Siegel’s Theorem, Edge Coloring, and a Holant Dichotomy

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Theorem (Siegel’s Theorem)

Any smooth algebraic curve of genus \( g > 0 \) defined by a polynomial \( f(x, y) \in \mathbb{Z}[x, y] \) has only finitely many integer solutions.
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Theorem (Faltings’ Theorem–Mordell Conjecture)

Any smooth algebraic curve of genus $g > 1$ defined by a polynomial $f(x, y) \in \mathbb{Z}[x, y]$ has only finitely many rational solutions.
Pell’s Equation (genus 0)

\[ x^2 - 61y^2 = 1 \]
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Smallest solution:

(1766319049, 226153980)
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Next smallest solution:

$$(288065397114519999215772221121510725946342952839946398732799, 9150698914859994783783151874415159820056535806397752666720)$$

$$x^2 - 991y^2 = 1$$
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Edge Coloring
Theorem (Vizing’s Theorem)

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$\Delta(G)$ is an obvious lower bound.
Tait (1880) showed that, for bridgeless 3-regular planar graphs, the statement that a proper 3-edge coloring always exists is equivalent to the Four Color (Conjecture) Theorem.
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For 3-regular (non-planar) graphs, 3-edge coloring is NP-complete (Holyer (1981)).

But this reduction is not parsimonious (see Welsh).
Problem $\#\kappa$-EdgeColoring:
Input: A graph $G$.
Output: The number of valid edge colorings of $G$, using $\kappa$ colors.
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$\#\kappa$-EdgeColoring is $\#P$-hard over planar $r$-regular graphs for $\kappa \geq r \geq 3$. 
Problem \( \#\kappa\text{-EdgeColoring} \):
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\( \#\kappa\text{-EdgeColoring} \) is \( \#P \)-hard over planar \( r \)-regular graphs for \( \kappa \geq r \geq 3 \).

- \( \kappa = r \), and
- \( \kappa > r \).
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- \(\kappa = r\), and 
- \(\kappa > r\).

This is proved in the framework of complexity dichotomy theorems.
Three Frameworks for Counting Problems

1. Graph Homomorphisms
2. Constraint Satisfaction Problems (CSP)
3. Holant Problems
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2. Constraint Satisfaction Problems (CSP)
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In each framework, there has been remarkable progress in the classification program of the complexity of counting problems.
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A signature grid $\Omega = (G, \mathcal{F}, \pi)$ consists of a graph $G = (V, E)$, where $\pi$ assigns a function $f_v \in \mathcal{F}$ to each $v \in V$. 

Over the Boolean domain $\{0, 1\}$, the Holant problem on instance $\Omega$ is to evaluate $\text{Holant}_\Omega = \sum_{\sigma} \prod_{v \in V} f_v(\sigma|E(v))$. 

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Over the Boolean domain $\{0, 1\}$, the Holant problem on instance $\Omega$ is to evaluate

$$\text{Holant}_\Omega = \sum_{\sigma} \prod_{v \in V} f_v(\sigma|_{E(v)}),$$

a sum over all edge assignments $\sigma : E \to \{0, 1\}$.
A function $f_v$ can be represented by listing its values in lexicographical order as in a truth table, which is a vector in $\mathbb{C}^{2^n}$, or as a tensor in $(\mathbb{C}^2)^{\otimes n}$. Holographic Transformations can change one function to another. E.g. The $n$-ary Equality function is $[1 \ 0] \otimes n + [0 \ 1] \otimes n$. Under the Holographic Transformation by $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $H \otimes n \{[1 \ 0] \otimes n + [0 \ 1] \otimes n\} = [1 \ 1] \otimes n + [1 \ -1] \otimes n$ is a (constant multiple of) the Parity function.
Constraint Functions

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\begin{bmatrix}
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is a (constant multiple of) the Parity function.
Equivalence with $\mathfrak{sl}(2; \mathbb{C})$ representation

$\mathfrak{sl}(2; \mathbb{C}) = \mathfrak{su}(2)_{\mathbb{C}}$. 

