NP-hardness of Nuclear Norm for Tensors

Shmuel Friedland

Univ. Illinois at Chicago

Joint work with Lek-Heng Lim

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Overview

- Norms
- Spectral and nuclear norms for matrices and tensors
- Weak membership and weak validity problems in unit ball of norms
- Approximation of norms
- NP-hardness of tensor and nuclear norms

A primer on norms

$\mathbb{F} = \mathbb{C}, \mathbb{R}$, - basic fields, $\mathbb{F}^m$ column space of vectors $\mathbf{x} = (x_1, \ldots, x_m)^\top$

$\nu : \mathbb{F}^n \to [0, \infty)$ a norm if

$\nu(\mathbf{x}) > 0$ if $\mathbf{x} \neq \mathbf{0}$, $\nu(\mathbf{x}+\mathbf{y}) \leq \nu(\mathbf{x}) + \nu(\mathbf{y})$, $\nu(a\mathbf{x}) = |a|\nu(\mathbf{x})$

$B_\nu := \{\mathbf{x} \in \mathbb{F}^n, \nu(\mathbf{x}) \leq 1\}$-unit ball, $S_\nu := \{\mathbf{x} \in \mathbb{F}^n, \nu(\mathbf{x}) = 1\}$-unit sphere

$\nu^\vee$-the dual norm: $\nu^\vee(\mathbf{x}) = \max\{\Re(\mathbf{y}^*\mathbf{x}), \mathbf{y} \in B_\nu\} = \max\{|\mathbf{y}^*\mathbf{x}|, \mathbf{y} \in S_\nu\}$

$= \max\{Re(\mathbf{y}^*\mathbf{x}), \mathbf{y} \in \mathrm{Ext}\ B_\nu\}$, Ext B_ν-extreme points of B_ν

$(\nu^\vee)^\vee = \nu$

If $\nu(\mathbf{x}) = \max\{\Re\mathbf{y}^*\mathbf{x}, \mathbf{y} \in S\}\ \forall \mathbf{x} \in \mathbb{F}^n$ and a compact balanced $S \in \mathbb{F}^n$

$(aS = S\ \forall a \in \mathbb{F}, |a| = 1)$, then $B_{\nu^\vee} = \mathrm{conv}\ S$

Eucledian norm $\|\mathbf{x}\| := \sqrt{\mathbf{x}^*\mathbf{x}}$ is self dual

$B(\mathbf{x}, r) := \{\mathbf{y} \in \mathbb{F}^n, \|\mathbf{y} - \mathbf{x}\| \leq r\}$, $r \geq 0$

$\|\mathbf{x}\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ for $p \in [1, \infty]$, $\|\mathbf{x}\|_2 = \|\mathbf{x}\|$

Spectral and Nuclear Norm for Matrices

$\mathbb{F}^{m\times n}$ - space of $m \times n$ matrices $A = [a_{ij}]_{i=j=1}^{m,n}$

$\langle A, B\rangle := \mathrm{Tr}(AB^*)$, $\|A\|_F = \sqrt{\mathrm{Tr}\, AA^*}$-Frobenius norm

$\|A\| = \sigma_1(A) := \max_{\|\mathbf{x}\|\le 1} \|A\mathbf{x}\|$ - spectral, or operator, or ℓ_2 norm of A

$\Omega_{m,n,\mathbb{F}} := \{\mathbf{u}\mathbf{v}^* \in \mathbb{F}^{m\times n}, \|\mathbf{u}\| = \|\mathbf{v}\| = 1\}$ balanced compact set

$\|A\| = \max\{\Re(\mathrm{Tr}(A\mathbf{v}\mathbf{u}^*)) = \Re(\mathbf{u}^* A\mathbf{v}), \mathbf{u}\mathbf{v}^* \in \Omega_{m,n,\mathbb{F}}\}$

SVD decomposition:

$A = U\Sigma V^*, UU^* = I_m, VV^* = I_n, \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_{\min(m,n)})$

$A = \sum_{i=1}^r \sigma_i \mathbf{u}_i\mathbf{v}_i^*, \sigma_1 \ge \ldots \ge \sigma_r > 0, r = \mathrm{rank}\, A$

