an orthonormal basis of eigenvectors.

2. The Fermat polynomial: If \( v_i = e_i \), for \( i = 1, \ldots, n \), then

\[ f_T(x_1, \ldots, x_n) = x_1^d + x_2^d + \cdots + x_n^d, \]

\[ T = e_1^\otimes d + e_2^\otimes d + \cdots + e_n^\otimes d. \]
Pick a topic $h \in \{1, 2, \ldots, k\}$ with distribution $(w_1, \ldots, w_k) \in \Delta_{k-1}$. Given $h = j$, $x_1, \ldots, x_d$ are i.i.d random variables taking values in $\{1, 2, \ldots, n\}$ with distribution $\mu_j = (\mu_{j1}, \ldots, \mu_{jn}) \in \Delta_{n-1}$. 
An Application: Exchangeable Single Topic Models

Pick a topic $h \in \{1, 2, \ldots, k\}$ with distribution $(w_1, \ldots, w_k) \in \Delta_{k-1}$. Given $h = j$, $x_1, \ldots, x_d$ are i.i.d random variables taking values in $\{1, 2, \ldots, n\}$ with distribution $\mu_j = (\mu_{j1}, \ldots, \mu_{jn}) \in \Delta_{n-1}$.

Then, the joint distribution of $x_1, \ldots, x_d$ is an $n \times n \times \cdots \times n$ symmetric tensor $T \in S^d(\mathbb{R}^n)$ whose entries sum to 1. Moreover,

$$T = \sum_{j=1}^{k} \mathbb{P}(h = j) \prod_{i=1}^{d} \mathbb{P}(x_i|h = j) = \sum_{j=1}^{k} w_j \mu_j \otimes^d.$$
Pick a topic $h \in \{1, 2, ..., k\}$ with distribution $(w_1, ..., w_k) \in \Delta_{k-1}$. Given $h = j$, $x_1, ..., x_d$ are i.i.d random variables taking values in $\{1, 2, ..., n\}$ with distribution $\mu_j = (\mu_{j1}, ..., \mu_{jn}) \in \Delta_{n-1}$.

Then, the joint distribution of $x_1, ..., x_d$ is an $n \times n \times \cdots \times n$ symmetric tensor $T \in S^d(\mathbb{R}^n)$ whose entries sum to 1. Moreover,

$$T = \sum_{j=1}^{k} \mathbb{P}(h = j) \prod_{i=1}^{d} \mathbb{P}(x_i | h = j) = \sum_{j=1}^{k} w_j \mu_j \otimes^d.$$ 

Given $T$, to recover the parameters $w, \mu$, use a transformation $T \mapsto T_{od}$ and decompose $T_{od}$ [Anandkumar et al.].
Eigenvectors of Tensors

Consider a symmetric tensor \( T \in S^d(\mathbb{K}^n) \).

**Example \((d = 2)\)**

\( T \) is an \( n \times n \) matrix and \( w \in \mathbb{K}^n \) is an eigenvector if

\[
T w = \begin{pmatrix}
\vdots \\
\sum_{j=1}^n T_{i,j} w_j \\
\vdots
\end{pmatrix} = \lambda w.
\]

**Example \((d = 3)\)**

\( T \) is an \( n \times n \times n \) tensor and \( w \in \mathbb{K}^n \) is an eigenvector if

\[
T w^2 := \begin{pmatrix}
\vdots \\
\sum_{j,k=1}^n T_{i,j,k} w_j w_k \\
\vdots
\end{pmatrix} = \lambda w.
\]
Eigenvectors of Symmetric Tensors

Definition

▶ Given a symmetric tensor $T \in S^d(\mathbb{K}^n)$, an eigenvector of $T$ with eigenvalue $\lambda$ is a vector $w \in \mathbb{C}^n$ such that

$$TW^{d-1} := \left( \sum_{i_2,\ldots,i_d=1}^n T_{i_2,i_2,\ldots,i_d} w_{i_2}\ldots w_{i_d} \right) = \lambda w.$$ 

Two eigenvector-eigenvalue pairs $(w, \lambda)$ and $(w', \lambda')$ are equivalent if there exists $t \in \mathbb{K} \setminus \{0\}$ such that $t^{d-2}\lambda = \lambda'$ and $tw = w'$.