Let $f$ be a symmetric constraint function $[f_0, f_1, ..., f_n]$. For any $U \in \mathfrak{su}(2)$, $U \otimes f$ is also a symmetric constraint function $U \otimes f = [f'_0, f'_1, ..., f'_n]$. This gives a representation $\phi_n : (f_0, f_1, ..., f_n) \mapsto (f'_0, f'_1, ..., f'_n)$. 

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\[ \varphi_n : (f_0, f_1, \ldots, f_n) \mapsto (f'_0, f'_1, \ldots, f'_n). \]
Let \( p_n(x, y) = \sum_{i=0}^{n} a_i \binom{n}{i} x^{n-i} y^i \). Then

\[
q_n(x, y) = p_n((x, y) U) = \sum_{i=0}^{n} a'_i \binom{n}{i} x^{n-i} y^i.
\]

This gives another representation

\[
\psi_n : (a_0, a_1, \ldots, a_n) \mapsto (a'_0, a'_1, \ldots, a'_n).
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Another representation

Let $p_n(x, y) = \sum_{i=0}^{n} a_i {n \choose i} x^{n-i} y^i$. Then

$$q_n(x, y) = p_n((x, y) U) = \sum_{i=0}^{n} a'_i {n \choose i} x^{n-i} y^i.$$ 

This gives another representation

$$\psi_n : (a_0, a_1, \ldots, a_n) \mapsto (a'_0, a'_1, \ldots, a'_n).$$

Theorem

*The two representations $\varphi_n$ and $\psi_n$ are the same.*
Consider a 3-regular graph $G$. Let $AD_3$ denote the following local constraint function:

$$AD_3(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [\kappa] \text{ are all distinct} \\ 0 & \text{otherwise} \end{cases}$$
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Now place $AD_3$ at each vertex $v$, with incident edges $x, y, z$. Then we evaluate the sum of product

$$\text{Holant}(G; AD_3) = \sum_{\sigma : E(G) \rightarrow [\kappa]} \prod_{v \in V(G)} AD_3 \left( \sigma \mid_{E(v)} \right).$$
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Clearly $\text{Holant}(G; \text{AD}_3)$ computes $\#\kappa$-EdgeColoring.
Holant Problems

In general, we consider all local constraint functions

\[ f(x, y, z) = \begin{cases} 
    a & \text{if } x = y = z \in [\kappa] \\
    b & \text{if } |\{x, y, z\}| = 2 \\
    c & \text{if } |\{x, y, z\}| = 3 
\end{cases} \]

And the Holant problem is to compute

\[ \text{Holant}(G; f) = \sum_{\sigma: E(G) \rightarrow [\kappa]} \prod_{v \in V(G)} f \left( \sigma \big|_{E(v)} \right). \]
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Succinct signatures \( f = \langle a, b, c \rangle \), where \( a, b, c \in \mathbb{C} \).
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Succinct signatures \( f = \langle a, b, c \rangle \), where \( a, b, c \in \mathbb{C} \).
Thus \( AD_3 = \langle 0, 0, 1 \rangle \).
L. Lovász:

Let $A = (A_{i,j}) \in \mathbb{C}^{\kappa \times \kappa}$ be a symmetric complex matrix.

The graph homomorphism problem is:
**INPUT:** An undirected graph $G = (V, E)$.
**OUTPUT:**

$$Z_A(G) = \sum_{\xi: V \rightarrow [\kappa]} \prod_{(u,v) \in E} A_{\xi(u), \xi(v)}.$$
Examples of Graph Homomorphism

Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

then $Z_A(G)$ counts the number of VERTEX COVERS in $G$. 
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Let

\[ A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} \]

then \( Z_A(G) \) counts the number of vertex \( \kappa \)-COLORINGS in \( G \).
Theorem (C., Xi Chen and Pinyan Lu)

There is a complexity dichotomy for $Z_A(\cdot)$:
For any symmetric complex valued matrix $A \in \mathbb{C}^{\kappa \times \kappa}$, the problem of computing $Z_A(G)$, for any input $G$, is either in $P$ or $\#P$-hard.

