Cactus Varieties of Cubic Forms:
Apolar Local Artinian Gorenstein Rings

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Brief

1. Waring rank and Cactus rank
2. Theorem: Dimensions of Cactus varieties for cubic forms
3. Apolarity
4. Local apolar schemes
5. Proof of the Theorem
Waring rank

\[ F \in S_d = K[x_0, \ldots, x_n]_d \text{ homog. of deg } d \text{ (char } K \neq 2, 3) \]

Definition

**Waring rank:** \( \min r \in \mathbb{N} \text{ s.t. } F = L_1^d + \cdots + L_r^d \text{ with } L_i \in S_1 \)

Veronese: \( \nu_d : \mathbb{P}(S_1) \to \mathbb{P}(S_d), [L] \mapsto [L^d] \)

- \( \min r \in \mathbb{N} \text{ s.t. } [F] \in r\text{-th secant space to } \nu_d(\mathbb{P}(S_1)) \)
- The shortest length of a ***smooth*** finite scheme \( \Gamma \subset \mathbb{P}(S_1) \text{ s.t. } [F] \in \langle \nu_d(\Gamma) \rangle, \Gamma = \{[L_1], \ldots, [L_r]\} \)
Cactus rank

Remove “smooth”

Definition

**Cactus Rank**: \( \min r \in \mathbb{N} \) s.t. \( \exists \) finite (Gor.) scheme \( \Gamma \) of length \( r \) s.t. \( [F] \in \langle \nu_d(\Gamma) \rangle \)

\[
\text{Sec}_r(\nu_d(\mathbb{P}(S_1))) = \bigcup_{\Gamma \in \text{Hilb}_r\mathbb{P}(S_1), \Gamma \text{smooth}} \langle \nu_d(\Gamma) \rangle \quad \text{Secant variety to Veronese}
\]

\[
\text{Cactus}_r(\nu_d(\mathbb{P}(S_1))) = \bigcup_{\Gamma \in \text{Hilb}_r\mathbb{P}(S_1)} \langle \nu_d(\Gamma) \rangle \quad \text{Cactus variety}
\]
Cactus rank

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Cactus variety
Theorem

We focus on **CUBIC** forms

**Theorem**

*When* \( r \leq 17 \), *then*

\[
\dim \text{Cactus}_r(X_3, n) = \dim \text{Sec}_r(X_3, n).
\]

*When* \( 18 \leq r \leq 2n + 2 \), *then* \( \dim \text{Cactus}_r(X_3, n) > \dim \text{Sec}_r(X_3, n) \) *and*

\[
\dim \text{Cactus}_r(X_3, n) =
\begin{cases}
\min \left\{ \frac{1}{48} r^3 - \frac{3}{8} r^2 + rn + \frac{5}{3} r - 2, \left( \frac{n+3}{3} \right) - 1 \right\}, & \text{if } r \geq 18 \text{ even}, \\
\min \left\{ \frac{1}{48} r^3 - \frac{7}{16} r^2 + rn + \frac{119}{48} r - \frac{65}{16}, \left( \frac{n+3}{3} \right) - 1 \right\}, & \text{if } r \geq 18 \text{ odd}.
\end{cases}
\]

[CNJ] : \( r \leq 13 \) Any local scheme is smoothable (=flat limit of smooth)
Theorem

We focus on CUBIC forms

Theorem

When $r \leq 17$, then

$$\dim \text{Cactus}_r(X_3,n) = \dim \text{Sec}_r(X_3,n).$$

When $18 \leq r \leq 2n + 2$, then $\dim \text{Cactus}_r(X_3,n) > \dim \text{Sec}_r(X_3,n)$ and

$$\dim \text{Cactus}_r(X_3,n) = \begin{cases} 
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\end{cases}$$

[CNJ]: $r \leq 13$ Any local scheme is smoothable (≃flat limit of smooth)
Apolarity

\[ S = K[x_0, \ldots, x_n], \quad T = K[y_0, \ldots, y_n] \]

\[ y^\alpha(x^{[\beta]}) = \begin{cases} x^{[\beta-\alpha]} & \text{if } \beta \geq \alpha, \\ 0 & \text{otherwise.} \end{cases} \]

\[ S_1 \text{ and } T_1 \text{ are dual spaces.} \]

\[ T \text{ is naturally the coordinate ring of } \mathbb{P}(S_1). \]