▶ For the corresponding $f \in \mathbb{K}[x_1, \ldots, x_n]$, $w \in \mathbb{C}^n$ is an eigenvector with eigenvalue $\lambda$ if

$$\nabla f(w) = d\lambda w.$$
Eigenvectors of Symmetric Tensors

Definition

- Given a symmetric tensor $T \in S^d(\mathbb{K}^n)$, an eigenvector of $T$ with eigenvalue $\lambda$ is a vector $w \in \mathbb{C}^n$ such that

$$TW^{d-1} := \left( \sum_{i_2,\ldots,i_d=1}^{n} T_{i_1,i_2,\ldots,i_d} w_{i_2} \ldots w_{i_d} \right) = \lambda w.$$

Two eigenvector-eigenvalue pairs $(w, \lambda)$ and $(w', \lambda')$ are equivalent if there exists $t \in \mathbb{K} \setminus \{0\}$ such that $t^{d-2}\lambda = \lambda'$ and $tw = w'$.

- For the corresponding $f \in \mathbb{K}[x_1, \ldots, x_n]$, $w \in \mathbb{C}^n$ is an eigenvector with eigenvalue $\lambda$ if

$$\nabla f(w) = d\lambda w.$$

Therefore, the eigenvectors of $f$ are given by the vanishing of the $2 \times 2$ minors of the matrix $[\nabla f(x)|x]$. 
Eigenvectors of Symmetric Tensors

Example

Let

\[
T = e_1 \otimes^3 + e_2 \otimes^3 + e_3 \otimes^3 \quad \text{and} \quad f(x, y, z) = x^3 + y^3 + z^3.
\]

Then, \((x, y, z)^T\) is an eigenvector of \(f\) if and only if the \(2 \times 2\) minors of the matrix

\[
\begin{bmatrix}
x \\
3x^2 \\
y \\
3y^2 \\
z \\
3z^2
\end{bmatrix}
\]

vanish. Therefore,

\[
x^2y - xy^2 = x^2z - xz^2 = y^2z - yz^2 = 0.
\]

This is equivalent to

\[
xy(x - y) = xz(x - z) = yz(y - z) = 0.
\]

The solutions are (up to scaling):

\[
\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}.\]
If $T = \sum_{i=1}^{n} \lambda_i v_i \otimes d$ is an odeco tensor, i.e. $v_1, \ldots, v_n$ are orthonormal, then the vectors $v_k$, $k = 1, \ldots, n$ are eigenvectors of $T$ with corresponding eigenvalues $\lambda_k$, $k = 1, \ldots, n$: 

$$Tv_k^{d-1} = \sum_{i=1}^{n} \lambda_i v_i (v_k \cdot v_i)^{d-1} = \lambda_k v_k.$$
Eigenvectors of Odeco Tensors

If $T = \sum_{i=1}^{n} \lambda_i v_i \otimes d$ is an odeco tensor, i.e. $v_1, ..., v_n$ are orthonormal, then the vectors $v_k$, $k = 1, ..., n$ are eigenvectors of $T$ with corresponding eigenvalues $\lambda_k$, $k = 1, ..., n$:

$$Tv_k^{d-1} = \sum_{i=1}^{n} \lambda_i v_i (v_k \cdot v_i)^{d-1} = \lambda_k v_k.$$

Is there an easy way of finding these vectors, i.e. finding the orthogonal decomposition of an odeco tensor?
If $T = \sum_{i=1}^{n} \lambda_i v_i \otimes d$ is an odeco tensor, i.e. $v_1, ..., v_n$ are orthonormal, then the vectors $v_k$, $k = 1, ..., n$ are eigenvectors of $T$ with corresponding eigenvalues $\lambda_k$, $k = 1, ..., n$:

$$T v_k^{d-1} = \sum_{i=1}^{n} \lambda_i v_i (v_k \cdot v_i)^{d-1} = \lambda_k v_k.$$

- Is there an easy way of finding these vectors, i.e. finding the orthogonal decomposition of an odeco tensor?

- Are these all of the eigenvectors of an odeco tensor?
Robust Eigenvectors

Definition
A unit vector $u \in \mathbb{R}^n$ is a robust eigenvector of a tensor $T \in S^d(\mathbb{R}^n)$ if there exists $\epsilon > 0$ such that for all $\theta \in B_\epsilon(u) = \{ u' : \| u - u' \| < \epsilon \}$, repeated iteration of the map

$$\theta \mapsto \frac{T \theta^{d-1}}{\| T \theta^{d-1} \|}, \quad (1)$$

starting from $\theta$ converges to $u$.