Given $A$, whether $Z_A(\cdot)$ is in $P$ or $\#P$-hard can be decided in polynomial time in the size of $A$.

Dichotomy Theorem for Graph Homomorphism

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Further generalized to all counting CSP.

Theorem (C., Xi Chen)

Every finite set $F$ of complex valued constraint functions on any finite domain set $[\kappa]$ defines a counting CSP problem $\#CSP(F)$ that is either computable in $P$ or $\#P$-hard.

The decision version of this is open (Feder-Vardi Dichotomy Conjecture).
Theorem (Main Theorem)

For any $\kappa$, any 3-regular graph $G$ and any $f = \langle a, b, c \rangle$, the problem $\text{Holant}(G; f)$ is either computable in polynomial time or is $\#P$-hard.
Dichotomy Theorem for Holant($G; f$)

**Theorem (Main Theorem)**

For any $\kappa$, any 3-regular graph $G$ and any $f = \langle a, b, c \rangle$, the problem Holant($G; f$) is either computable in polynomial time or is $\#P$-hard.

$\#\kappa$-EdgeColoring is the special case for $f = \langle 0, 0, 1 \rangle$. 
• On domain size $\kappa = 3$, Holant($G; \langle 5, 2, -4 \rangle$) is computable in P.
Non-trivial Examples of Tractable Holant Problems

- On domain size $\kappa = 3$, Holant$(G; \langle 5, 2, -4 \rangle)$ is computable in P.

$$f = \langle 5, 2, -4 \rangle = \frac{1}{3} \left[ (-1, 2, 2)^{\otimes 3} + (2, -1, 2)^{\otimes 3} + (2, 2, -1)^{\otimes 3} \right].$$

Holographic transformation by the orthogonal matrix $T = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$. 
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• In general Holant$(G; \langle \kappa^2 - 6\kappa + 4, -2(\kappa - 2), 4 \rangle)$ is computable in P.
Non-trivial Examples of Tractable Holant Problems

- On domain size $\kappa = 3$, $\text{Holant}(G; \langle 5, 2, -4 \rangle)$ is computable in $\mathbb{P}$.

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  Holographic transformation by the orthogonal matrix $T = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$.

- In general $\text{Holant}(G; \langle \kappa^2 - 6\kappa + 4, -2(\kappa - 2), 4 \rangle)$ is computable in $\mathbb{P}$.

- Suppose $\kappa = 4$. For any $\lambda \in \mathbb{C}$,

  $$\text{Holant}(G; \lambda \langle -3 - 4i, 1, -1 + 2i \rangle)$$

  is computable in $\mathbb{P}$. 


Tutte Polynomial

Definition

For an undirected graph $G = (V, E)$, the Tutte polynomial of $G$ is

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{\kappa(A) - \kappa(E)} (y - 1)^{\kappa(A) + |A| - |V|},$$

where $\kappa(A)$ denotes the number of connected components of the spanning subgraph $(V, A)$. 

The chromatic polynomial is

$$\chi(G; \lambda) = (-1)^{|V| - \kappa(G)} \lambda^{|E| - \kappa(G)} T(G; 1 - \lambda, 0),$$

(1)

Theorem (Vertigan)

For any $x, y \in \mathbb{C}$, the problem of computing the Tutte polynomial at $(x, y)$ over planar graphs is #P-hard unless $(x - 1)(y - 1) \in \{1, 2\}$ or $(x, y) \in \{(1, 1), (-1, -1), (\omega, \omega^2), (\omega^2, \omega)\}$, where $\omega = e^{2\pi i / 3}$. In each of these exceptional cases, the computation can be done in polynomial time.
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A plane graph (a), its medial graph (c), and the two graphs superimposed (b).
A plane graph (a), its directed medial graph (c), and the two graphs superimposed (b).
A graph is \textbf{Eulerian} if every vertex has an even degree.
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A directed graph is **Eulerian** if \( \text{deg}_{\text{in}}(v) = \text{deg}_{\text{out}}(v) \), at every vertex \( v \).
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We don’t require connectedness.
Eulerian Subgraphs

A graph is **Eulerian** if every vertex has an even degree.

