The Effect of Restrictive Interactions on Epidemics

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Contact process with avoidance

Joint work with
David Sivakoff and Matthew Wascher

Overview

1. The contact process
2. The contact process with avoidance
3. Results
4. Proof sketch for $\mathbb{Z}$
5. Proof sketch for $\mathbb{Z}_n$
6. Proof sketch for the star graph

The Contact Process

Graph: $G = (V, E)$

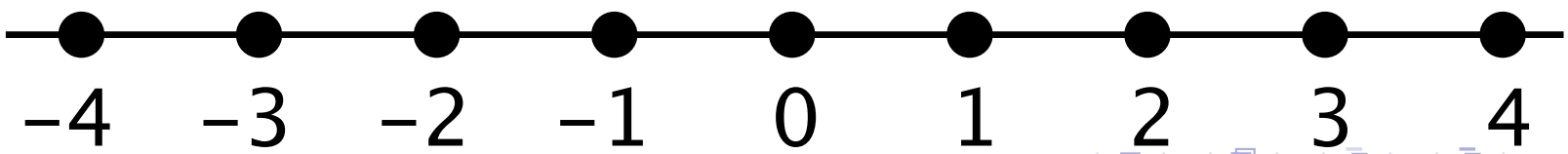
Let $\xi_t^A \subseteq V$ denote the set of 'infected' vertices at time $t \geq 0$, where $\xi_0^A = A$ is the set of initially infected vertices.

$$\xi_t^A(v) := \mathbb{1}_{\{v \in \xi_t^A\}}$$

Contact Process

Let $\lambda > 0$ be the infection rate. Then for each $v \in V$ and $t \geq 0$:

$$\xi_t^A(v) \text{ transitions } \begin{cases} 1 \to 0 & \text{at rate } 1, \\ 0 \to 1 & \text{at rate } \lambda \left| \mathcal{N}(v) \cap \xi_t^A \right|. \end{cases}$$



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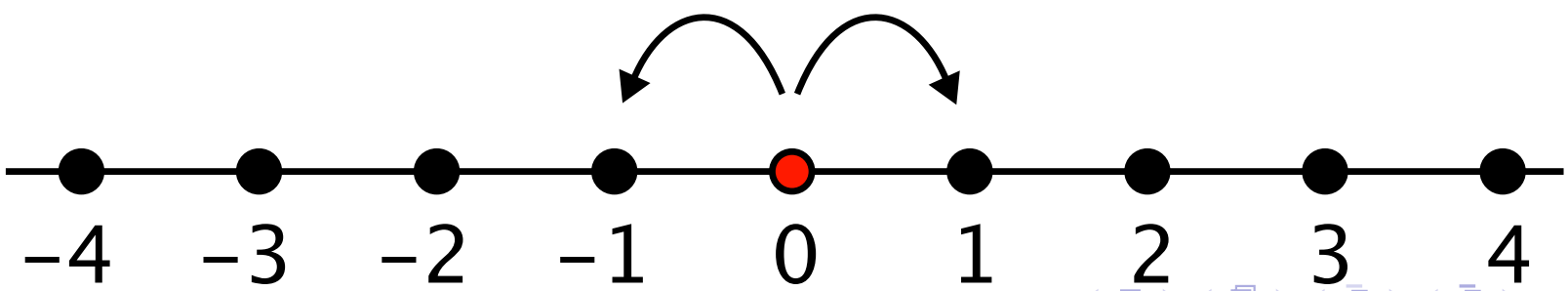

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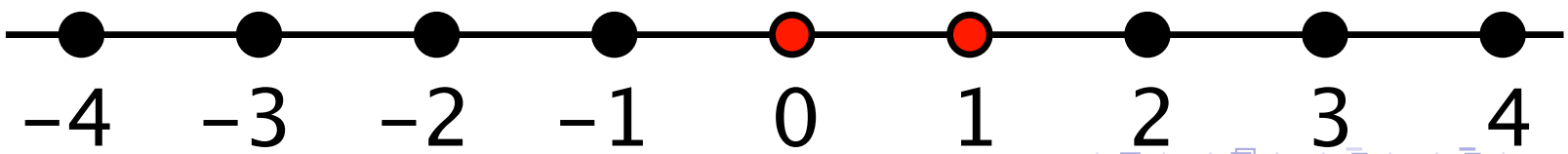

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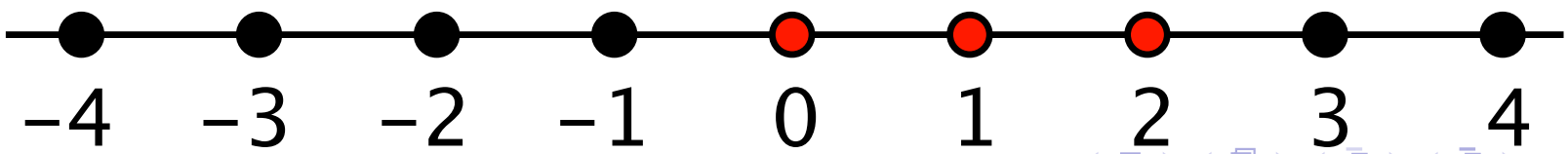

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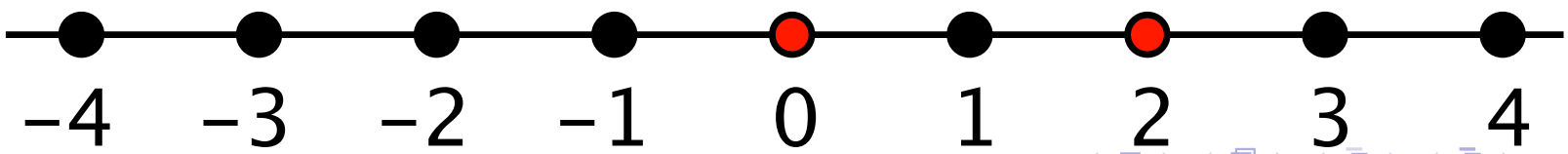

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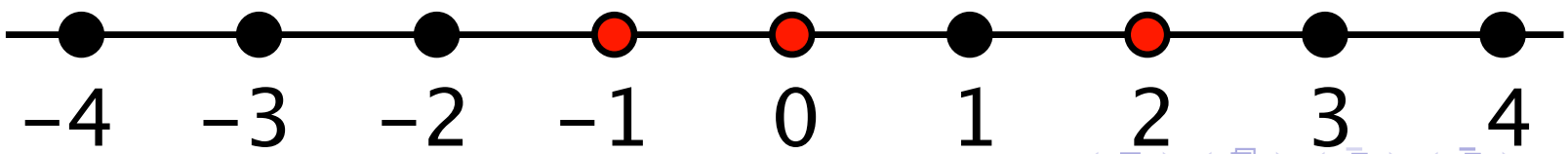

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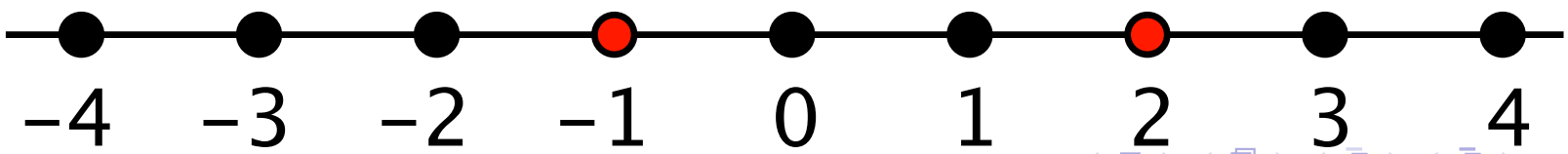