**Definition**

\[ f \in S. \text{ Apolar ideal: } f^\perp = \{ \varphi \in T \mid \varphi(f) = 0 \}. \]

\[ \Gamma \subset \mathbb{P}(S_1) \text{ is apolar to } F \in S \text{ if } I_\Gamma \subset F^\perp \subset T. \]

**Lemma (Apolarity Lemma)**

\[ \Gamma \subset \mathbb{P}(S_1) \text{ is apolar to } F \in S_d \text{ iff } [F] \in \langle \nu_d(\Gamma) \rangle \subset \mathbb{P}(S_d). \]
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Apolarity

\[ \text{crk}(F) = \min r \in \mathbb{N} \text{ s.t. } \exists \Gamma \in \text{Hilb}_r(\mathbb{P}(S_1)): [F] \in \langle \nu_d(\Gamma) \rangle \]

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\[ \text{crk}(F) = \min r \in \mathbb{N} \text{ s.t. } \exists \Gamma \in \text{Hilb}_r(\mathbb{P}(S_1)): l_{\Gamma} \subset F^\perp \]
Local apolar scheme

Properties: $T_f := T/f^\perp$ local Artinian Gorenstein ring ([IK]):

- Local: The image of $T_1$ in $T_f$ generates the only max ideal $m$;
- Artinian: $T_f$ is finitely generated as $K$-mod;
- Gor: $T_f$ has 1-dim’l socle (the annihilator of the max ideal).

If $F$ is homog. $\Rightarrow T/F^\perp$ graded.
Local apolar scheme

\[ F \in S_d \Rightarrow \Gamma \text{ apolar scheme locally Gor. } \Rightarrow \]

\[ \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_s, \text{ with } \Gamma_i \text{ local A.G.} \]

\[ F = F_1 + \cdots + F_s \text{ s.t. } \Gamma_i \text{ apolar to } F_i \]

More generally: any Local AG scheme \( \Gamma_i \) is the AFFINE apolar scheme of a poly \( g_i \in S \) (unique up to a unit in the ring of diff. operators): for any LAG \( T^0/I \), \( \exists g \in S^0 \text{ s.t. } I = g^\perp \).
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More generally: any Local AG scheme \( \Gamma_i \) is the AFFINE apolar scheme of a poly \( g_i \in S \) (unique up to a unit in the ring of diff. operators): for any LAG \( T^0/l, \exists g \in S^0 \text{ s.t. } l = g^\perp \).
Proof of the Thm

\( F \) defines \( \Gamma \):

\[ F \in S_d \Rightarrow \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_s, \text{ with } \Gamma_i \text{ local A.G.} \]

\( \Gamma \) is defined by \( g_i \)'s:

\[ \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_s \leftarrow g_1 + \cdots + g_s \text{ s.t. } \Gamma_i \text{ AFFINE apolar to } g_i \]

Plan of the proof of the Theorem:

1. Understand the link between \( F \) and the \( g_i \)'s.
Proof of the Thm

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$$g = g^{(0)} + \cdots + g^{(d)} + g^{d+1} + \cdots + g^{(l)}$$

where $\deg d$ tail of $g$.

Proposition

$F \in S_d$, $f = F(1, x_1, \ldots, x_n)$. Let $\Gamma$ be a scheme of minimal length among local schemes supported at $[l] = [1 : 0 : \ldots : 0]$ that are apolar to $F \Rightarrow \Gamma$ is the AFFINE apolar scheme to a poly $g \in K[x_1, \ldots, x_n]$ whose $\deg d$ tail equals $f$.

So $g$ may be chosen s.t. $f$ is its tail (: $\Gamma$ is also defined by many $g$’s that do not have $f$ as a tail)
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Natural apolar scheme of $F$ at $[l]$: $Z_{F,l} := V(f^\perp)$

Lemma (Buczyński)

$Z$ local supp at $[l]$ apolar to $F \Rightarrow \exists Z' \subset Z$ apolar to $F$ s.t. $Z' = Z_{G,l}$ for some $G \in S$. Moreover $F = \Psi(G)$ for some $\Psi \in T$.