A directed graph is **Eulerian** if \( \text{deg}_{\text{in}}(v) = \text{deg}_{\text{out}}(v) \), at every vertex \( v \).

We don’t require connectedness.

Suppose \( G \) is a connected plane graph and \( \vec{G}_m \) its directed medial graph. For any \( \kappa \), the **Eulerian partitions** \( \pi(\vec{G}_m) \) are \( \kappa \)-labelings of edges of \( \vec{G}_m \), such that each color set forms an Eulerian digraph.

**Theorem (Ellis-Monagahan)**

\[
\kappa \ T(G; \kappa + 1, \kappa + 1) = \sum_{c \in \pi(\vec{G}_m)} 2^{\mu(c)},
\]

where \( \mu(c) \) is the number of monochromatic vertices in the coloring \( c \).
Directed Medial Graph Local Configuration
Eulerian Local Configuration
Directed Medial Graph Local Configuration
Directed Medial Graph Local Configuration
The sum $\sum_{c \in \pi(\vec{G}_m)} 2^{\mu(c)}$ can be expressed as a Holant problem:

$$E(w, x, y) = \begin{cases} 
2 & \text{if } w = x = y = z \in [\kappa] \\
1 & \text{if } w = x \neq y = z \in [\kappa] \\
0 & \text{if } w = y \neq x = z \in [\kappa] \\
1 & \text{if } w = z \neq x = y \in [\kappa] \\
0 & \text{all other cases.}
\end{cases}$$

Denote by $E = \langle 2, 1, 0, 1, 0 \rangle$. 

The Eulerian Signature
The sum $\sum_{c \in \pi(\tilde{G}_m)} 2^{\mu(c)}$ can be expressed as a Holant problem:

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2 & \text{if } w = x = y = z \in [\kappa] \\
1 & \text{if } w = x \neq y = z \in [\kappa] \\
0 & \text{if } w = y \neq x = z \in [\kappa] \\
1 & \text{if } w = z \neq x = y \in [\kappa] \\
0 & \text{all other cases.}
\end{cases}$$

Denote by $E = \langle 2, 1, 0, 1, 0 \rangle$.

To be Eulerian, at every vertex $v \in V(G)$, either it is monochromatic, or RRBB cyclically, since the local orientation in $\tilde{G}_m$ is “in, out, in, out”. Then

$$\sum_{c \in \pi(\tilde{G}_m)} 2^{\mu(c)} = \text{Holant}_{G_m}(\langle 2, 1, 0, 1, 0 \rangle)$$
An Arity 4 Gadget

Quaternary gadget $f$. All vertices are assigned the $\text{AD}_\kappa$ signature.

Think of $\kappa = 3$.

$$f(\begin{array}{c} w \\ x \\ y \\ z \end{array}) = \begin{cases} 0 & \text{if } w = x = y = z \in [\kappa] \\ 1 & \text{if } w = x \neq y = z \in [\kappa] \\ 1 & \text{if } w = y \neq x = z \in [\kappa] \\ 0 & \text{if } w = z \neq x = y \in [\kappa] \\ 0 & \text{all other cases.} \end{cases}$$

Denote by $f = \langle 0, 1, 1, 0, 0 \rangle$. 
A gadget with two parallel edges

Again think of $\kappa = 3$.

$$f_0\left(\begin{array}{c} w \\ x \\ z \\ y \end{array}\right) = \begin{cases} 1 & \text{if } w = x = y = z \in [\kappa] \\ 0 & \text{if } w = x \neq y = z \in [\kappa] \\ 0 & \text{if } w = y \neq x = z \in [\kappa] \\ 1 & \text{if } w = z \neq x = y \in [\kappa] \\ 0 & \text{all other cases.} \end{cases}$$

Denote by $f_0 = \langle 1, 0, 0, 1, 0 \rangle$. 
Theorem

\textsc{\#}$\kappa$-\textsc{EdgeColoring} is \#P-hard over planar $\kappa$-regular graphs for $\kappa \geq 3$. 