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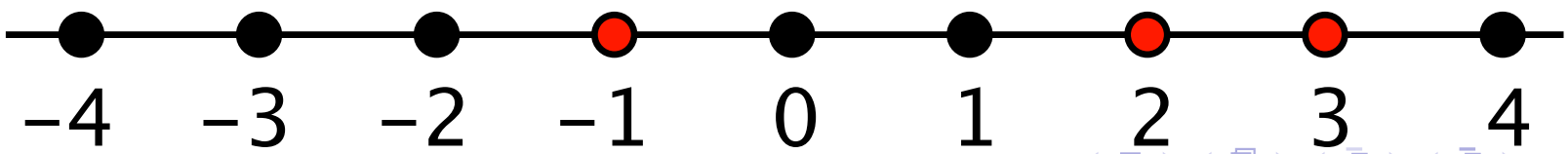

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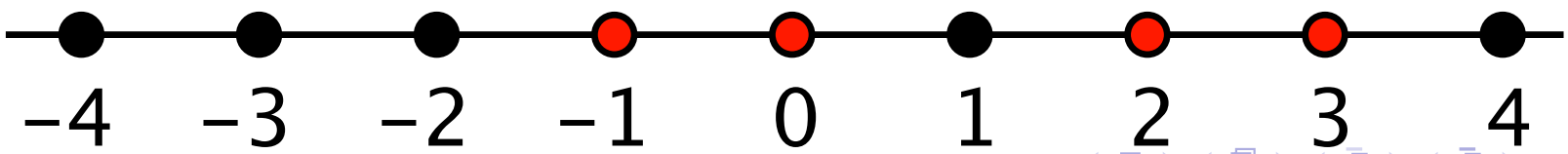
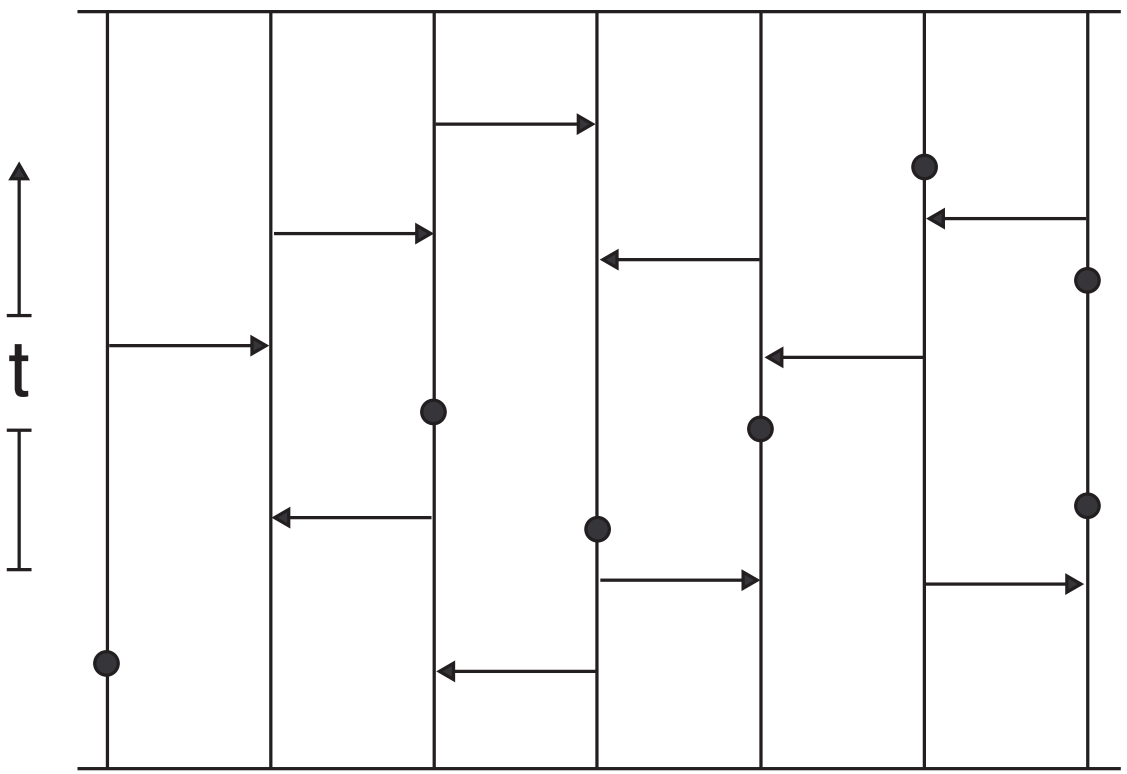


Graphical Representation



- At times $(T_k^v)_{k\geq 1}$ of a Poisson process with rate 1, vertex v becomes uninfected. Represented by a dot at v.
- At times $(T_k^{(u,v)})_{k\geq 1}$ of a Poisson process with rate λ, if u is infected then v becomes infected. Represented by an arrow $u \to v$.

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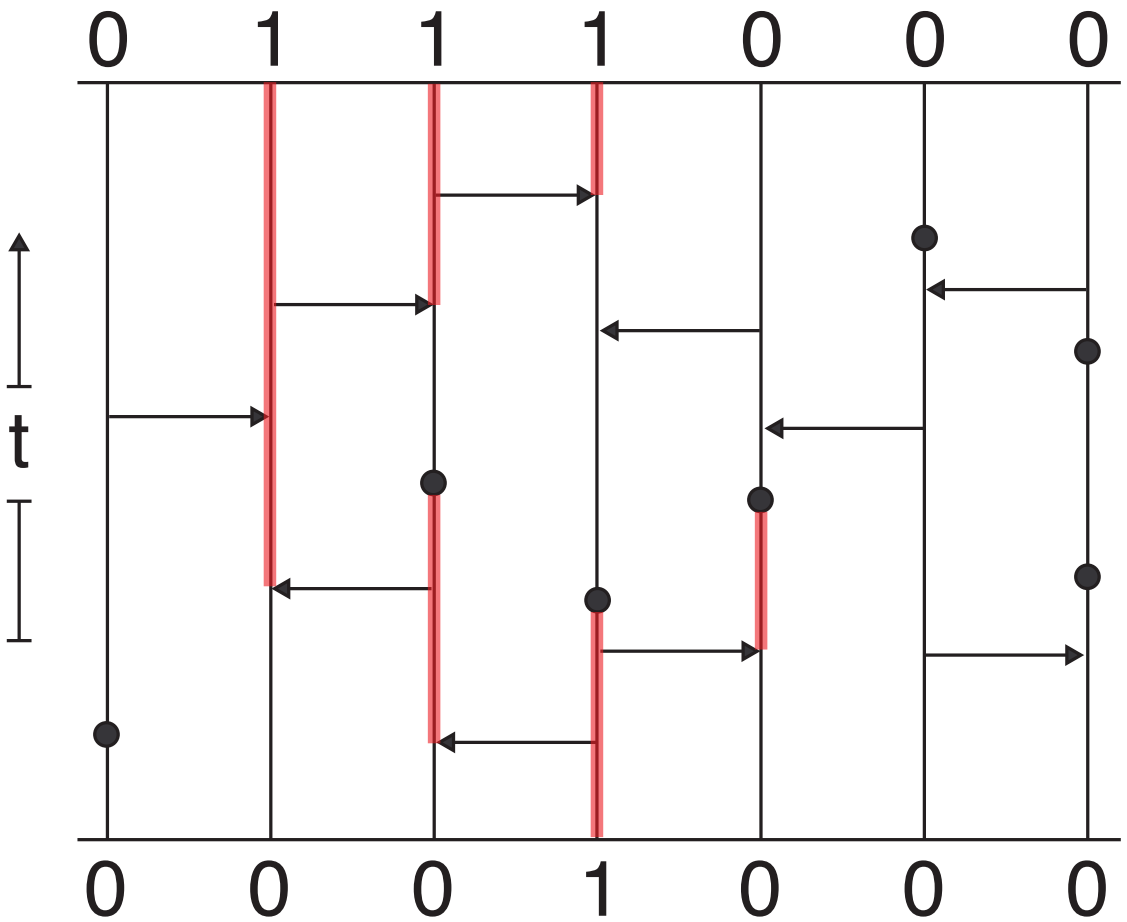