Nuclear norm: $\|A\|_1 := \sum_{i=1}^{\min(m,n)} \sigma_i(A)$, F-norm: $\|A\|_F^2 = \sum \sigma_i^2$

If A is real valued then $\|A\|, \|A\|_1$ over real same as over complex

Complexity of computation of $\|A\|, \|A\|_1$ is $O(mn), O(\min(m,n)mn)$

Importance of matrix nuclear norm in missing entry completion:

Netflex problem

Minimal characterization of matrix nuclear norm

$B_{nuc} := \{A \in \mathbb{F}^{m\times n} : \ \|A\|_1 = \sum_{i=1}^r \sigma_i \le 1\}$

$A = \|A\|_1 \sum_{i=1}^r \frac{\sigma_i}{\|A\|_1} \mathbf{u}_i \mathbf{v}_i^*$

The set of extreme points of B_{nuc} is $\Omega_{m,n,\mathbb{F}}$

Characterization of spectral norm gives $\|\cdot\|^\vee = \|\cdot\|_1$

$\|A\|_1 = \min\{\sum_{i=1}^N \|\mathbf{x}_i\|\|\mathbf{y}_i^*\|, \sum_{i=1}^N \mathbf{x}_i\mathbf{y}_i^* = A\}$

Proof $\|A\|_1 = \|\sum_{i=1}^N \mathbf{x}_i\mathbf{y}_i^*\| \le \sum_{i=1}^N \|\mathbf{x}_i\mathbf{y}_i^*\|_1 = \sum_{i=1}^N \|\mathbf{x}_i\|\|\mathbf{y}_i^*\|$

Caratheodory: $\dim \mathbb{F}^{m\times n} = mn \Rightarrow$ it is sufficinient $N = mn + 1$

Alternating Minimization Method (AMM) for computing $\|A\|_1$:

Choose $\mathbf{y}_1, \ldots, \mathbf{y}_N \in \mathbb{F}^n \setminus \{\mathbf{0}\}$ in general position (at random)

$L(A, \mathbf{y}_1, \ldots, \mathbf{y}_N) := \{X := [\mathbf{x}_1 \ldots \mathbf{x}_N] \in \mathbb{F}^{m\times N}, A = \sum_{i=1}^N \mathbf{x}_i\mathbf{y}_i^*\}$

Find $\min_{X\in L(A,\mathbf{y}_1,\ldots,\mathbf{y}_N)} \|X\|_y, = [\mathbf{x}_{1,1} \ldots \mathbf{x}_{N,1}], \|X\|_y := \sum_{i=1}^n \|\mathbf{x}_i\|\|y_i\|$

Now repeat this minization with respect to $\mathbf{y}_1, \ldots, \mathbf{y}_n$ and so on

Notations

Indices: $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, $[m] := \{1, \ldots, m\}$

$J = \{j_1, \ldots, j_k\} \subset [d]$

Tensors: $\otimes_{i=1}^d \mathbb{F}^{m_i} = \mathbb{F}^{m_1 \times \ldots \times m_d} = \mathbb{F}^{\mathbf{m}}$

Contraction of $\mathcal{T} = [t_{i_1,\ldots,i_d}] \in \mathbb{F}^{\mathbf{m}}$ with $\mathcal{X} = [x_{i_{j_1},\ldots,i_{j_k}}] \in \otimes_{j_p \in J} \mathbb{F}^{m_{j_p}}$:

$\mathcal{T} \times \mathcal{X} = \sum_{i_{j_p} \in [m_{j_p}], j_p \in J} t_{i_1,\ldots,i_d} x_{i_{j_1},\ldots,i_{j_k}} \in \otimes_{l \in [d] \setminus J} \mathbb{F}^{m_l}$

Example $\mathcal{T} \times (\mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_{k-1} \otimes \mathbf{x}_{k+1} \otimes \ldots \otimes \mathbf{x}_d) =$
$\sum_{i_j \in [m_j], j \in [d] \setminus \{k\}} t_{i_1,\ldots,i_d} \prod_{j \in [d] \setminus \{k\}} x_{i_j,j}$

is a vector in $\mathbb{F}^{m_k}$

$\|\mathcal{T}\| = \sqrt{\mathcal{T} \times \bar{\mathcal{T}}}$ - Hilbert-Schmidt norm of $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$

