Cactus Varieties of Cubic Forms: Apolar Local Artinian Gorenstein Rings

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Brief

1. Waring rank and Cactus rank
2. Theorem: Dimensions of Cactus varieties for cubic forms
3. Apolarity
4. Local apolar schemes
5. Proof of the Theorem

Waring rank

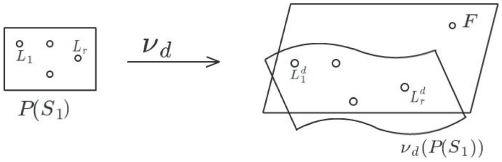
$F \in S_d = K[x_0, \dots, x_n]_d$ homog. of deg d ($\mathrm{char} K \neq 2, 3$)

Definition

Waring rank: $\min r \in \mathbb{N}$ s.t. $F = L_1^d + \cdots + L_r^d$ with $L_i \in S_1$

Veronese: $\nu_d : \mathbb{P}(S_1) \to \mathbb{P}(S_d), [L] \mapsto [L^d]$

- $\min r \in \mathbb{N}$ s.t. $[F] \in r$-th secant space to $\nu_d(\mathbb{P}(S_1))$
- The shortest length of a smooth finite scheme $\Gamma \subset \mathbb{P}(S_1)$ s.t. $[F] \in \langle \nu_d(\Gamma) \rangle, \Gamma = \{[L_1], \dots, [L_r]\}$



Cactus rank

Remove "smooth"

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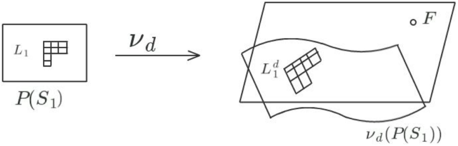

$\mathrm{Sec}_r(\nu_d(\mathbb{P}(S_1))) = \overline{\cup_{\Gamma \in Hilb_r \mathbb{P}(S_1), \Gamma \mathrm{smooth}} \langle \nu_d(\Gamma) \rangle}$ Secant variety to Veronese

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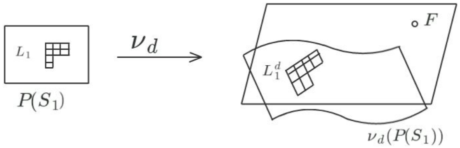

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Theorem

We focus on **CUBIC** forms

Theorem

When $r \leq 17$*, then*

$$\dim \mathrm{Cactus}_r(X_{3,n}) = \dim \mathrm{Sec}_r(X_{3,n}).$$

When $18 \leq r \leq 2n+2$*, then* $\dim \mathrm{Cactus}_r(X_{3,n}) > \dim \mathrm{Sec}_r(X_{3,n})$ *and*

$$\dim \mathrm{Cactus}_r(X_{3,n}) =$$

$$= \begin{cases} \min\left\{\frac{1}{48}r^3 - \frac{3}{8}r^2 + rn + \frac{5}{3}r - 2, \binom{n+3}{3} - 1\right\}, \text{ if } r \geq 18 \text{ even}, \\ \min\left\{\frac{1}{48}r^3 - \frac{7}{16}r^2 + rn + \frac{119}{48}r - \frac{65}{16}, \binom{n+3}{3} - 1\right\}, \text{ if } r \geq 18 \text{ odd}. \end{cases}$$

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Apolarity

$S = K[x_0, \ldots, x_n], \; T = K[y_0, \ldots, y_n]$

$$y^\alpha(x^{[\beta]}) = \begin{cases} x^{[\beta-\alpha]} & \text{if } \beta \geq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

S_1 and T_1 are dual spaces.

T is naturally the coordinate ring of $\mathbb{P}(S_1)$.

Definition

$f \in S$. **Apolar ideal:** $f^\perp = \{\varphi \in T \mid \varphi(f) = 0\}$.

$\Gamma \subset \mathbb{P}(S_1)$ is apolar to $F \in S$ if $I_\Gamma \subset F^\perp \subset T$.

Lemma (Apolarity Lemma)

$\Gamma \subset \mathbb{P}(S_1)$ is apolar to $F \in S_d$ iff $[F] \in \langle \nu_d(\Gamma) \rangle \subset \mathbb{P}(S_d)$.