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Monotonicity

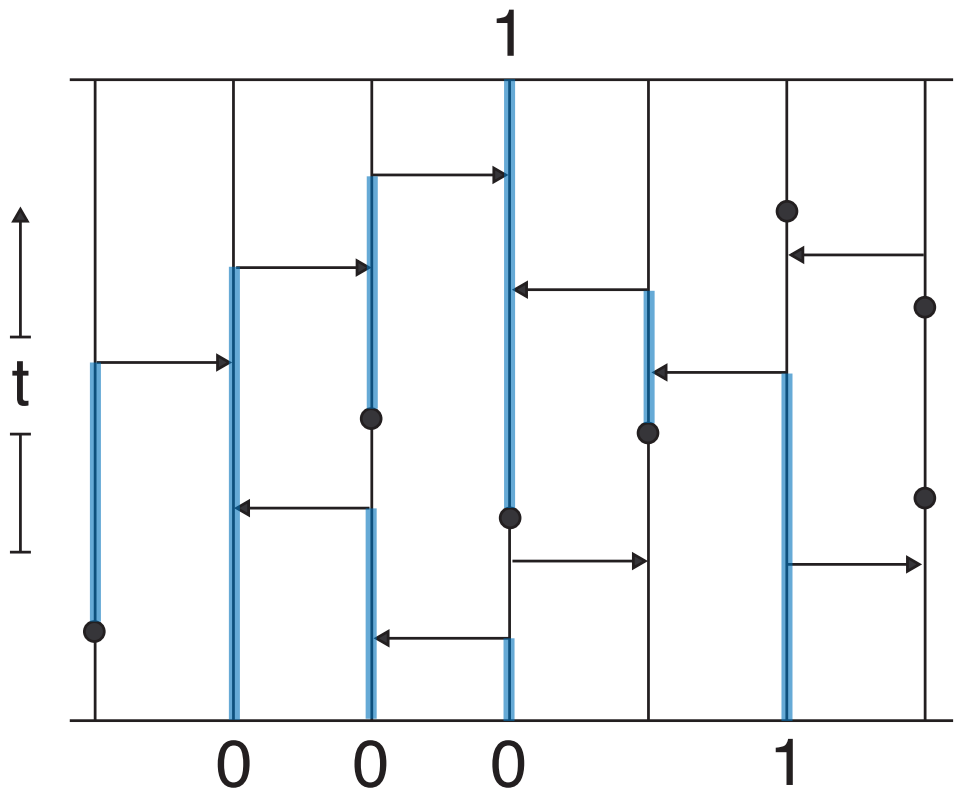
The contact process is monotonic in the following sense:

- If $A \subseteq B$ then $\xi_t^A \subseteq_{st} \xi_t^B$ for all t.
- If $\lambda_1 \leq \lambda_2$ then $\xi_t^{A,1} \subseteq_{st} \xi_t^{A,2}$

This type of monotonicity is also called **attractiveness**.

This can be demonstrated via coupling using the Harris construction.

Monotonicity



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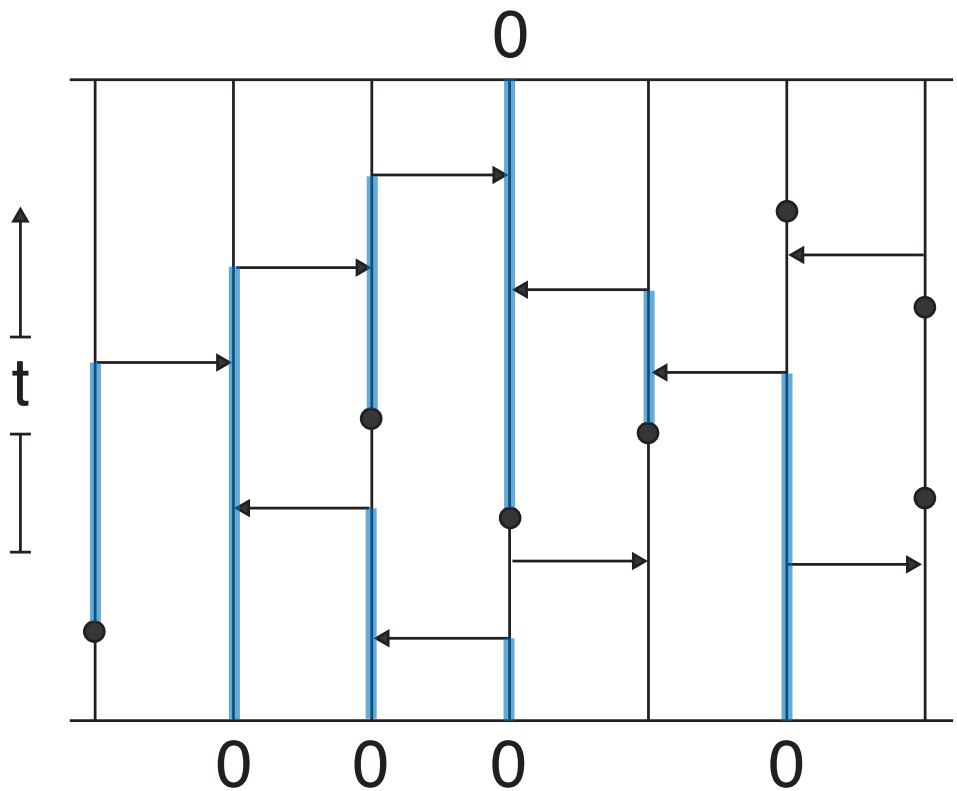


Contact Process: Critical infection rate, λ_c

For an infinite graph, such as the lattice $\mathbb{Z}^d$, define the critical infection rate:

$$\lambda_c := \sup\{\lambda \ : \ P(\xi_t^0 \neq \emptyset \text{ for all } t) = 0\}.$$

Monotonicity in λ, apparent from the graphical construction, implies that for $\lambda > \lambda_c$ and any $A \neq \emptyset$,

$$P(\xi_t^A \neq \emptyset \text{ for all } t) > 0.$$

On $\mathbb{Z}$ with nearest-neighbor edges, $1.539 \leq \lambda_c < 2$ [see, e.g., Liggett 1999]

Contact Process: Critical infection rate, λ_c

For a finite graph (e.g., $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$) $\xi_t^A = \emptyset$ is the unique absorbing state, so for every $\lambda > 0$,

$$P(\xi_t^A = \emptyset \text{ for some } t) = 1.$$

Identify λ_c by the (asymptotic in n) time to reach the absorbing state.