$\Rightarrow$ we can choose $\Gamma = Z_{G,l}$ for some $G$ s.t. $F = \Psi(G)$.

If $F = \Psi(G)$ then $f$ and $\psi(g)$ have the same deg $d$ tail.

$\text{Diff}(\psi(g)) \subset \text{Diff}(g) \Rightarrow Z_{\psi(g)} \subset \Gamma$.

$\Gamma$ minimal $\Rightarrow$ suff. to see that $Z_{\psi(g)}$ apolar $F$.

$(\psi'(\psi(g)) = 0 \Rightarrow \Psi'(\Psi(G)) = 0.)$
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$(\psi'(\psi(g)) = 0 \Rightarrow \Psi'(\Psi(G)) = 0.)$
Parameterize the set of poly’s $g \in K[x_1, \ldots x_n]$ whose affine apolar scheme has given length.  

Parameterize the family of cubic tails $f \in K[x_1, \ldots, x_n]$ of $g$’s.

$$C_{r,l} = \bigcup_{\text{supp } Z_l = [l^d], l(Z_l) \leq r} \langle Z_l \rangle$$

$$= \{ [F] \in \mathbb{P}(S_d) | f = g_{\leq d} \text{ for some } g \in S_{loc} \text{ with } \dim \text{Diff}(g) \leq r \}$$

$$W_{r,n} = \bigcup_{l \in S_1} C_{r,l}$$

$$\text{Cactus}_r(\nu_d(\mathbb{P}(S_1))) = \bigcup_{r_1 + \cdots + r_s = r} J(W_{r_1,n}, \ldots, W_{r_s,n})$$
Proof of the Thm

2 Parameterize the set of poly’s $g \in K[x_1, \ldots x_n]$ whose affine apolar scheme has given length. \textit{Proposition} Parameterize the family of cubic tails $f \in K[x_1, \ldots, x_n]$ of $g$’s.

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$$W_{r,n} = \bigcup_{l \in S_1} C_{r,l}$$

$$\text{Cactus}_r(\nu_d(\mathbb{P}(S_1))) = \overline{\bigcup_{r_1 + \cdots + r_s = r} J(W_{r_1,n}, \ldots, W_{r_s,n})}$$
Proof of the Thm

2 Parameterize the set of poly’s $g \in K[x_1, \ldots x_n]$ whose affine apolar scheme has given length. \[\text{Proposition} \iff \text{Parameterize the family of cubic tails } f \in K[x_1, \ldots, x_n] \text{ of } g's.\]

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C_{r,l} = \bigcup_{\text{supp}Z_l = [l^d], l(Z_l) \leq r} \langle Z_l \rangle
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\[
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Proof of the Thm

Plan of the proof of the Theorem:

1. Understand the link between $F$ and the $g_i$’s.

2. Parameterize the set of poly’s $g \in K[x_1, \ldots x_n]$ whose affine apolar scheme has given length.

   Proposition

   Parameterize the family of cubic tails $f \in K[x_1, \ldots , x_n]$ of $g$’s.

   Find a discrete invariant for LAG schemes, parameterize the cubic tails of all polynomials that define a scheme with given invariant.

3. Show which invariants have the biggest family of cubic tails.
   (Invariant: Hilbert function with symmetric decomposition)
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2. Parameterize the set of poly’s \( g \in K[x_1, \ldots x_n] \) whose affine apolar scheme has given length.

Find a discrete invariant for local AG schemes, parameterize the cubic tails of all polynomials that define a scheme with given invariant.

\[ f \in S, \deg(f) = d, f^\perp \subset T \]

\[ T_f := T / f^\perp \overset{\sim}{\to} \text{Diff}(f) = \{ \psi(f) \mid \psi \in T \} \]

\[ \psi \mapsto \psi(f) \]
Proof of the Thm

**Iarrobino**

\( T_f \) is local \( \Rightarrow \) one max ideal \( m \).