Many other gadgets can be reduced using the same method. For instance, we have

\begin{align*}
N_{1} & \rightarrow N_{2} \\
N_{s} & \rightarrow N_{s+1}
\end{align*}

Let $f_{s}$ be the signature for the $s$th gadget. Then

$f_{s} = M_{s} f_{0}$, where

\[
M = \begin{pmatrix}
0 & \kappa - 1 & 0 & 0 & 0 \\
1 & \kappa - 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and $f_{0} = [1 0 0 1 0]^{T}$.
Theorem

**#κ-EdgeColoring** is **#P-hard** over planar κ-regular graphs for κ ≥ 3.

We reduce Pl-Holant(⟨2, 1, 0, 1, 0⟩) to Pl-Holant(AD₃).
## #P-hardness of #κ-EdgeColoring

### Theorem

\textbf{#κ-EdgeColoring is #P-hard over planar κ-regular graphs for κ ≥ 3.}

We reduce Pl-Holant(⟨2, 1, 0, 1, 0⟩) to Pl-Holant(AD₃).

Let $f_s$ be the signature for the $s$th gadget. Then $f_s = M^s f_0$, where

$$M = \begin{bmatrix}
0 & \kappa - 1 & 0 & 0 & 0 \\
1 & \kappa - 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

and $f_0 = [1 \ 0 \ 0 \ 1 \ 0]^T$. 
#P-hardness of \#\(\kappa\)-EdgeColoring

**Theorem**

\#\(\kappa\)-EdgeColoring is #P-hard over planar \(\kappa\)-regular graphs for \(\kappa \geq 3\).

We reduce Pl-Holant(\(\langle 2, 1, 0, 1, 0 \rangle\)) to Pl-Holant(AD\(_3\)).

Let \(f_s\) be the signature for the \(s\)th gadget. Then \(f_s = M^s f_0\), where

\[
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0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

and \(f_0 = [1 \ 0 \ 0 \ 1 \ 0]^T\). One can easily verify that \(f_1 = f\).
By the spectral decomposition $M = P \Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 - \kappa & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} \kappa - 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$
Eigenvalues and Eigenvectors

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Let $x = (\kappa - 1)^{2s}$, then

$$f_{2s} = P\Lambda^{2s}P^{-1}f_0 = P \begin{bmatrix} x & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} P^{-1}f_0 = \begin{bmatrix} \frac{x-1}{\kappa} + 1 \\ \kappa \times 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$
Eigenvalues and Eigenvectors

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$$P = \begin{bmatrix}
1 & 1 - \kappa & 0 & 0 & 0 \\
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0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} P^{-1} f_0 = \begin{bmatrix}
x - 1 + 1 \\
\frac{x-1}{\kappa} + 1 \\
0 \\
1 \\
0
\end{bmatrix}.$$

Note that if $x = 1 + \kappa$, then it is the Eulerian Signature $\mathcal{E} = \langle 2, 1, 0, 1, 0 \rangle$. 
Consider an instance $\Omega$ of $\text{Pl-Holant}(\langle 2, 1, 0, 1, 0 \rangle)$ on domain size $\kappa$. 

Evaluating this polynomial at $x = 1 + \kappa$ gives the value of $\text{Pl-Holant}(\Omega)$. 

Using our oracle for $\text{Pl-Holant}(\text{AD}_\kappa)$, we can evaluate this polynomial at $n + 1$ distinct points $x = (\kappa - 1)^{2s}$ for $0 \leq s \leq n$. Then via polynomial interpolation, we can recover the coefficients of this polynomial efficiently.
Consider an instance $\Omega$ of Pl-Holant(⟨2, 1, 0, 1, 0⟩) on domain size $\kappa$.