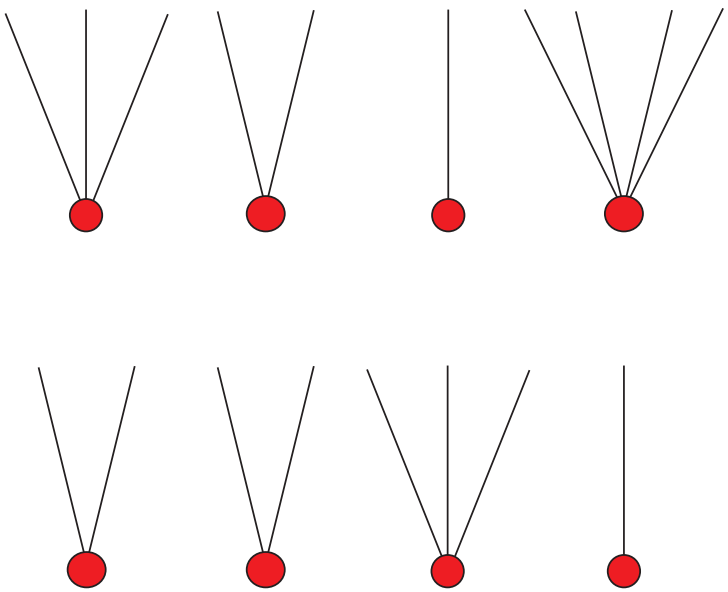
For example, consider $\mathbb{Z}_n$.
Let $\tau = \inf\{t : \xi_t^{\mathbb{Z}_n} = \emptyset\}$.
For $\lambda_c = \lambda_c(\mathbb{Z})$, then there exist $c, C > 0$ such that

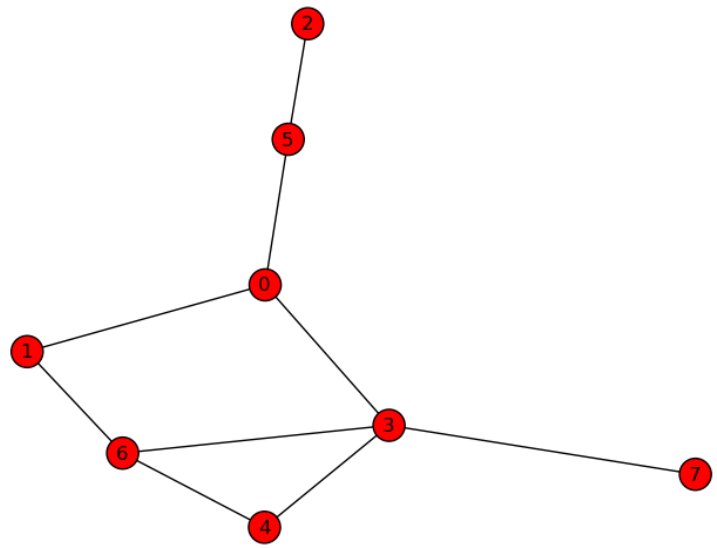
- $P(\tau \leq e^{cn}) \to 0$ if $\lambda > \lambda_c$,
- $P(\tau \geq C \log n) \to 0$ if $\lambda < \lambda_c$.

Random graphs by example: The configuration model

To generate a random graph with a prescribed degree sequence:

- Draw N independent values from the degree distribution $\{p_k\}$: $d_1, \ldots, d_N$ (conditional on $d_1 + \cdots + d_N$ even)
- The i^{th} vertex has d_i half-edges attached to it.
- Pair up half-edges at random.





Contact process on finite random graphs

Start with all vertices infected. How long before everyone is healthy ($\xi_t^V = \emptyset$)?

Suppose G_n has $p_k \propto k^{-\gamma}$.

- Pastor-Satorras and Vespigniani (2001, 2002) used mean-field theory to predict:
  - If $\gamma \leq 3$ then the process survives for a long time for any $\lambda > 0$.
  - If $\gamma > 3$ then the process survives for a long time only if $\lambda > \lambda_c > 0$.
- Berger, Borgs, Chayes and Saberi (2005) show $\lambda_c = 0$ on preferential attachment graphs ($\gamma = 3$). Exponential survival on stars.
- C. and Durrett (2009) show $\lambda_c = 0$ on configuration model graphs, $\gamma > 2$.
- Peterson (2011) shows the mean field predictions hold ($\lambda_c > 0$ for $\gamma > 3$) on the complete graph with random vertex weights.

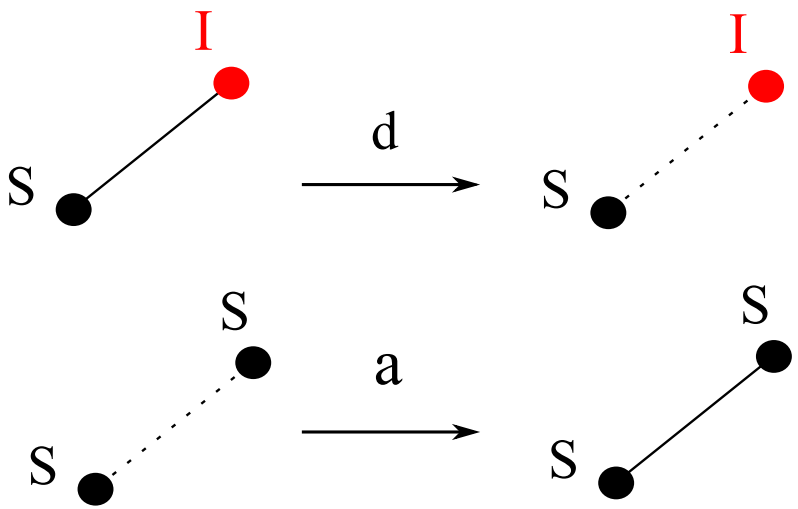
Contact process on finite random graphs

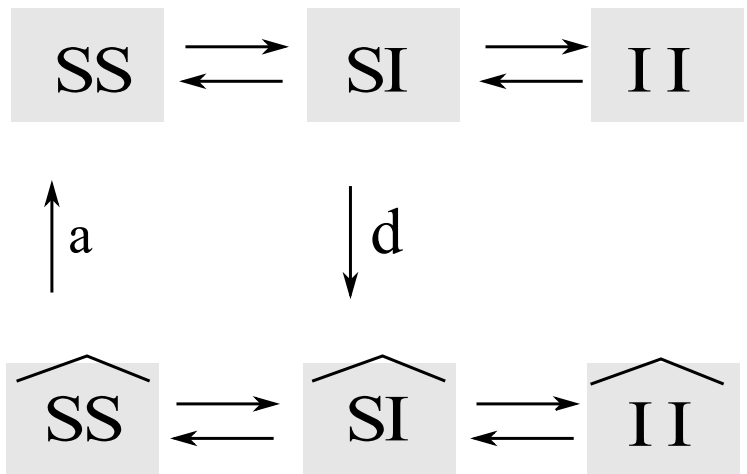
Exponential survival and metastable densities for random graphs:

- Configuration model with power-law degree $\gamma > 2$ [Mountford, Valesin, & Yao, 2013]
- Configuration model with $\gamma \leq 2$ [Can & Shapira, 2015]
- Preferential attachment [Can, 2017]
- Stationary dynamic graphs: $\lambda_c > 0$ if network mixes quickly and $\gamma > 4$ [Jacob & Mörters, 2017]
- General degree distribution: $\lambda_c > 0$ iff $\mathrm{E}e^{t \cdot \deg(v)} < \infty$ for some $t > 0$ [Huang & Durrett, 2018; Bhamidi, Nam, Nguyen & Sly, 2019]

Contact process with avoidance behavior

SIS with temporary link inactivation:
[Guo, Trajanovsky, van de Bovenkamp, Wang & Van Mieghem, 2013;
Tunc, Shkarayev & Shaw, 2013;
Shkarayev, Tunc & Shaw, 2014]