- **\( m \)-adic filtration:**
  \[
  T_f = m^0 \supset \cdots \supset m^{d+1} = 0
  \\
  T_f^* = \bigoplus_{i=0}^{d} \frac{m^i}{m^{i+1}}
  
  \]

- **Łöewy filtration:**
  \[
  T_f = (0 : m^{d+1}) \supset \cdots \supset
  \\
  (0 : m) \supset 0
  
  \]

**Interpr. in terms of partial of \( f \)**

- **\( m^i \mapsto \)** Partial of order at least \( i \) of \( f \) (deg \( \leq d - i \))
  
  (Order of a partial \( f' \) of \( f = \) largest order of \( \psi \in T \) s.t. \( f' = \psi(f) \))

- **\( (0 : m^i) \mapsto \)**
  \[
  \text{Diff}(f)_{i-1} = \text{partials of deg at most } i - 1 \text{ of } f
  
  \]
Proof of the Thm

Order filtration: \( \text{Diff}(f) = \text{Diff}(f)_0 \supseteq \text{Diff}(f)_1 \supseteq \cdots \supseteq \text{Diff}(f)_d \)

Degree filtration: \( \text{Diff}(f) = \text{Diff}(f)_d \supseteq \text{Diff}(f)_{d-1} \supseteq \cdots \supseteq \text{Diff}(f)_0 \)

Different filtrations \( f \) not homog

but

\[
\frac{(0 : m^i)}{(0 : m^{i-1})} \cong \left( \frac{m^{i-1}}{m^i} \right)^\vee
\]

\[
\frac{\text{Diff}(f)_{i+1}}{\text{Diff}(f)_i} \cong \frac{\text{Diff}(f)_{i+1}}{\text{Diff}(f)_i}
\]

and for the double filtration:

\( \text{Diff}(f)_i^a = \) Partials of deg at most \( i \) and order at least \( d - i - a \)

\[
Q_{a,i}^\vee \cong \frac{\text{Diff}(f)_i^a}{\text{Diff}(f)_{i-1}^a + \text{Diff}(f)_{i-1}^{a-1}} \cong \frac{\text{Diff}(f)^i_a}{\text{Diff}(f)_{i-1}^a + \text{Diff}(f)_{i-1}^a}
\]

So the Hf of the two filtrations are dual to each other.
Proof of the Thm

In particular

\[ H_f(i) = \dim_K(Diff(f))_i - \dim_K(Diff(f))_{i-1} \]

has symmetric decomposition:

\[ H = \sum_{a \geq 0} \Delta_a \]

each \( \Delta_a \) symm. around \( (d - a)/2 \), i.e. \( \Delta_a(i) = \Delta_a(d - a - i) \)

where \( \dim(Q^\vee_{a,i}) = \Delta_{f,a}(i) \).

Every partial sum \( \sum_{a=0}^{\alpha} \Delta_{Q_a} \) is the Hf of a \( K \)-alg. generated in deg 1.

(Various restrictions on the possible decompositions (Mgc))
Proof of the Thm

Our goal was to parameterize the set of poly’s $f \in K[x_1, \ldots, x_n]$ with a given length for their subschem $Z_f$

Example

$f = x_1^3 + x_2^2$, Length($Z_f$) = 5:

$$1(f) = f = x_1^3 + x_2^2$$

$$y_1(f) = x_1^2$$
$$y_2(f) = x_2$$

$$y_1^2(f) = x_1$$
$$y_2^2(f) = 1$$

$$y_1^3(f) = 1$$

$H = 1 \ 2 \ 1 \ 1$

$\Delta_0 = 1 \ 1 \ 1 \ 1$

$\Delta_1 = 0 \ 1 \ 0 \ 0$

Next step: Characterize poly’s $f$ with given symm decomp. for $H_f$. 
Our goal was to parameterize the set of poly’s $f \in K[x_1, \ldots, x_n]$ with a given length for their subschem $Z_f$

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Next step: Characterize poly’s $f$ with given symm decomp. for $H_f$. 
Proof of the Thm

A general cubic form $F \in S = \mathbb{C}[y_0, \ldots, y_n]$ with even local cactus rank $2m, m \leq n$ is projectively equivalent to some

$$f_3 + x_0 f_2 + x_0^2 f_1 + x_0^3 f_0$$

where

$$f_3 \in \mathbb{C}[x_1, \ldots, x_{m-1}]_3,$$

$$f_2 \in \langle x_1, \ldots, x_n \rangle \cdot \langle x_1, \ldots, x_{m-1} \rangle,$$

$$f_1 \in \langle x_1, \ldots, x_n \rangle,$$

$$f_0 \in \mathbb{C}.$$

The forms of local cactus rank $2n$ form a family of codimension $\binom{n-1}{2} + 1$ in the space of cubic forms $\mathbb{C}[x_0, \ldots, x_n]_3$. 
Proof of the Thm