$\langle \mathcal{T}, \mathcal{S} \rangle := \mathcal{T} \times \bar{\mathcal{S}}$ inner product in $\mathbb{C}^{\mathbf{m}}$

Tensor nuclear and spectral norms - 3-tensor

$\mathcal{A} \in \mathbb{F}^{l\times m\times n}, \quad \Omega_{m,n,l,\mathbb{F}} := \{\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z} \in \mathbb{F}^{m\times n\times l}, \|\mathbf{x}\|\|\mathbf{y}\|\|\mathbf{z}\| = 1\}$

$\|\mathcal{A}\|_{\sigma,\mathbb{F}} := \max_{\mathbf{x},\mathbf{y},\mathbf{z}\neq 0} \frac{\Re\langle\mathcal{A},\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\rangle}{\|\mathbf{x}\|\|\mathbf{y}\|\|\mathbf{z}\|} =$

$\max\{|\langle\mathcal{A},\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}\rangle|, \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z} \in \Omega_{m,n,l,\mathbb{F}}\}$ -spectral norm

$\|\mathcal{A}\|_{*,\mathbb{F}} := \min\left\{\sum_{i=1}^r \|\mathbf{x}_i\|\|\mathbf{y}_i\|\|\mathbf{z}_i\| : \mathcal{A} = \sum_{i=1}^r \mathbf{x}_i\otimes\mathbf{y}_i\otimes\mathbf{z}_i,\ r\in\mathbb{N}\right\}$

$\|\cdot\|_{*,\mathbb{F}} = \|\cdot\|_{\sigma,\mathbb{F}}^{\vee}, \ \mathrm{Ext} B_{\|\cdot\|_{*,\mathbb{F}}} = \Omega_{m,n,l,\mathbb{F}}$.

Hillar-Lim: Spectral norm is NP-hard to compute

Theorem: Nuclear norm is NP-hard to compute

Problem: For real tensors do we we have equalities (as for matrices)

$\|\mathcal{A}\|_{\sigma,\mathbb{R}} = \|\mathcal{A}\|_{\sigma,\mathbb{C}}, \quad \|\mathcal{A}\|_{*,\mathbb{R}} = \|\mathcal{A}\|_{*,\mathbb{C}}$?

For $\mathcal{A} \geq 0$ first equality holds - triangle inequality

Tensor nuclear and spectral norms - d-tensor

$\mathbb{F}^{m_1\times\ldots\times m_d} = \otimes_{j=1}^d \mathbb{F}^{m_j}$ space of d-mode tensors

$B(m,\mathbb{F}) := \{\mathbf{x}\in\mathbb{F}^m,\ \|\mathbf{x}\|\le 1\},\quad S(m,\mathbb{F}) := \{\mathbf{x}\in\mathbb{F}^m,\ \|\mathbf{x}\| = 1\}$

$\|\mathcal{A}\|_{\sigma,\mathbb{F}} := \max\left\{|\langle\mathcal{A},\otimes_{j\in[d]}\mathbf{x}_j\rangle|,\ \mathbf{x}_j\in S(m_j,\mathbb{F}^{m_j}), j\in[d]\right\}$

$\|\mathcal{A}\|_{*,\mathbb{F}} := \min\left\{\sum_{i=1}^r\prod_{j=1}^d \|\mathbf{x}_{i,j}\| : \mathcal{A} = \sum_{i=1}^r \otimes_{j=1}^d \mathbf{x}_{i,j}\ r\in\mathbb{N}\right\}$

Spectral and nuclear norms are dual

$\|\mathcal{A}\|_\sigma := \|\mathcal{A}\|_{\sigma,\mathbb{C}}$, $\|\mathcal{A}\|_* := \|\mathcal{A}\|_{*,\mathbb{C}}$

$\|\mathcal{A}\|_{\sigma,\mathbb{F}} \le \|\mathcal{A}\|_\sigma$ for $\mathcal{A}\in\mathbb{R}^{m_1\times\ldots\times m_d}$, $\quad \|\mathcal{A}\|_{*,\mathbb{F}} \ge \|\mathcal{A}\|_{\sigma,\mathbb{F}}$

For $\mathbf{x} = (x_1,\ldots,x_N)^T\in\mathbb{C}^N$ let $|\mathbf{x}| := (|x_1|,\ldots,|x_N|)^T$.