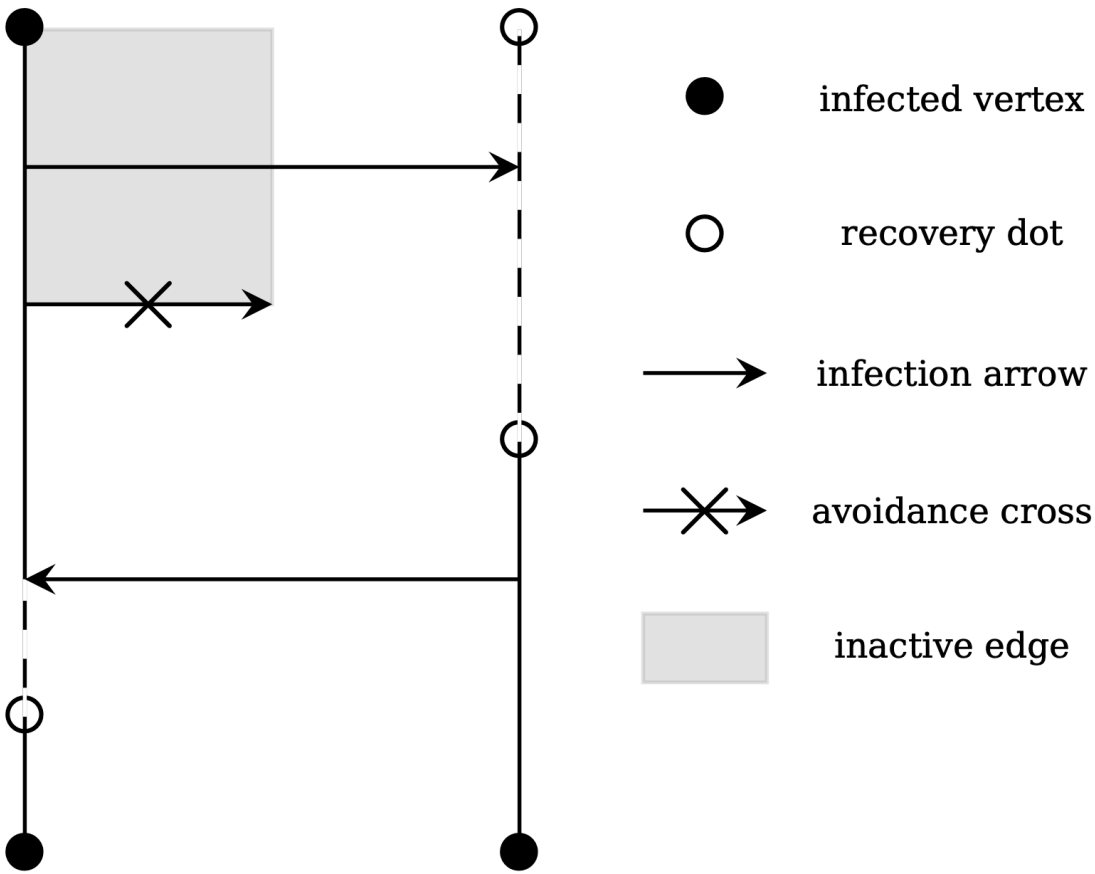
Our model

Contact process with avoidance [link inactivation]

Fix a directed graph G. States of vertices are $\xi_t \in \{0,1\}^V$ and states of edges are $\eta_t \in \{0,1\}^E$.

- $\xi_t(v)$ transitions $1 \to 0$ at rate 1.
- $\xi_t(v)$ transitions $0 \to 1$ at rate $\lambda \sum_{(w,v)\in E} \eta_t((w,v))\xi_t(w)$.
- $\eta_t((w,v))$ transitions $1 \to 0$ at rate α if $\xi_t(v) = 0$ and $\xi_t(w) = 1$.
- $\eta_t((w,v))$ transitions $0 \to 1$ instantaneously when $\xi_t(w) = 0$.

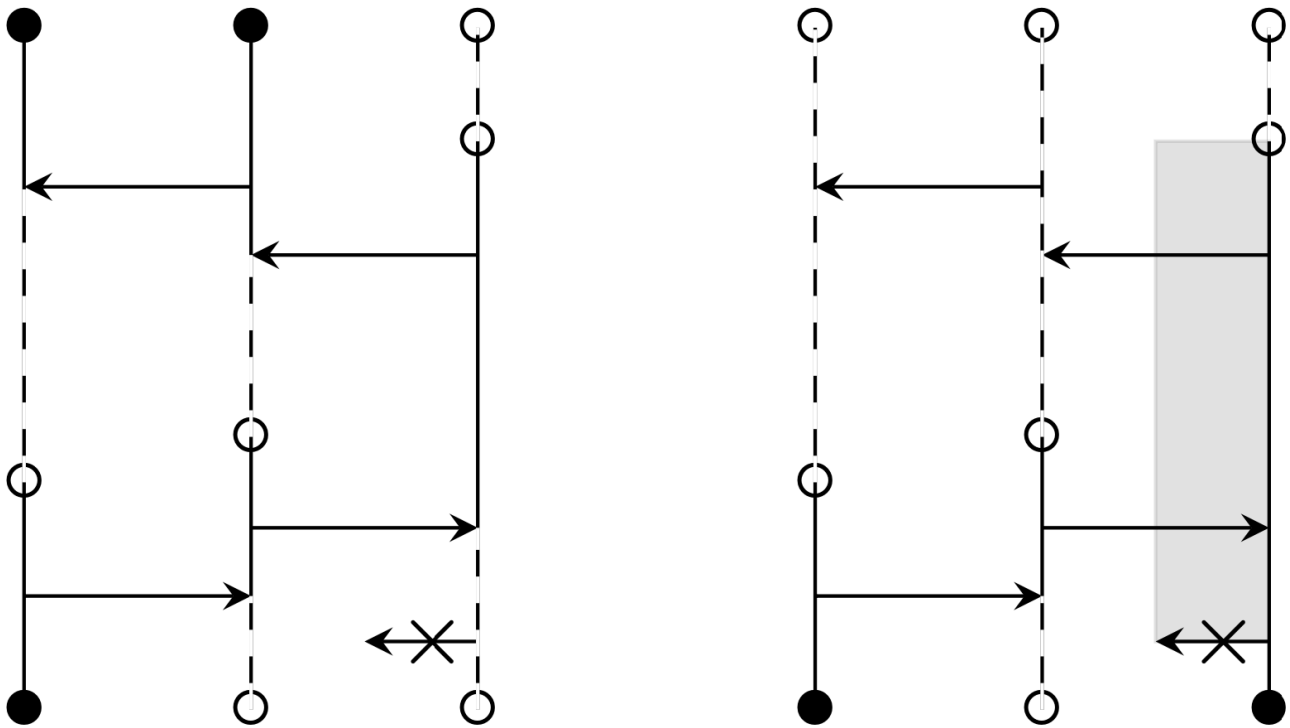
Graphical Representation, Harris construction



- As in contact process: dots (rate 1) induce recovery, arrows (rate λ) induce infection.
- At times $(\tilde{T}_k^{(u,v)})_{k\geq 1}$ of a Poisson process with rate α, if u is infected and v is not, then v avoids u. Represented by an arrow with $\times$ from $u \to v$. Avoidance lasts until the next recovery dot at u, and arrows $u \to v$ are ignored during avoidance.