Theorem (Anandkumar et al.)
Let $T$ have an orthogonal decomposition $T = \sum_{i=1}^n \lambda_i v_i^\otimes d$ as in the definition.

1. The set of $\theta \in \mathbb{R}^n$ which do not converge to some $v_i$ under repeated iteration of (1) has measure 0.

2. The set of robust eigenvectors of $T$ is equal to $\{ v_1, v_2, ..., v_n \}$. 
The Tensor Power Method

The tensor power method consists of repeated iteration of the map

\[ u \mapsto \frac{T u^{d-1}}{\|T u^{d-1}\|} , \]

or equivalently,

\[ u \mapsto \frac{\nabla f(u)}{\|\nabla f(u)\|} . \]

Algorithm

Input: An odeco tensor \( T \).

Output: An orthogonal representation of \( T \).

Repeat

Find \( v_i \leftarrow \) power method output starting from a random \( u \in \mathbb{R}^n \).
Recover \( \lambda_i = T \cdot v_i^d \).
\( T \leftarrow T - \lambda_i v_i^{\otimes d} \).

Return \( v_1, ..., v_n \) and \( \lambda_1, ..., \lambda_n \).
The Number of Eigenvectors of a Tensor

Recall: Given a tensor $T \in S^d(\mathbb{C}^n)$ with corresponding polynomial $f$, the eigenvectors $x \in \mathbb{C}^n$ are the solutions to the equations given by the $2 \times 2$ minors of the matrix

$$[\nabla f(x) | x].$$

Theorem (Sturmfels and Cartwright)

If a tensor $T \in S^d(\mathbb{C}^n)$ has finitely many eigenvectors, then their number is $\frac{(d-1)^n-1}{d-2}$. 
Eigenvectors of Odeco Tensors

Odeco tensors are nice because we can characterize all of their eigenvectors.

**Theorem**
Let $f \in S^d(\mathbb{C}^n)$ be an odec tensor with $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \lambda_i (v_i \cdot x)^d$, where $V$ is the orthogonal matrix with columns $v_1, \ldots, v_n$. Then, $f$ has $\frac{(d-1)^n-1}{d-2}$ eigenvectors. Explicitly, they are

$$
\begin{bmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{bmatrix}
= V
\begin{bmatrix}
  \lambda_1^{-\frac{1}{d-2}} \\
  \eta_2 \lambda_2^{-\frac{1}{d-2}} \\
  \vdots \\
  \eta_k \lambda_k^{-\frac{1}{d-2}} \\
  0 \\
  \vdots \\
  0
\end{bmatrix},
$$

where $k = 1, \ldots, n$ and $\eta_2, \ldots, \eta_k$ are $(d-2)$-nd roots of unity, up to permutation.
Example \((d = 3, \ n = 3)\)

Let

\[ T = e_1 \otimes^3 + e_2 \otimes^3 + e_3 \otimes^3. \]

Then, \(V = I\), the identity matrix and the eigenvectors of \(T\) are:

\[
\begin{align*}
  k = 1 & \quad (1:0:0)^T, (0:1:0)^T, (0:0:1)^T \\
  k = 2 & \quad (1:1:0)^T, (1:0:1)^T, (0:1:1)^T \\
  k = 3 & \quad (1:1:1)^T.
\end{align*}
\]
The Set of Odeco Tensors

- **Parametric representation:**
The set of orthogonally decomposable tensors can be parametrized by $\mathbb{R}^n \times O_n(\mathbb{R})$:

$$\lambda, V \mapsto \sum_{i=1}^{n} \lambda_i v_i \otimes d.$$  

- **Implicit representation:**
The set of orthogonally decomposable tensors can also be represented as the solutions to a set of equations.

**Definition**
The *odeco variety* is the Zariski closure of the set of all odeco tensors in $S^d(\mathbb{C}^n)$.

**Goal:** find equations defining this variety.
The Odeco Variety

Let $T \in S^d(\mathbb{C}^n)$. Let $\mathcal{F}$ be the set of the following equations:

$$
\mathcal{F} = \langle \sum_{s=1}^{n} T_{i_1,\ldots,i_{d-1},s} T_{j_1,\ldots,j_{d-1},s} - T_{k_1,\ldots,k_{d-1},s} T_{l_1,\ldots,l_{d-1},s} \rangle,
$$

where $i_1,\ldots,i_{d-1}, j_1,\ldots,j_{d-1}, k_1,\ldots,k_{d-1}, l_1,\ldots,l_{d-1} \in \{1, 2, \ldots, n\}$ are such that $\{i_1,\ldots,i_{d-1}, j_1,\ldots,j_{d-1}\} = \{k_1,\ldots,k_{d-1}, l_1,\ldots,l_{d-1}\}$.