Suppose ⟨2, 1, 0, 1, 0⟩ appears $n$ times in $\Omega$. 
Consider an instance $\Omega$ of $\text{Pl-Holant}(\langle 2, 1, 0, 1, 0 \rangle)$ on domain size $\kappa$.

Suppose $\langle 2, 1, 0, 1, 0 \rangle$ appears $n$ times in $\Omega$.

We construct from $\Omega$ a sequence of instances $\Omega_{2s}$ of $\text{Pl-Holant}(\text{AD}_\kappa)$ indexed by $s \geq 0$, by replacing each occurrence of $\langle 2, 1, 0, 1, 0 \rangle$ with the gadget $f_{2s}$. 

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An Interpolation

Consider an instance $\Omega$ of $\text{Pl-Holant}(\langle 2, 1, 0, 1, 0 \rangle)$ on domain size $\kappa$.

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As a polynomial in $x = (\kappa - 1)^{2s}$, $\text{Pl-Holant}_{\Omega_{2s}}$ is independent of $s$ and has degree at most $n$ with integer coefficients.
Consider an instance $\Omega$ of Pl-Holant($\langle 2, 1, 0, 1, 0 \rangle$) on domain size $\kappa$.

Suppose $\langle 2, 1, 0, 1, 0 \rangle$ appears $n$ times in $\Omega$.

We construct from $\Omega$ a sequence of instances $\Omega_{2s}$ of Pl-Holant($AD_{\kappa}$) indexed by $s \geq 0$, by replacing each occurrence of $\langle 2, 1, 0, 1, 0 \rangle$ with the gadget $f_{2s}$.

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Evaluating this polynomial at $x = 1 + \kappa$ gives the value of $\text{Pl-Holant}_\Omega$. 
We try to prove for every $f = \langle a, b, c \rangle$, Holant$(G; f)$ is either tractable or #P-hard.
Overview of Proof of Dichotomy \( \text{Holant}(G; f) \)

We try to prove for every \( f = \langle a, b, c \rangle \), \( \text{Holant}(G; f) \) is either tractable or \#P-hard.

1. We construct a special unary signature from the given \( \langle a, b, c \rangle \).
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Along the way, we may find certain $\langle a, b, c \rangle$ does not allow us to achieve these steps.
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Along the way, we may find certain \( \langle a, b, c \rangle \) does not allow us to achieve these steps.

Instead, in those cases, we can directly prove that these problems are either in \( P \) or \#P-hard (without the help of additional signatures).
We say that \( \lambda_1, \lambda_2, \ldots, \lambda_\ell \in \mathbb{C} - \{0\} \) satisfy the lattice condition if for all \( x \in \mathbb{Z}^\ell - \{0\} \) with \( \sum_{i=1}^\ell x_i = 0 \), we have

\[
\prod_{i=1}^\ell \lambda_i^{x_i} \neq 1.
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Taking the logarithms, this is really a condition about linear independence \( \{\log \lambda_i\} \) over \( \mathbb{Q} \).
Theorem

If there exists an infinite sequence of planar $\mathcal{F}$-gates defined by an initial signature $s \in \mathbb{C}^{n \times 1}$ and a recurrence matrix $M \in \mathbb{C}^{n \times n}$ satisfying the following conditions,

1. $M$ is diagonalizable with $n$ linearly independent eigenvectors;
2. $s$ is not orthogonal to exactly $\ell$ of these linearly independent row eigenvectors of $M$ with eigenvalues $\lambda_1, \ldots, \lambda_\ell$;
3. $\lambda_1, \ldots, \lambda_\ell$ satisfy the lattice condition;

then

$$\text{Pl-Holant}(\mathcal{F} \cup \{f\}) \leq_T \text{Pl-Holant}(\mathcal{F})$$