Non-monotonicity in the Harris construction

Figure: Smaller initial infected set, larger final infected set



Results for $\mathbb{Z}$

Define upper and lower thresholds for survival of infection on $\mathbb{Z}$:

Assume initial configuration $\xi_0 = \{0\}$ and $\eta_0 = E$. For each α, let

$$\lambda_c^-(\alpha) := \inf\{\lambda : P(\xi_t \neq \emptyset \text{ for all } t) > 0\}$$
$$\lambda_c^+(\alpha) := \sup\{\lambda : P(\xi_t \neq \emptyset \text{ for all } t) = 0\}$$

Conjecture: $\lambda_c^-(\alpha) = \lambda_c^+(\alpha)$, and increases linearly with α.

Theorem (C., Sivakoff, & Wascher, 2022)

There exist constants $a, b > 0$ such that for all $\alpha > 0$,

$$1 + \alpha \leq \lambda_c^-(\alpha) \leq \lambda_c^+(\alpha) \leq a + b\alpha.$$

Results for the n-cycle, $\mathbb{Z}_n$

Assume initial configuration $\xi_0 = \mathbb{Z}_n$ and $\eta_0 = E$.
Let $\tau = \inf\{t : \xi_t = \emptyset\}$.

Theorem (C., Sivakoff, & Wascher, 2022)

Let a, b be as in the previous theorem and fix $\alpha > 0$.

- If $\lambda < 1 + \alpha$, then there exists $C > 0$ such that $P(\tau \leq C \log n) \to 1$.
- If $\lambda > a + b\alpha$, then there exists $c > 0$ such that $P(\tau \geq e^{cn}) \to 1$.

Results for star graph on n vertices

Assume initial configuration $\xi_0 = V$ and $\eta_0 = E$.
Let $\tau = \inf\{t : \xi_t = \emptyset\}$.

Theorem (C., Sivakoff, & Wascher, 2022)

For $\lambda > 0$ and $\alpha > 0$, let

$$\Delta = 2\left[(\lambda + \alpha + 1) - \sqrt{(\lambda + \alpha + 1)^2 - 4\alpha}\right]^{-1}.$$

Then

$$n^{\Delta - o(1)} \leq \tau \leq O(n^{\Delta}) \qquad \text{in Probability.}$$

Note: If $\alpha = 0$ and $\lambda > 0$, then $\tau \geq e^{cn}$ with high probability.

Note: $\Delta > 1$ for all $\alpha > 0$ and $\lambda > 0$.

Next step

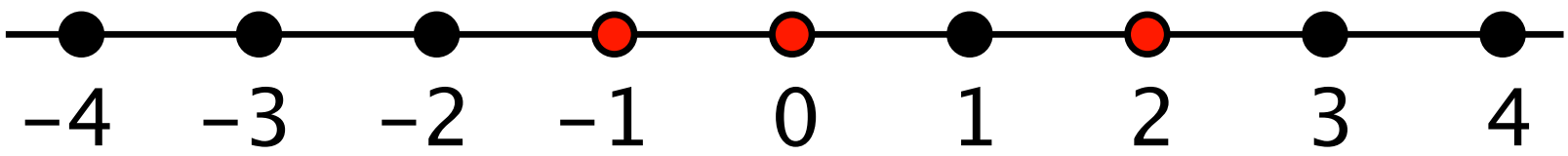
If G_n has asymptotic degree distribution $p_k \propto k^{-\gamma}$, do there exist γ and α (large) and $\lambda > 0$ (small) such that $\tau = n^{O(1)}$? $e^{o(n)}$?

Proof Ideas: $\mathbb{Z}$ and $\mathbb{Z}_n$

For $\lambda < 1 + \alpha$, consider location of rightmost infected vertex after k jumps, R_k:

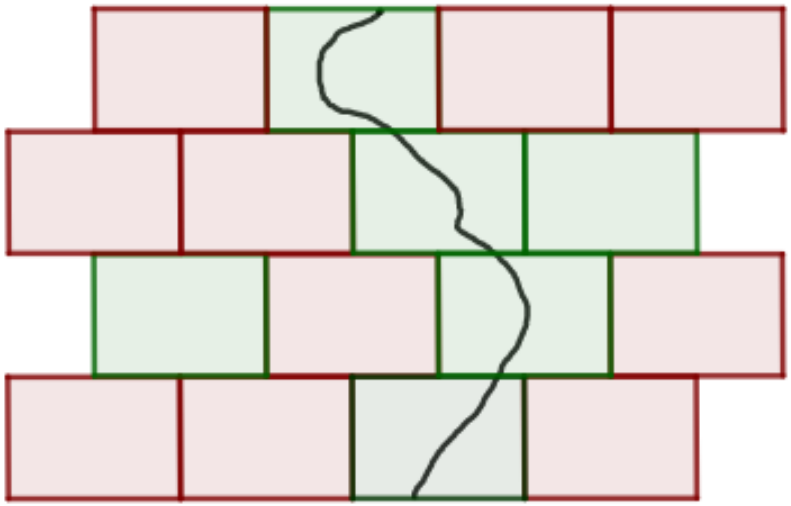
- $P(R_{k+1} = R_k + 1) = \frac{\lambda}{1+\alpha+\lambda} < \frac{1}{2}$,
- $P(R_{k+1} \leq R_k - 1) = \frac{1+\alpha}{1+\alpha+\lambda} > \frac{1}{2}$.

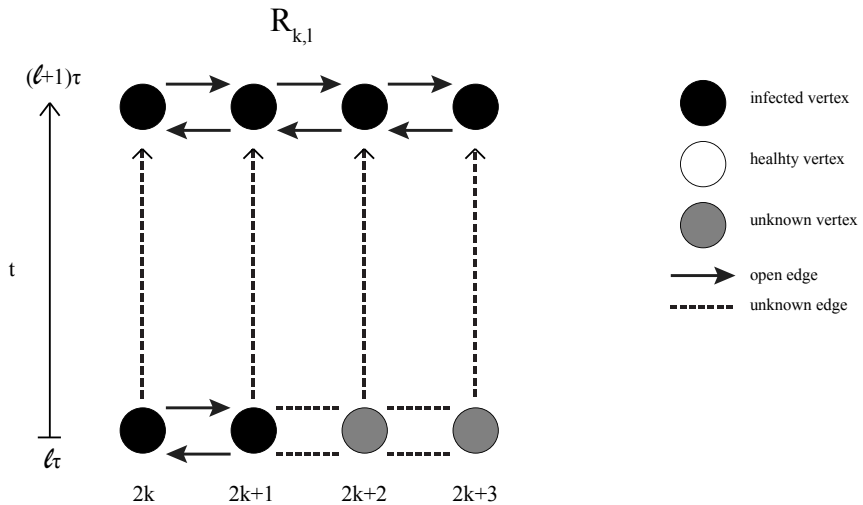
Dominated by biased random walk, so $R_k \to -\infty$.



Proof Ideas: $\mathbb{Z}$ and $\mathbb{Z}_n$

For large λ, subdivide space-time into $4 \times \tau$ sized blocks.





- A Good block allows passage of infection from either of the two blocks below.
- By choosing $\lambda > a + b\alpha$ and τ appropriately, P(A block is Good) can be made large so that the good blocks stochastically dominate a supercritical Oriented Site Percolation.

Bound for λ_α^+

While comparing the space-time blocks of the Harris construction with the “sites” of a supercritical oriented site percolation model, the mai challenge is that the regions are not independent, and one can’t ignore the dependence because it is not clear whether that would help or hinder the survival.

We need to define the blocks carefully so that $P(\mathit{ablockisgood})$ is high regardless of interferences coming from the outside of that block.