Conjecture

The odeco variety is given by $\mathcal{V}(\mathcal{F})$ for general $n$.

Lemma

The equations $\mathcal{F}$ vanish on the set of orthogonally decomposable tensors.

Proposition

The odeco variety is equal to $\mathcal{V}(\mathcal{F})$ for $n = 2$, i.e. in the case of $2 \times 2 \times \cdots \times 2$ tensors.
Nonsymmetric Tensor Decomposition

Let $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n = (\mathbb{R}^n)^\otimes d$. A decomposition of $T$ is an expression of the form

$$T = \sum_{i=1}^{r} \lambda_i a_i \otimes b_i \otimes c_i \otimes \cdots .$$

A tensor $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ is orthogonally decomposable, or odeco, if we can decompose it as

$$T = \sum_{i=1}^{n} \lambda_i a_i \otimes b_i \otimes c_i \otimes \cdots ,$$

so that $a_1, \ldots, a_n \in \mathbb{R}^n$ are orthonormal, $b_1, \ldots, b_n \in \mathbb{R}^n$ are orthonormal, $c_1, \ldots, c_n \in \mathbb{R}^n$ are orthonormal, etc.
Nonsymmetric Odeco Tensors

Example

1. If $T \in \mathbb{R}^n \otimes \mathbb{R}^n$ is a matrix, then $T$ has singular value decomposition

   $$T = U \Sigma V^T = \sum_{i=1}^{k} \sigma_i u_i v_i^T,$$

   where $u_1, ..., u_k$ are orthonormal and $v_1, ..., v_k$ are orthonormal.

2. The tensor $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ given by

   $$T = \sum_{i=1}^{n} \lambda_i e_i \otimes e_i \otimes \cdots \otimes e_i$$

   is odeco.
Singular Vector Tuples

Example
Given a matrix $T \in \mathbb{R}^n \otimes \mathbb{R}^n$, $(u, v)$ is a singular vector tuple of $T$ if there exist $\lambda_1$ and $\lambda_2$ such that

$$Tu = \begin{bmatrix} \vdots \\ \sum_j T_{ij}u_j \end{bmatrix} = \lambda_1 v$$

and

$$T^T v = \begin{bmatrix} \vdots \\ \sum_i T_{ij}v_i \end{bmatrix} = \lambda_2 u.$$

Definition
Given a tensor $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ a singular vector tuple is a $d$-tuple $(x_1, \cdots, x_d) \in \mathbb{C}^n \times \cdots \times \mathbb{C}^n$ such that for every $1 \leq k \leq d$,

$$\begin{bmatrix} \ Sum_{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_d} T_{i_1i_{k-1}i_{k+1}\ldots i_d}x_{i_1}x_{i_1}x_{i_1(k-1)}x_{i_{k+1}(k+1)}\ldots x_{i_d} \\ \vdots \end{bmatrix} = \lambda_k x_k,$$

for some $\lambda_k \in \mathbb{C}$. 
Examples

1. If $T \in \mathbb{R}^n \otimes \mathbb{R}^n$ is a generic matrix with singular value decomposition

$$T = U \Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T,$$

$(u_1, v_1), \ldots, (u_n, v_n)$ are all of the singular vector pairs of $T$.

2. Let $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ be given by $T = \sum_{i=1}^{n} \lambda_i e_i \otimes \cdots \otimes e_i$. Therefore, $(x_1, \cdots, x_d)$ is a singular vector tuple of $T$ if and only if

$$\begin{bmatrix}
\lambda_1 x_11 \cdots x_{1(k-1)} x_{1(k+1)} \cdots x_{1d} \\
\vdots \\
\lambda_n x_{n1} \cdots x_{n(k-1)} x_{n(k+1)} \cdots x_{nd}
\end{bmatrix} = \lambda_k x_{.k},$$

i.e. the matrix

$$\begin{bmatrix}
\lambda_1 x_11 \cdots x_{1(k-1)} x_{1(k+1)} \cdots x_{1d} & x_{1k} \\
\vdots & \vdots \\
\lambda_n x_{n1} \cdots x_{n(k-1)} x_{n(k+1)} \cdots x_{nd} & x_{nk}
\end{bmatrix}$$

has rank 1 for every $1 \leq k \leq d$. 
The number of singular vector tuples of a tensor