for any signature $f$ that is orthogonal to the $n - \ell$ of these linearly independent eigenvectors of $M$ to which $s$ is also orthogonal.
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To prove our dichotomy we use a combinatorial construction with $n = 9$ and $\ell = 5$. 
<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{y}{x}$</td>
<td>and replacing</td>
</tr>
<tr>
<td>$M$</td>
<td>After setting</td>
</tr>
<tr>
<td>$\tilde{\lambda}$</td>
<td>The characteristic polynomial of</td>
</tr>
<tr>
<td>$p$</td>
<td>satisfy</td>
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<tr>
<td>$x$</td>
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</tbody>
</table>
The characteristic polynomial of $M$ is $\lambda_M(x, \kappa) = (x - \kappa^3)^4 f(x, \kappa)$, where 

$$f(x, \kappa) = x^5 - \kappa^6 (2\kappa - 1)x^3 - \kappa^9 (\kappa^2 - 2\kappa + 3)x^2 + (\kappa - 2)(\kappa - 1)\kappa^{12} x + (\kappa - 1)^3 \kappa^{15}.$$
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After setting

$$\tilde{f}(x, \kappa) = \frac{1}{\kappa^{15}} f(\kappa^3 x, \kappa) = x^5 - (2\kappa - 1)x^3 - (\kappa^2 - 2\kappa + 3)x^2 + (\kappa - 2)(\kappa - 1)x + (\kappa - 1)^3$$
The characteristic polynomial of \( M \) is 
\[
\lambda_M(x, \kappa) = (x - \kappa^3)^4 f(x, \kappa),
\]
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\]
and replacing \( \kappa \) by \( y + 1 \) we get 
\[
p(x, y) = x^5 - (2y + 1)x^3 - (y^2 + 2)x^2 + (y - 1)yx + y^3.
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and replacing $\kappa$ by $y + 1$ we get

$$p(x, y) = x^5 - (2y + 1)x^3 - (y^2 + 2)x^2 + (y - 1)yx + y^3.$$ 

We want to prove that for all integer $y \geq 4$, the roots of $p(x, y)$ satisfy the lattice condition.
Irreducible over $\mathbb{Q}[x]$?

We suspect that for any integer $y \geq 4$, $p(x, y)$ is in fact irreducible in $\mathbb{Q}[x]$. 

\[
p(x, y) = \begin{cases} 
(x - 1)(x^4 + x^3 + 2x^2 - x + 1)y = -1 \\
(x + 1)(x^4 - x^3 - 2x^2 - x + 1)y = 1 \\
(x - 1)(x^2 - x - 4)(x^2 + 2x + 2)y = 2 \\
(x - 3)(x^4 + 3x^3 + 2x^2 - 5x - 9)y = 3 
\end{cases}
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We suspect that for any integer $y \geq 4$, $p(x, y)$ is in fact irreducible in $\mathbb{Q}[x]$. Can’t prove that.
Irreducible over \( \mathbb{Q}[x] \)?

We suspect that for any integer \( y \geq 4 \), \( p(x, y) \) is in fact irreducible in \( \mathbb{Q}[x] \).

Can’t prove that.

We know five integer solutions \( (x, y) \in \mathbb{Z}^2 \), so for these five values of \( y \in \mathbb{Z} \), \( p(x, y) \) is reducible as a polynomial in \( x \):

\[
p(x, y) = \begin{cases} 
(x - 1)(x^4 + x^3 + 2x^2 - x + 1) & y = -1 \\
x^2(x^3 - x - 2) & y = 0 \\
(x + 1)(x^4 - x^3 - 2x^2 - x + 1) & y = 1 \\
(x - 1)(x^2 - x - 4)(x^2 + 2x + 2) & y = 2 \\
(x - 3)(x^4 + 3x^3 + 2x^2 - 5x - 9) & y = 3.
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We will show there are no other integer solutions \((x, y) \in \mathbb{Z}^2\).
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This means, for all integer \(y \geq 4\), \(p(x, y)\) is either irreducible or is a product of two irreducible polynomials of degree 2 and 3 respectively.
Effective Siegel’s Theorem?

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Note that, by Gauss Lemma, for any integer \(y\), the monic polynomial \(p(x, y)\) in \(x\) is irreducible over \(\mathbb{Z}\) iff it is irreducible over \(\mathbb{Q}\).
Effective Siegel’s Theorem?