Proof Ideas: Star graph

When the center is infected, the leaves are independent Markov chains on $\{1A, 0A, 0D, 1D\}$ with generator

$$\begin{array}{c} \\ 1A \\ 0A \\ 0D \\ 1D \end{array} \begin{bmatrix} -1 & 1 & 0 & 0 \\ \lambda & -(\lambda+\alpha) & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

which has eigenvalues $0 > -\gamma_1 > -1 > -\gamma_2$, where $\gamma_1 = 1/\Delta$.

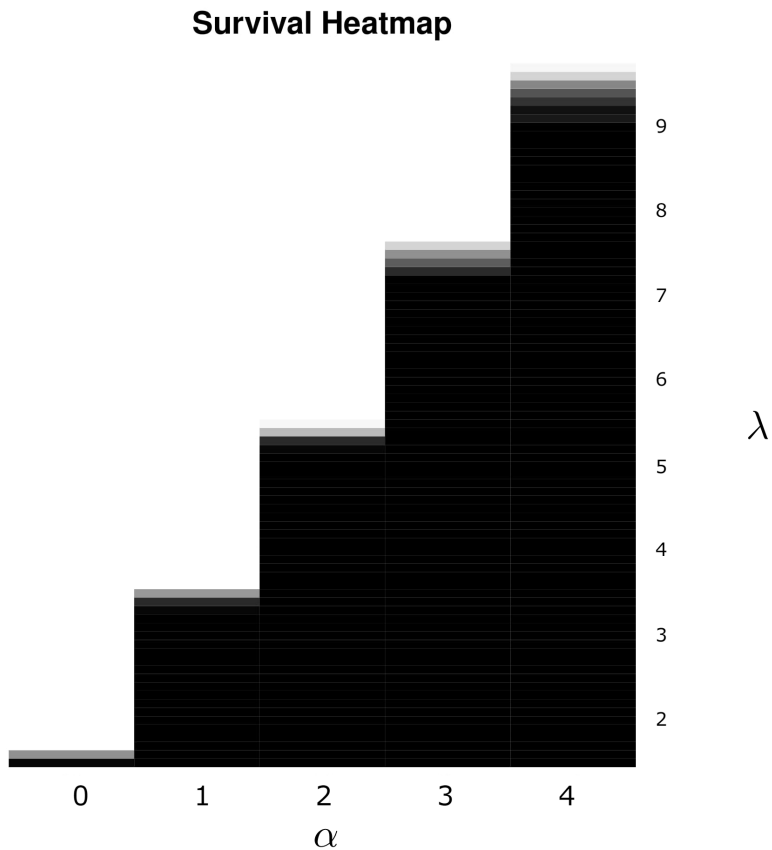
Center becomes healthy at time $T \sim \mathrm{Exp}(1)$.

Number of non-$0D$ leaves at time T is $\approx ne^{-\gamma_1 T}$, which is < 1 if $T > \Delta \log n$.

$P(T > \Delta \log n) = n^{-\Delta}$, so the center must be reinfected about n^{Δ} times before this happens.

If this doesn't happen, then there are many $1A$ leaves at time T, so the center is reinfected quickly.

Simulation summary for CPA on cycle graphs



Simulation study of a model for badging and pool-testing system

Ongoing joint work with
Bud Mishra and Inavamsi Enaganti (NYU)

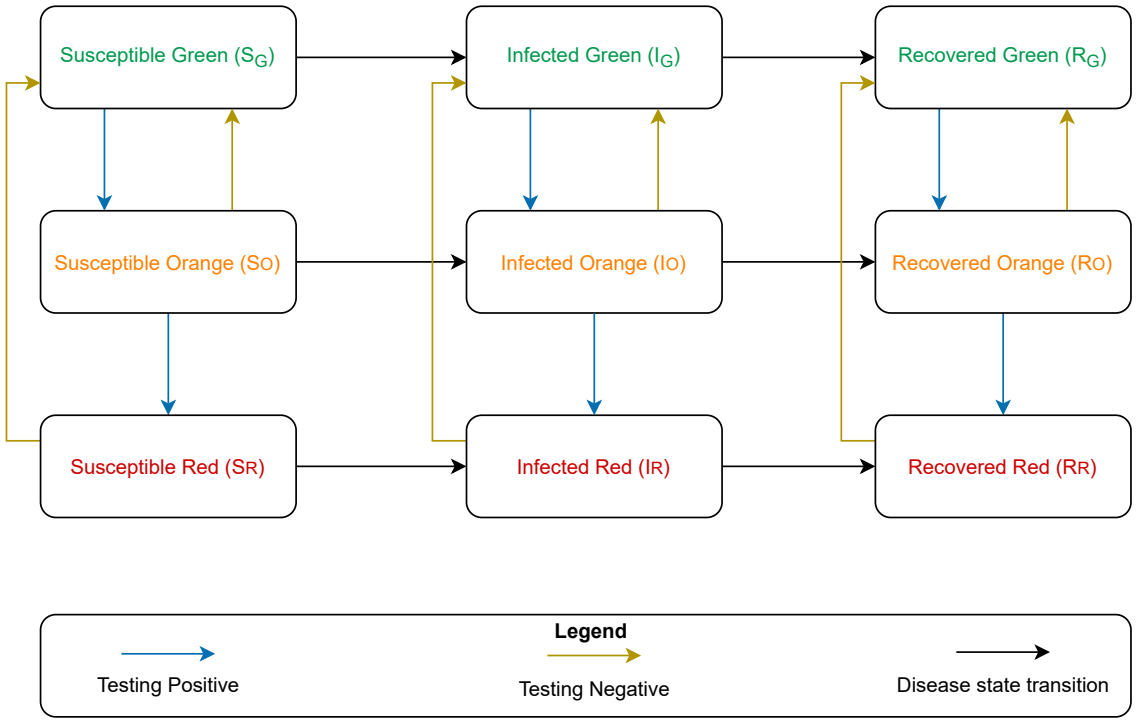
Motivation

- To track pandemic involving highly contagious diseases (with some infected individuals being asymptomatic carriers), ideally, every individual would be tested frequently and regularly, unless there is a complete lockdown.
- But, there are many impediments and constraints, which can make it impossible or un- realistic to test every individual frequently within short intervals, e.g., lack of availability of sufficiently many testing machines, cost of reliable test procedures, the time necessary for obtaining results for such tests, huge population size and lack of awareness about the tests (particularly in some of the developing countries).

The badging system

- Three kinds of badges: green, orange, and red.
- People with green badges would have no restriction on their movements and accesses. People having red badges may be highly restricted, and people with orange badges are in-between.
- The extent of restriction would be parameterized.
- Testing rates may be ordered. People with red (resp. orange) badges will be tested at a lower frequency compared to the people having orange (resp. red) badges.
- Green (resp. orange) badges downgrades to orange (resp. red) after testing positive. Orange and red badges upgrade to green badges after testing negative.
- Pool testing. Limited number of pools.
- Anonymous communication to the users.

Compartment model of the badging system



Each of the 9 compartments correspond to a disease state of an individual and the badge of the individual. The transitions in black denote the change in the disease state of an individual. The Blue and Yellow transitions correspond to change in the Badge of an individual based on the result of a test being positive or negative respectively.

ODE system

Three main aspects need to be modeled:

- the structure of the network that captures the physical proximity and interactions among people,
- the disease spreading mechanism,
- rules for issuing different badges and associated rubrics.