**Theorem (Friedland and Ottaviani)**

Let $T \in \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ be a generic tensor. Then, $T$ has finitely many singular vector tuples and they correspond to nonzero singular values. Their number is the coefficient of the monomial $\prod_{i=1}^{d} t_i^{n-1}$ in the polynomial

$$
\prod_{i=1}^{d} \frac{(\sum_j t_j - t_i)^n - t_i^n}{\sum_j t_j - 2t_i}.
$$
Singular Vectors of Odeco Tensors

**Theorem**

Let $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ be odeco with decomposition $T = \sum_{i=1}^{n} \lambda_i a_i \otimes b_i \otimes c_i \otimes \cdots$. Let $A = (a_1 \mid a_2 \mid \cdots \mid a_n)$, $B = (b_1 \mid b_2 \mid \cdots \mid b_n)$, etc., so that $A$, $B$, $C$, etc., are orthogonal matrices. Then, the singular vector tuples of $T$ are given as follows:

**Type I**

\[
A = \begin{bmatrix}
\lambda_1^{\frac{1}{d-2}} \\
\chi_{12} \eta_2 \lambda_2^{\frac{1}{d-2}} \\
\vdots \\
\chi_{1k} \eta_k \lambda_k^{\frac{1}{d-2}} \\
0 \\
\vdots \\
0
\end{bmatrix} \otimes
B = \begin{bmatrix}
\lambda_1^{\frac{1}{d-2}} \\
\chi_{22} \eta_2 \lambda_2^{\frac{1}{d-2}} \\
\vdots \\
\chi_{2k} \eta_k \lambda_k^{\frac{1}{d-2}} \\
0 \\
\vdots \\
0
\end{bmatrix} \otimes
C = \begin{bmatrix}
\lambda_1^{\frac{1}{d-2}} \\
\chi_{32} \eta_2 \lambda_2^{\frac{1}{d-2}} \\
\vdots \\
\chi_{3k} \eta_k \lambda_k^{\frac{1}{d-2}} \\
0 \\
\vdots \\
0
\end{bmatrix} \otimes \cdots,
\]

where $1 \leq k \leq n$, $\chi_{ij}$ is a 2-nd root of unity, $\eta_i$ is a $(d-2)$-nd root of unity, up to permutation.

**Type II**

\[
Ax_1 \otimes Bx_2 \otimes Cx_3 \otimes \cdots,
\]

where the matrix $X = (x_{ij})_{ij}$ has at least two zeros in each row and no column is identical to 0.
Let $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ be given by $T = \sum_{i=1}^{n} \lambda_i e_i \otimes \cdots \otimes e_i$. Therefore, $(x_1, \cdots, x_d)$ is a singular vector tuple of $T$ if and only if

$$
\begin{bmatrix}
\lambda_1 x_{11} \cdots x_{1(k-1)} x_{1(k+1)} \cdots x_{1d} \\
\vdots \\
\lambda_n x_{n1} \cdots x_{n(k-1)} x_{n(k+1)} \cdots x_{nd}
\end{bmatrix} = \lambda_k x_{k},
$$

i.e. the matrix

$$
\begin{bmatrix}
\lambda_1 x_{11} \cdots x_{1(k-1)} x_{1(k+1)} \cdots x_{1d} & x_{1k} \\
\vdots & \vdots \\
\lambda_n x_{n1} \cdots x_{n(k-1)} x_{n(k+1)} \cdots x_{nd} & x_{nk}
\end{bmatrix}
$$

has rank 1 for every $1 \leq k \leq d$. 
The Set of Odeco Tensors

Definition
The *odeco variety* is the Zariski closure of the set of all odeco tensors in $\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$.

Goal: find equations defining this variety.

Conjecture
The prime ideal of the set of odeco tensors is given by

$$I = \langle \sum_{s=1}^{n} T_{a_1\ldots a_{r-1}s a_r \ldots a_d} T_{b_1\ldots b_{r-1}s b_r \ldots b_d} - T_{c_1\ldots c_{r-1}s c_r \ldots c_d} T_{d_1\ldots d_{r-1}s d_r \ldots d_d} \rangle,$$

where $a, b, c, d \in \{1, \ldots, n\}^{d-1}$, $\{a_i, b_i\} = \{c_i, d_i\}$ and $r \in \{1, \ldots, n\}$.

Lemma
*The ideal $I$ vanishes on the set of odeco tensors.*
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