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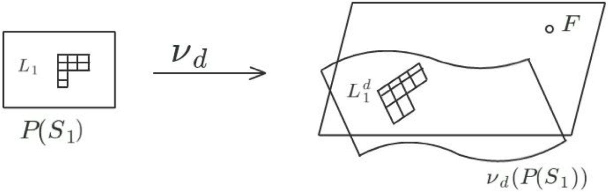

- $\mathrm{crk}(F) = \min r \in \mathbb{N}$ s.t. $\exists \Gamma \in \mathrm{Hilb}_r(\mathbb{P}(S_1)) : [F] \in \langle \nu_d(\Gamma) \rangle$
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Local apolar scheme

Properties: $T_f := T/f^\perp$ local Artinian Gorenstein ring ([IK]):

- Local: The image of T_1 in T_f generates the only max ideal m;
- Artinian: T_f is finitely generated as K-mod;
- Gor: T_f has 1-dim'l socle (the annihilator of the max ideal).

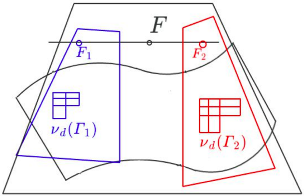
If F is homog. $\Rightarrow$ $T/F^\perp$ graded.

Local apolar scheme

$F \in S_d \Rightarrow \Gamma$ apolar scheme locally Gor. $\Rightarrow$

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_s, \text{ with } \Gamma_i \text{ local A.G.}$$

$$F = F_1 + \cdots + F_s \text{ s.t. } \Gamma_i \text{ apolar to } F_i$$



More generally: any Local AG scheme Γ_i is the AFFINE apolar scheme of a poly $g_i \in S$ (unique up to a unit in the ring of diff. operators): for any LAG $T^0/I, \exists g \in S^0 s.t. I = g^{\perp}$.

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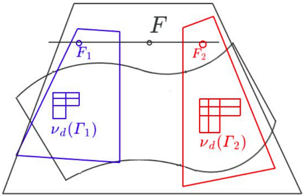

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Proof of the Thm

F defines Γ:

$F \in S_d \Rightarrow \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_s$, with Γ_i local A.G.

Γ is defined by g_i's:

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Proposition

$F \in S_d$, $f = F(1, x_1, \ldots, x_n)$. Let Γ be a scheme of minimal length among local schemes supported at $[l] = [1 : 0 : \ldots : 0]$ that are apolar to $F \Rightarrow \Gamma$ is the AFFINE apolar scheme to a poly $g \in K[x_1, \ldots, x_n]$ whose deg d tail equals f.

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Natural apolar scheme of F at $[l]$: $Z_{F,l} := V(f^{\perp})$

Lemma (Buczyński)

Z local supp at $[l]$ apolar to $F \Rightarrow \exists Z' \subset Z$ apolar to F s.t. $Z' = Z_{G,l}$ for some $G \in S$. Moreover $F = \Psi(G)$ for some $\Psi \in T$.

$\Rightarrow$ we can choose $\Gamma = Z_{G,l}$ for some G s.t. $F = \Psi(G)$.

If $F = \Psi(G)$ then f and $\psi(g)$ have the same deg d tail.

$Diff(\psi(g)) \subset Diff(g) \Rightarrow Z_{\psi(g)} \subset \Gamma$.

Γ minimal $\Rightarrow$ suff. to see that $Z_{\psi(g)}$ apolar F

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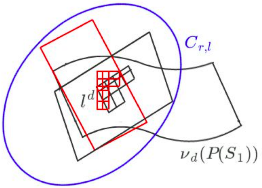
Proof of the Thm

2. Parameterize the set of poly's $g \in K[x_1, \dots x_n]$ whose affine apolar scheme has given length. $\overset{\text{Propostion}}{\Longleftrightarrow}$ Parameterize the family of cubic tails $f \in K[x_1, \dots, x_n]$ of g's.

$$C_{r,l} = \bigcup_{\mathrm{supp} Z_l = [l^d], l(Z_l) \le r} \langle Z_l \rangle$$



$= \{[F] \in \mathbb{P}(S_d) | f = g_{\le d}$ for some $g \in S_{loc}$ with $\dim Diff(g) \le r\}$