Claim $\|\mathcal{A}\|_\sigma \le \||\mathcal{A}|\|_\sigma$

For $\mathcal{A}\ge 0$ in characterization of $\||\mathcal{A}|\|_{\sigma,\mathbb{R}}$ use $\mathbf{x}_j\ge 0$

Even for matrices one may have $\|\mathcal{A}\|_* > \||\mathcal{A}|\|_*$

AMM for spectral and nuclear norms

Maximal characterization $\max\left\{|\langle\mathcal{A},\otimes_{j\in[d]}\mathbf{x}_j\rangle|,\ \mathbf{x}_j\in S(m_j,\mathbb{F}^{m_j}), j\in[d]\right\}$

applied to $\mathbf{x}_j$ yields AMM algo, usually converges to a local minimum

$(\mathbf{x}_1^\star,\ldots,\mathbf{x}_d^\star)$ - a fixed point of corresponding map

yields Newton method: Friedland-Venu 2014

Let $N(>\prod_{j=1}^d m_j)$, $\mathbf{x}_{k,j}\in\mathbb{F}^{m_j}\setminus\{\mathbf{0}\}, k\in[N]$ in general pos. $j\in[d]\setminus\{i\}$

$L(\mathcal{A},\mathbf{x},i):=\{X_i=[\mathbf{x}_{1,i}\ldots\mathbf{x}_{N,i}]\in\mathbb{F}^{m_i\times N},\mathcal{A}=\sum_{k=1}^N\otimes_{j=1}^d\mathbf{x}_{k,j}\}$

Min. convex function $\|X_i\|_x:=\sum_{k=1}^N\|\mathbf{x}_{k,i}\|\otimes_{j\in[d]\setminus\{i\}}\|\mathbf{x}_{k,j}\|$ on $L(\mathcal{A},\mathbf{x},i)$

Alternate all variables

4-tensors and bi-partite density matrices

$\mathbb{C}^{m\times m} \supset \mathbb{H}^{m\times m} \supset \mathbb{H}_{+}^{m\times m} \supset \mathbb{H}_{+,1}^{m\times m}$

Hermitian, positive definite and density matrices

$A = [a_{ij}] \in \mathbb{F}^{m\times n}], B = [b_{kl}] \in \mathbb{F}^{p\times q}$, Kronecker product $A \otimes B$

$$\begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \vdots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} = [c_{(i,k)(j,l)}] \in \mathbb{F}^{(mp)\times(nq)}, c_{(i,k)(j,l)} = a_{ij}b_{kl}$$

Viewing $\mathcal{C} := [c_{(i,k)(j,l)}] \in \mathbb{F}^{m\times p\times n\times q}$ we get $\mathbb{F}^{m\times n} \otimes \mathbb{F}^{p\times q} \sim \mathbb{F}^{m\times p\times n\times q}$

$\mathcal{C} = [c_{i,k,j,l}] \in \mathbb{C}^{m\times n\times m\times n}$ is called:

Bi-symmetric: $c_{i,k,j,l} = c_{j,l,i,k}$ for all i, j, k, l - $C = [c_{(i,k)(j,l)}]$ symmetric

Bi-hermitian: $c_{i,k,j,l} = \bar{c}_{j,l,i,k}$ for all i, j, k, l - $C = [c_{(i,k)(j,l)}]$ hermitian

Positive definite: C is hermitian positive semi-definite,

bi-partite density matrix: $\mathrm{Tr}\, C = 1$

Bi-partite separable states and nuclear norm

Separable states in $\mathbb{C}^{m\times n\times m\times n}$:

$\mathrm{S}(m,n) := \mathrm{conv}\,((\mathbf{x}\otimes\mathbf{x}^*)\otimes(\mathbf{y}\otimes\mathbf{y}^*), \mathbf{x}\in\mathrm{S}(m,\mathbb{C}), \mathbf{y}\in\mathrm{S}(n,\mathbb{C})) \subset \mathbb{H}_{mn,+,1}$

For $\mathcal{A} = [a_{ikjl}] \in \mathbb{C}^{m\times n\times m\times n}$ define $\mathrm{tr}(\mathcal{A}) := \sum_{i,j} a_{ijij}$ $(= \mathrm{Tr}\, C)$

Note $\mathrm{Tr} \otimes_{j=1}^4 x_j = (\mathbf{x}_3^\top \mathbf{x}_1)(\mathbf{x}_4^\top \mathbf{x}_1) \le \prod_{i=1}^4 \|\mathbf{x}_i\|$

THM: $|\,\mathrm{Tr}\,\mathcal{A}| \le \|\mathcal{A}\|_*$ equality iff $\mathcal{A} = t\mathcal{B}$, $\mathcal{B}\in\mathrm{S}(m,n)$

Cor. A bipartite density matrix is separable iff its nuclear norm is 1

Gurvits 2003: Weak membership in $\mathrm{S}(m,n)$ is NP-hard $\Rightarrow$:

Membership in the unit ball of nuclear norm on $\mathbb{C}^{m\times n\times m\times n}$ NP-hard

Friedland-Lim: Weak membership is NP-hard

Clique number and spectral norm of 4-tensors

G-graph on n vertices, $A(G), \kappa(G)$-adjacency matrix, clique number

Motzkin-Strassen 1965: $1 - \frac{1}{\kappa(G)} = \max \mathbf{x}^\top A(G)\mathbf{x}$, $\mathbf{x}$ probab. vector

COR: It is NP-hard to approximate $\kappa(G)$ up to order $\frac{1}{n^2}$

F-L: G induces 4-nonnegative symmetric positive definite tensor

$\mathcal{B}(G) \in \mathbb{C}^{n\times n\times n\times n}$ whose spectral norm is $1 - \frac{1}{\kappa(G)}$

So spectral norm is NP-hard to approximate within arbitrary δ

1. We show that this is equivalent to NP-hardness of

weak membership in unit ball of spectral norm

2. Weak membership in B_ν is polynomial iff

weak membership in $B_{\nu^\vee}$ is polynomial

Unit ball of a norm

Norm $\nu : \mathbb{R}^n \to [0, \infty)$, ν-ball $B_\nu := \{x \in \mathbb{R}^M,\ \nu(x) \le 1\}$

all norms in $\mathbb{R}^n$ are equivalent : $\exists$ rational $K(\nu) \ge k(\nu) > 0$:

$k(\nu)\|x\| \le \nu(x) \le K(\nu)\|x\|$ for all $\mathbf{x} \in \mathbb{R}^n$

$< k(\nu) >, < K(\nu) >$ number of bits encoding $k(\nu), K(\nu)$

$< B_\nu > := < k(\nu) > + < K(\nu) >$

$$\frac{1}{\prod_{i=1}^d m_i}\|\mathcal{A}\| \le \frac{1}{\sqrt{\prod_{i=1}^d m_i}}\|\mathcal{A}\| \le \|\mathcal{A}\|_{\ \sigma} \le \|\mathcal{A}\|, \mathcal{A} \in \mathbb{F}^{m_1 \times \ldots \times m_d}$$

$$\|\mathcal{A}\| \le \|\mathcal{A}\|_* \le \sqrt{\prod_{i=1}^d m_i}\ \|\mathcal{A}\| \le \prod_{i=1}^d m_i\ \|\mathcal{A}\|$$

$$< K(\|\cdot\|_\sigma) > + < k(\|\cdot\|_\sigma) > = < K(\|\cdot\|_*) > + < k(\|\cdot\|_*) > << \prod_{i=1}^d m_i >$$