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Proof of the Thm

A general cubic form $F \in S = \mathbb{C}[y_0, \ldots, y_n]$, with odd local cactus rank $2m + 1$, $m \leq n$ is projectively equivalent to some

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where

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The forms of local cactus rank $2n + 1$, $n > 3$ form a family of codimension $\binom{n-2}{2} - 1$ in the space of cubic forms $\mathbb{C}[x_0, \ldots, x_n]_3$. 
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A general cubic form \( F \in S = \mathbb{C}[y_0, \ldots, y_n] \), with odd local cactus rank \( 2m + 1 \), \( m \leq n \) is projectively equivalent to some

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Proof of the Thm

3. Show which invariants have the biggest family of cubic tails.

\[ C_{r,l} = \bigcup_{\text{supp}(Z_l) = \nu_3([l]), l(Z_l) \leq r} \langle Z_l \rangle \iff \text{Parameterize the family of cubic tails } f \in K[x_1, \ldots, x_n] \text{ of } g's: \]

\[ V_r(3, n) = \{ f \mid f = g_{\leq 3} \text{ for some } g \in K[x_1, \ldots, x_n], \dim \text{Diff}(g) \leq r \} \]

\[ V(3, \Delta, n) = \{ f_{\leq 3} \mid f \in K[x_1, \ldots, x_n], \Delta_f = \Delta \} \]

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For which Hf and which \( \Delta, \nu(3, \Delta, n) := \dim(V(3, \Delta, n)) \) attains its max given \( r \)?

\[ \implies \max(\nu(3, \Delta, n)) = \text{upper bound for } \nu_r(3, n) := \dim(V_r(3, \Delta)) \]
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For which Hf and which \( \Delta, v(3, \Delta, n) := \dim(V(3, \Delta, n)) \) attains its max given \( r \)?

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Proof of the Thm

Proposition

\[ r \geq 7, \quad v(3, \Delta, n) \text{ attains its max for} \]

\[
\begin{align*}
\Delta = (1, m - 1, m - 1, 1), & \quad r = 2m, \\
\Delta = (1, 1, 1, 1, 1), (0, m - 2, m - 2, 0), & \quad r = 2m + 1
\end{align*}
\]

and

\[ v(3, \Delta, n) = \begin{cases} 
M_e := \binom{m+2}{3} + 2m(n - m) + 3m - n - 1, & r = 2m, \\
M_o := \binom{m+2}{3} + 2m(n - m) + 3m - 2, & r = 2m + 1.
\end{cases} \]
Proof of the Thm

\[ \text{Cactus}_r(X_3, n) = \bigcup J(W_{r_1,n}, \ldots, W_{r_s,n}) \]

\[ W_{r,n} = \bigcup_{l \in S_1} C_{r,l} \] Local cactus variety

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max(\( v(3, \Delta, n) \)) \( \Rightarrow \) upper bound for \( v_r(3, n) \), i.e. for \( \dim(C_{r,l}) \)

The largest component of \( W_{2m,n} \) is the union as \( l \) varies, of projective varieties whose affine cones are isomorphic to \( V_{2m}(3, n) \), so a parameterization of \( W_{2m,n} \) has dimension \( v_{2m}(3, n) - 1 + n \).

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BUT parameterization may not be generically finite, the formulas are upper bounds for the dimension of these varieties.
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Cactus Varieties of Cubic Forms: Apolar Local Artinian Gorenstein Rings

J. Jelisiejew, P. Macias Marques, K. Ranestad

Alessandra Bernardi

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We have equality if we show that the parameterization is generically one to one.

When $b$ is even, for a general $[F] \in W_{b,n}$ there is a unique $l$ such that $Z_{F,l}$ has length $b$.

When $b$ is odd, and there is a unique $l$ such that $f = \pi_l(F)$ is the tail of a quartic polynomial $g_l$ whose apolar scheme $Z_{g_l}$ has length $b$.

So $\dim(\text{Cactus}_r(X_3,n)) = \dim W_{r,n}$. 
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THANKS!