$W_{r,n} = \bigcup_{l \in S_1} C_{r,l}$

$\mathrm{Cactus}_r(\nu_d(\mathbb{P}(S_1))) = \overline{\bigcup_{r_1 + \dots + r_s = r} J(W_{r_1,n}, \dots, W_{r_s,n})}$

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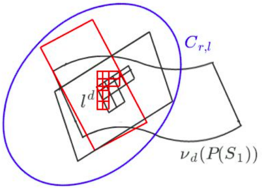

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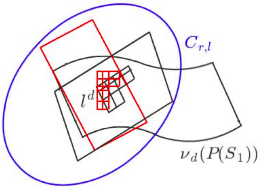

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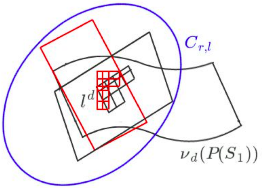

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   Find a discrete invariant for local AG schemes, parameterize the cubic tails of all polynomials that define a scheme with given invariant.

$f \in S$, $\deg(f) = d$, $f^\perp \subset T$

$$\begin{array}{rcl} T_f := T/f^\perp & \xrightarrow[\tau]{\sim} & Diff(f) = \{\psi(f) \,|\, \psi \in T\} \\ \psi & \mapsto & \psi(f) \end{array}$$

Proof of the Thm

Iarrobino

T_f is local $\Rightarrow$ one max ideal m.

- m-adic filtration:

  $T_f = m^0 \supset \cdots \supset m^{d+1} = 0$

  $T_f^* = \bigoplus_{i=0}^{d} \frac{m^i}{m^{i+1}}$

- Löewy filtration:

  $T_f = (0 : m^{d+1}) \supset \cdots \supset (0 : m) \supset 0$

Interpr. in terms of partial of f

- $m^i \overset{\mathcal{T}}{\mapsto}$Partial of order at least i of f (deg $\leq d - i$) (Order of a partial f' of f = largest order of $\psi \in T$ s.t. $f' = \psi(f)$)

- $(0 : m^i) \overset{\mathcal{T}}{\mapsto}$ $Diff(f)_{i-1}$ =partials of deg at most $i - 1$ of f

Proof of the Thm

Order filtration: $Diff(f) = Diff(f)^0 \supseteq Diff(f)^1 \supseteq \cdots \supseteq Diff(f)^d$

Degree filtration: $Diff(f) = Diff(f)_d \supseteq Diff(f)_{d-1} \supseteq \cdots \supseteq Diff(f)_0$

Different filtrations (f not homog)

but

$$\frac{(0:m^i)}{(0:m^{i-1})} \cong \left(\frac{m^{i-1}}{m^i}\right)^\vee \qquad \frac{Diff(f)_{i+1}}{Diff(f)_i} \simeq \frac{Diff(f)^{i+1}}{Diff(f)^i}$$

and for the double filtration:

$Diff(f)_i^a$ =Partials of deg at most i and order at least $d-i-a$

$$Q_{a,i}^\vee \simeq \frac{Diff(f)_i^a}{Diff(f)_{i-1}^a + Diff(f)_i^{a-1}} \simeq \frac{Diff(f)_a^i}{Diff(f)_a^{i-1} + Diff(f)_{a-1}^i}$$

So the Hf of the two filtrations are dual to each other.

Proof of the Thm

In particular

$$H_f(i) = \dim_K(Diff(f))_i - \dim_K(Diff(f)_{i-1})$$

has symmetric decomposition:

$$H = \sum_{a \geq 0} \Delta_a$$

each Δ_a symm. around $(d-a)/2$, i.e. $\Delta_a(i) = \Delta_a(d-a-i)$

where $\dim(Q^\vee_{a,i}) = \Delta_{f,a}(i)$.

Every partial sum $\sum_{a=0}^{\alpha} \Delta_{Q_a}$ is the Hf of a K-alg. generated in deg 1.