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Note that, by Gauss Lemma, for any integer \(y\), the monic polynomial \(p(x, y)\) in \(x\) is irreducible over \(\mathbb{Z}\) iff it is irreducible over \(\mathbb{Q}\).

**Lemma**

Let \(f(x) \in \mathbb{Q}[x]\) be a polynomial of degree \(n \geq 2\). If the Galois group of \(f\) over \(\mathbb{Q}\) is \(S_n\) or \(A_n\) and the roots of \(f\) do not all have the same complex norm, then the roots of \(f\) satisfy the lattice condition.
Irreducible Quintic

Lemma

For any integer \( y \geq 1 \), the polynomial \( p(x, y) \) has three distinct real roots and two nonreal complex conjugate roots in \( x \).

Proof by discriminant.
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Lemma

For any integer \( y \geq 4 \), if \( p(x, y) \) is irreducible in \( \mathbb{Q}[x] \), then the roots of \( p(x, y) \) satisfy the lattice condition.

Proof.

Three distinct real roots do not have the same complex norm.
An irreducible polynomial of prime degree \( n \) with exactly two nonreal roots has \( S_n \) as its Galois group over \( \mathbb{Q} \).
Hence they satisfy the lattice condition.
What if it is Reducible?

Some more Galois Theory is needed if it is a product of two irreducible polynomials of degree 2 and 3.
What if it is Reducible?

Some more Galois Theory is needed if it is a product of two irreducible polynomials of degree 2 and 3.

They still satisfy the lattice condition.
**Lemma**

The only integer solutions to $p(x, y) = 0$ are

$$(1, -1), (0, 0), (-1, 1), (1, 2), \text{ and } (3, 3).$$
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Let $p(a, b) = 0$ with $a \neq 0$. 
But Why No Other Integer Solutions?

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One can show that \( a \mid b^2 \).
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One can show that \( a | b^2 \).

Consider

\[ g_1(x, y) = y - x^2 \quad \text{and} \quad g_2(x, y) = \frac{y^2}{x} + y - x^2 + 1. \]

(This particular choice is due to Aaron Levin.) Whenever \( p(a, b) = 0 \) with \( a \neq 0 \), \( g_1(a, b) \) and \( g_2(a, b) \) are integers. However, we show that if \( a \leq -3 \) or \( a \geq 17 \), then either \( g_1(a, b) \) or \( g_2(a, b) \) is not an integer.
Puiseux Series

The Puiseux series expansions for \( p(x, y) \) are

\[
y_1(x) = x^2 + 2x^{-1} + 2x^{-2} - 6x^{-4} - 18x^{-5} + O(x^{-6})
\]

\[
y_2(x) = x^{3/2} - \frac{1}{2}x + \frac{1}{8}x^{1/2} - \frac{65}{128}x^{-1/2} - x^{-1} - \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2})
\]

\[
y_3(x) = -x^{3/2} - \frac{1}{2}x - \frac{1}{8}x^{1/2} + \frac{65}{128}x^{-1/2} - x^{-1} + \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2})
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If we substitute say $y_2(x)$ in $g_2(x, y_2(x))$, we get $O(x^{-1/2})$, where the multiplier in the $O$-notation is bounded both above and below by a nonzero constant in absolute value.
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So for large $x$, it is non-zero and non-integral.
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\[ y_3(x) = -x^{3/2} - \frac{1}{2}x - \frac{1}{8}x^{1/2} + \frac{65}{128}x^{-1/2} - x^{-1} + \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2}) \]

If we substitute say $y_2(x)$ in $g_2(x, y_2(x))$, we get $O(x^{-1/2})$, where the multiplier in the $O$-notation is bounded both above and below by a nonzero constant in absolute value.

So for large $x$, it is non-zero and non-integral.

Hence there are no large integral solutions.
Some papers can be found on my web site
http://www.cs.wisc.edu/~jyc

THANK YOU!