For $\epsilon > 0$: $S(B_\nu, \epsilon)$ closed ϵ-neighborhood of B_ν

$S(B_\nu, -\epsilon)$-a closed subset of B_ν s.t. $S(S(B_\nu, -\epsilon), \epsilon) = B_\nu$

Weak membership and validity problems

Given $\mathbf{y} \in \mathbb{R}^n$ and rational $\delta > 0, \gamma, \mathbf{c}$

Membership problem (MEM) for B_ν: determine if $\mathbf{y}$ in B_ν

Weak membership problem (WMEM) for B_ν

assert either $\mathbf{y} \in S(B_\nu, \delta)$ or $\mathbf{y} \notin S(B_\nu, -\delta)$

(Membership implies weak membership)

Weak validity problem (WVAL) problem for B_ν:

assert either $\mathbf{c}^T\mathbf{x} \le \gamma + \epsilon$ for all $\mathbf{x} \in S(B_\nu, -\epsilon)$,

or $\mathbf{c}^T\mathbf{x} \ge \gamma - \epsilon$ for some $\mathbf{x} \in S(B_\nu, \epsilon)$

Yudin-Nemirovski: If there exists a deterministic algorithm solving

WMEM problem for $B_\nu, \mathbf{y}, \delta$ in Poly($< B_\nu > + < \delta >$)

then there exists a deterministic algorithm solving WVAL problem for

$B_\nu, \mathbf{c}, \gamma, \delta$ in Poly($< B_\nu > + < \mathbf{c} > + < \gamma > + < \delta >$).

Equivalence of weak membership in B_ν and $B_{\nu^\vee}$ in $\mathbb{R}^n$

For compact $K \subset \mathbb{R}^n$, $\mathbf{c} \in \mathbb{R}^n$ set $M(K, \mathbf{c}) := \max_{\mathbf{x} \in K} \mathbf{c}^\mathsf{T} x$.

$\nu(\mathbf{x}) = M(B_{\nu^\vee}, \mathbf{x}), \quad K_{\nu^\vee} = \frac{1}{k_\nu}, \quad k_{\nu^\vee} = \frac{1}{K_\nu}$

$(1 + k_\nu \delta) B_\nu \subseteq S(B_\nu, \delta) \subseteq (1 + K_\nu \delta) B_\nu$ for $0 < \delta \in \mathbb{Q}$

$(1 - K_\nu \delta) B_\nu \subseteq S(B_\nu, -\delta) \subseteq (1 - k_\nu \delta) B_\nu \quad$ for $K_\nu \delta < 1$

$\left(1 - \frac{\delta}{K_\nu}\right) \nu(\mathbf{x}) \ge M(S(B_{\nu^\vee}, -\delta), \mathbf{x}) \ge \left(1 - \frac{\delta}{k_\nu}\right) \nu(\mathbf{x}) \quad$ for $\frac{\delta}{k_\nu} < 1$

$\left(1 + \frac{\delta}{K_\nu}\right) \nu(\mathbf{x}) \le M(S(B_{\nu^\vee}, \delta), \mathbf{x}) \le \left(1 + \frac{\delta}{k_\nu}\right) \nu(\mathbf{x})$

LEM: For $k_\nu \ge 2$ WVAL in $B_{\nu^\vee}$ implies WMEM in B_ν

PRF Let $x \in \mathbb{Q}^n$, $\delta \in (0, \frac{1}{2})$, $\gamma = 1$

If $\mathbf{x}^\top \mathbf{y} \le 1 + \delta \; \forall \mathbf{y} \in S(B_{\nu^\vee}, -\delta) \Rightarrow \mathbf{x} \in S(B_\nu, \delta)$

If $\mathbf{x}^\top \mathbf{y} > 1 - \delta$ for some $y \in S(B_{\nu_*}, \delta)$ then $\mathbf{x} \notin S(B_\nu, -\delta)$

WMEM in $B_{\nu^\vee} \Rightarrow$ WVAL in $B_{\nu^\vee} \Rightarrow$ WMEM in $B_\nu \Rightarrow$ WMEM in $B_\nu^\vee$