(Various restrictions on the possible decompositions (Mgc))

Proof of the Thm

Our goal was to parameterize the set of poly's $f \in K[x_1, \dots, x_n]$ with a given length for their subschem Z_f

Example

$f = x_1^3 + x_2^2$, Length$(Z_f) = 5$:

$1(f) = f = x_1^3 + x_2^2$

$y_1(f) = x_1^2$ $\quad$ $y_2(f) = x_2$

$y_1^2(f) = x_1$ $\quad$ $y_2^2(f) = 1$

$y_1^3(f) = 1$

H	$=$	1	2	1	1
Δ_0	$=$	1	1	1	1
Δ_1	$=$	0	1	0	0

Next step: Characterize poly's f with given symm decomp. for H_f.

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Proof of the Thm

A general cubic form $F \in S = \mathbb{C}[y_0, \ldots, y_n]$ with even local cactus rank $2m$, $m \leq n$ is projectively equivalent to some

$$f_3 + x_0 f_2 + x_0^2 f_1 + x_0^3 f_0$$

where

$$\begin{aligned} f_3 &\in \mathbb{C}[x_1, \ldots, x_{m-1}]_3, \\ f_2 &\in \langle x_1, \ldots, x_n \rangle \cdot \langle x_1, \ldots, x_{m-1} \rangle, \\ f_1 &\in \langle x_1, \ldots, x_n \rangle, \\ f_0 &\in \mathbb{C}. \end{aligned}$$

The forms of local cactus rank $2n$ form a family of codimension $\binom{n-1}{2} + 1$ in the space of cubic forms $\mathbb{C}[x_0, \ldots, x_n]_3$.

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A general cubic form $F \in S = \mathbb{C}[y_0, \ldots, y_n]$, with odd local cactus rank $2m+1$, $m \leq n$ is projectively equivalent to some

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The forms of local cactus rank $2n+1$, $n > 3$ form a family of codimension $\binom{n-2}{2} - 1$ in the space of cubic forms $\mathbb{C}[x_0, \ldots, x_n]_3$.

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3 Show which invariants have the biggest family of cubic tails.

$C_{r,l} = \bigcup_{supp(Z_l)=\nu_3([l]), l(Z_l)\leq r} \langle Z_l \rangle \Leftrightarrow$ Parameterize the family of cubic tails $f \in K[x_1, \dots, x_n]$ of g's:

$$V_r(3,n) = \{f \mid f = g_{\leq 3} \text{ for some } g \in K[x_1, \dots, x_n], \dim Diff(g) \leq r\}$$

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For which Hf and which Δ, $v(3,\Delta,n) := \dim(V(3,\Delta,n))$ attains its max given r? $\Longrightarrow \max(v(3,\Delta,n)) =$upper bound for $v_r(3,n) := \dim(V_r(3,\Delta))$

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Proof of the Thm

Proposition

$r \geq 7$, $v(3, \Delta, n)$ attains its $\max$ *for*

$$\begin{cases} \Delta = (1, m-1, m-1, 1), & r = 2m, \\ \Delta = (1,1,1,1,1), (0, m-2, m-2, 0), & r = 2m+1 \end{cases}$$

and

$$v(3, \Delta, n) = \begin{cases} M_e := \binom{m+2}{3} + 2m(n-m) + 3m - n - 1, r = 2m, \\ M_o := \binom{m+2}{3} + 2m(n-m) + 3m - 2, r = 2m+1. \end{cases}$$

Proof of the Thm

$\mathrm{Cactus}_r(X_{3,n}) = \overline{\bigcup_{r_1+\cdots+r_s} J(W_{r_1,n},\ldots,W_{r_s,n})}$

$W_{r,n} = \overline{\bigcup_{I\in S_1} C_{r,I}}$ Local cactus variety

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The largest component of $W_{2m,n}$ is the union as I varies, of projective varieties whose affine cones are isomorphic to $V_{2m}(3,n)$, so a parameterization of $W_{2m,n}$ has dimension $v_{2m}(3,n)-1+n$.

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BUT parameterization may not be generically finite, the formulas are upper bounds for the dimension of these varieties.

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When b is even, for a general $[F] \in W_{b,n}$ there is a unique l such that $Z_{F,l}$ has length b.

When b is odd, and there is a unique l such that $f = \pi_l(F)$ is the tail of a quartic polynomial g_l whose apolar scheme Z_{g_l} has length b.

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THANKS!