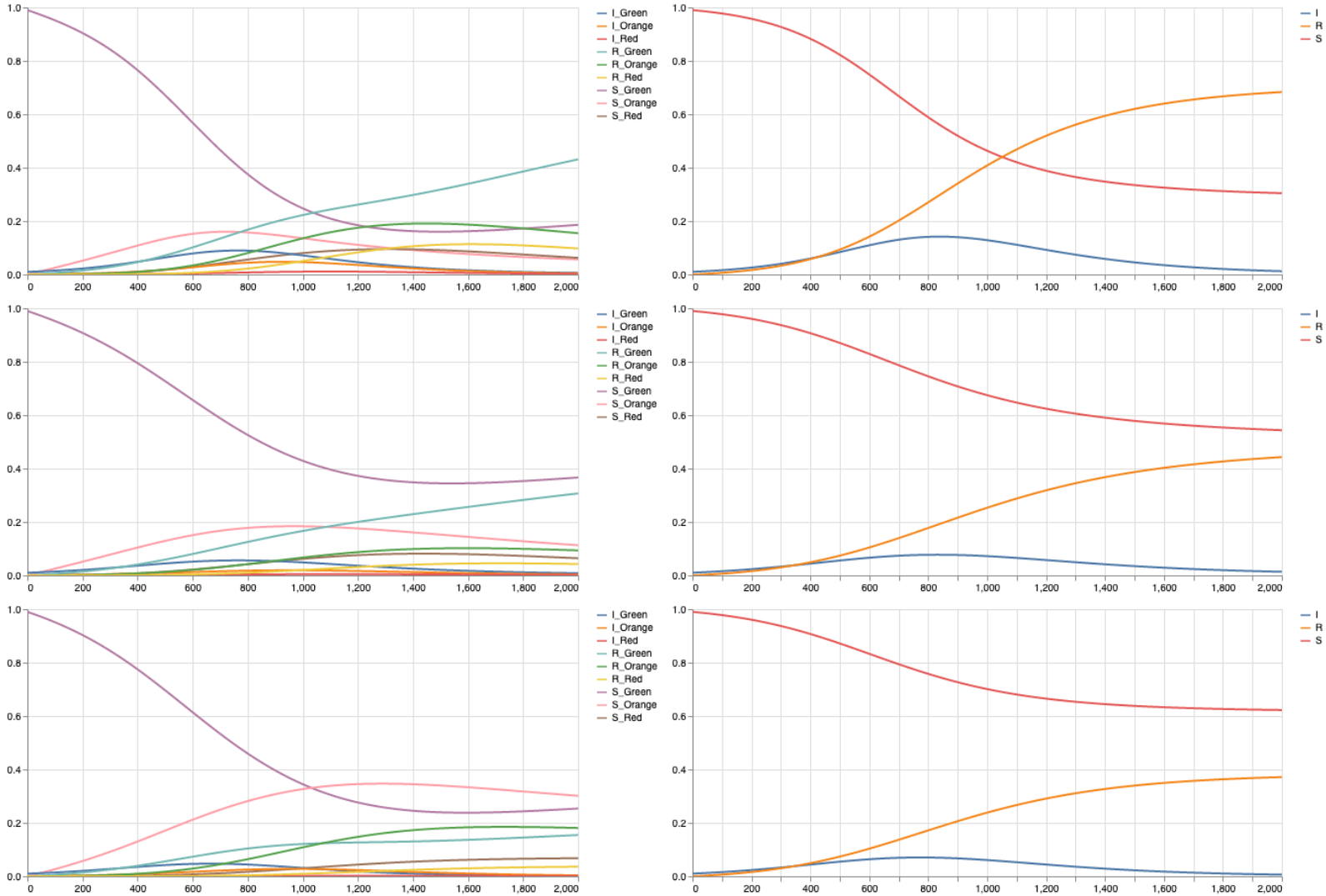
Assuming "complete graph" interaction, SIR epidemic model, and the 3 badge system, the state space is

$$X := \{D_B : D \in \{S, I, R\} \text{ and } B \in \{G, O, R\}\}$$

Parameters:

- β and γ for the standard SIR model,
- "allowable movement policy". Person with badge B has freedom level $\phi_B \in [0, 1]$. S_B and $I_{B'}$ interacts at rate $\beta\phi_B\phi_{B'}$, and increases I_B.
- testing rates: $t_G \geq t_O \geq t_R$.
- false positivity and negativity chances, pool size, number of pools.

ODE Plots

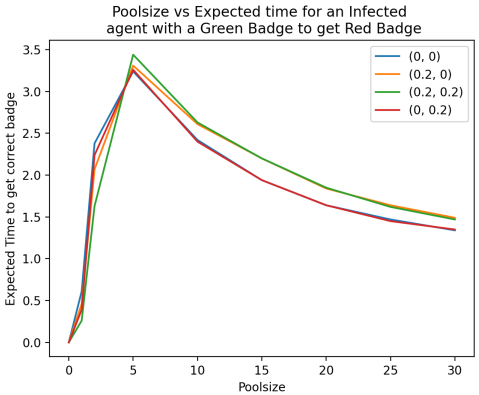


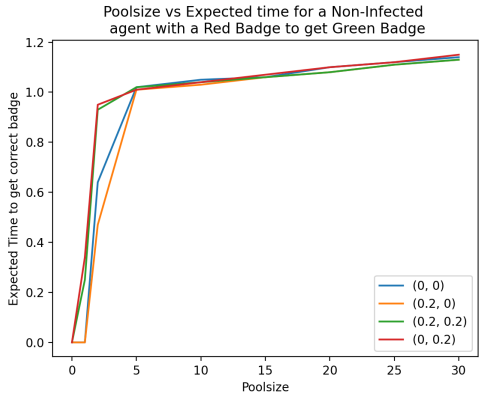
1st row: $1 = \phi_G = \phi_O = \phi_R$ and $t_G = t_O = t_R$.

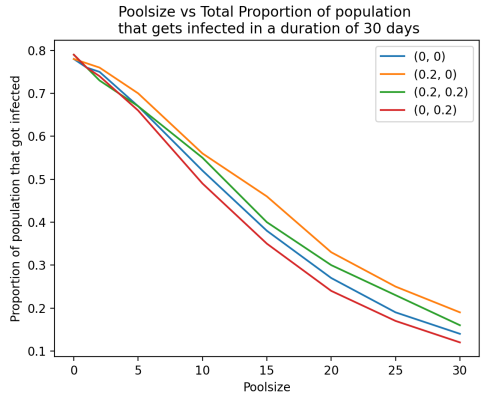
2nd row: $1 = \phi_G > \phi_O > \phi_R = 0$ and $t_G = t_O = t_R$.

3rd row: $1 = \phi_G > \phi_O > \phi_R = 0$ and $t_G > t_O > t_R$.

Agent-based model plots







- The ODE system captures the long term behaviors. Other questions (e.g., expected time to get a correct badge) can be addressed by looking at the associated ABM.
- The expected time needed for a person to get a correct badge is not monotone in pool size, and it stabilizes if the pool size is increased sufficiently.
- Total number of infected people is a decreasing function of pool size.
- Robustness against inherent false positive and false negative testing rates.

Good testing and badging policies

Mainly four kinds of cost:

- cost for quarantining or restricting people
- cost incurred by the infected people
- cost for the tests
- overhead

Good strategies are those for which the total cost is low.

- The total cost associated with a pandemic within a certain time period depends on many factors. The disease prevalence, fatality rates, economic costs because of restrictions on the movement of people which in turn depend on the pooling strategy, additional constraints, etc. share a very convoluted nonlinear relationship with the total cost.
- The constraints and priorities vary across different communities. Different communities may have different sets of constraints, and they can set parameters according to their needs, constraints, preferences, or abilities.
- The total cost is observed to be a non-monotone function of the parameters (even pool size).

Thanks!