Weak membership and norm approximation I

DEF: ν is polynomially approximable if for all $\|\mathbf{x}\| = 1, \epsilon \in (0, \kappa_\nu) \cap \mathbb{Q}$

$\exists$ pol. time algo in $n + \langle\delta\rangle + \langle K_\nu\rangle + \langle k_\nu\rangle$ for $\omega(\mathbf{x})$:

$\omega(\mathbf{x}) - \epsilon < \nu(\mathbf{x}) < \omega(\mathbf{x}) + \epsilon$

THM: THAE

(1) ν is polynomially approximable

(2) Weak membership in B_ν is polynomial

PRF: (1)$\Rightarrow$(2). $\mathbf{x} \in \mathbb{R}^n, 0 < \delta \in \mathbb{Q}$ given

$\|\mathbf{x}\| \le \frac{1}{K_\nu} \Rightarrow \nu(\mathbf{x}) \le 1 \Rightarrow \mathbf{x} \in S(B_\nu, \delta)$

$\|\mathbf{x}\| \ge \frac{1}{k_\nu} \Rightarrow \nu(\mathbf{x}) \ge 1 \Rightarrow \mathbf{x} \notin S(B_\nu, -\delta)$

$\|\mathbf{x}\| \in (\frac{1}{K_\nu}, \frac{1}{k_\nu}), \mathbf{y} = \frac{1}{\|\mathbf{x}\|}\mathbf{x}, \epsilon = \frac{k_\nu^2 \delta}{2}$

$\|\mathbf{x}\|\omega(\mathbf{y}) \le 1 + \frac{k_\nu \delta}{2} \Rightarrow \mathbf{x} \in S(B_\nu, \delta)$ otherwise $\mathbf{x} \notin S(B_\nu, -\delta)$

Weak membership and norm approximation II

(2)$\Rightarrow$(1). $\mathbf{x}$ given, $\|\mathbf{x}\| = 1 \to \nu(\mathbf{x}) \in [k_\nu, K_\nu]$

Set $K_{\nu,0} = K_\nu, k_{\nu,0} = k_\nu, i = 0$ and assume $\nu(\mathbf{x}) \in [k_{\nu,i}, K_{\nu,i}]$

$$a_i = \frac{k_{\nu,i}+K_{\nu,i}}{2}, \quad \delta_i = \frac{K_{\nu,i}-k_{\nu,i}}{2K_{\nu,i}(K_{\nu,i}+k_{\nu,i})}, \quad \mathbf{y} = \frac{1}{a_i}\mathbf{x}$$

If $\mathbf{y} \in S(B_\nu, \delta_i) \Rightarrow \nu(x) =\leq \frac{3}{4}K_{\nu,i} + \frac{1}{4}k_{\nu,i}$

set $k_{\nu,i+1} = k_{\nu,i}, K_{\nu,i+1} = \frac{3}{4}K_{\nu,i} + \frac{1}{4}k_{\nu,i}$

If $\mathbf{y} \notin S(B_\nu, -\delta_i) \Rightarrow \nu(x) =\geq \frac{3}{4}k_{\nu,i} + \frac{1}{4}K_{\nu,i}$

set $k_{\nu,i+1} = \frac{3}{4}k_{\nu,i} + \frac{1}{4}k_{\nu,i}, K_{\nu,i+1} = K_{\nu,i}$

Observe $\nu(\mathbf{x}) \in [k_{\nu,i+1}, K_{\nu,i+1}]$ and $K_{\nu,i+1} - k_{\nu,i+1} = \frac{3}{4}(K_{\nu,i} - k_{\nu,i})$

Repeat this procedure $O(\log\delta)$ times to get $\omega(\mathbf{x}) = \frac{1}{2}(K_{\nu,i} + k_{\nu,i})$

WMEM for nuclear tensor norm is NP-hard

Finding an ϵ approximation to spectral norm is NP-hard

Equivalent to WMEM in the unit ball of nuclear norm

Finding an WMEM in the uni spectral norm is NP-hard

WMEM for nuclear norm is NP-hard

WMEM of nuclear norm is equivalent to approximation of nuclear norm

Approximation of nuclear norm is NP-hard

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