Functional Law of Large Numbers and PDEs for Spatial Epidemic Models with Infection-age Dependent Infectivity

Etienne Pardoux (I2M, AMU)

joint work with G. Pang (Rice Univ. USA)

Simons Institute, 10/24/2022

The spatial model

- We consider a population of fixed size N distributed in $K = K^N$ locations $x_1, \ldots, x_K$ in $\mathcal{S} = [0,1]$. Let $\mathtt{I}_k^N, k = 1, \ldots, K^N$ be a partition of $[0,1]$ such that $x_k^N \in \mathtt{I}_k^N$ and $|\mathtt{I}_k^N| = (K^N)^{-1}$, $1 \le k \le K$. Individuals do not move between the locations. There are B_k^N individuals at location x_k, $B_1^N + \cdots + B_{K^N}^N = N$.
- In each location, individuals are categorized into three groups : S, I and R. We denote the numbers of indiv. in those compartments at time t and location x_k as $S_k^N(t)$, $I_k^N(t)$ and $R_k^N(t)$. For all $t \ge 0$, $S_k^N(t) + I_k^N(t) + R_k^N(t) = B_k^N$.
- We define the piecewise constant functions of x :

$$S^N(t,x) = \sum_{k=1}^{K} S_k^N(t) 1_{\mathtt{I}_k^N}(x), \; I^N(t,x) = \sum_{k=1}^{K} I_k^N(t) 1_{\mathtt{I}_k^N}(x),$$

$$R^N(t,x) = \sum_{k=1}^{K} R_k^N(t) 1_{\mathtt{I}_k^N}(x).$$

The spatial model

- We consider a population of fixed size N distributed in $K = K^N$ locations $x_1, \dots, x_K$ in $\mathcal{S} = [0,1]$. Let $\mathtt{I}_k^N, k = 1, \dots, K^N$ be a partition of $[0,1]$ such that $x_k^N \in \mathtt{I}_k^N$ and $|\mathtt{I}_k^N| = (K^N)^{-1}$, $1 \le k \le K$. Individuals do not move between the locations. There are B_k^N individuals at location x_k, $B_1^N + \cdots + B_{K^N}^N = N$.
- In each location, individuals are categorized into three groups : S, I and R. We denote the numbers of indiv. in those compartments at time t and location x_k as $S_k^N(t)$, $I_k^N(t)$ and $R_k^N(t)$. For all $t \ge 0$, $S_k^N(t) + I_k^N(t) + R_k^N(t) = B_k^N$.
- We define the piecewise constant functions of x :

$$S^N(t,x) = \sum_{k=1}^{K} S_k^N(t) 1_{\mathtt{I}_k^N}(x), \quad I^N(t,x) = \sum_{k=1}^{K} I_k^N(t) 1_{\mathtt{I}_k^N}(x),$$

$$R^N(t,x) = \sum_{k=1}^{K} R_k^N(t) 1_{\mathtt{I}_k^N}(x).$$

The spatial model

- We consider a population of fixed size N distributed in $K = K^N$ locations $x_1, \dots, x_K$ in $\mathcal{S} = [0,1]$. Let $\mathtt{I}_k^N, k = 1, \dots, K^N$ be a partition of $[0,1]$ such that $x_k^N \in \mathtt{I}_k^N$ and $|\mathtt{I}_k^N| = (K^N)^{-1}$, $1 \le k \le K$. Individuals do not move between the locations. There are B_k^N individuals at location x_k, $B_1^N + \cdots + B_{K^N}^N = N$.
- In each location, individuals are categorized into three groups : S, I and R. We denote the numbers of indiv. in those compartments at time t and location x_k as $S_k^N(t)$, $I_k^N(t)$ and $R_k^N(t)$. For all $t \ge 0$, $S_k^N(t) + I_k^N(t) + R_k^N(t) = B_k^N$.
- We define the piecewise constant functions of x :

$$S^N(t,x) = \sum_{k=1}^{K} S_k^N(t) 1_{\mathtt{I}_k^N}(x), \;\; I^N(t,x) = \sum_{k=1}^{K} I_k^N(t) 1_{\mathtt{I}_k^N}(x),$$

$$R^N(t,x) = \sum_{k=1}^{K} R_k^N(t) 1_{\mathtt{I}_k^N}(x)\,.$$

Varying infectivity

- In each location x_k, there are two types of individuals : those who are infected at time $t = 0$, there are $I_k^N(0)$ of those, who have been infected at times $\tau_{-j,k}^N < 0$, $1 \le j \le I_k^N(0)$, and those who are susceptible at time $t = 0$, and may get infected (i.e. jump from S to I) at times $\tau_{1,k}^N < \tau_{2,k}^N < \cdots$.
- Let $\{\lambda_{j,k},\ j \in \mathbb{Z}\backslash\{0\},\ 1 \le k \le K\}$ be an i.i.d. collection of random elements of $\mathrm{D}(\mathbb{R};\mathbb{R}_+)$, which satisfy $\lambda(t) = 0$ for $t < 0$. Let $\eta_{j,k} = \sup\{t > 0, \lambda_{j,k}(t) > 0\}$. At time $\tau_{j,k}^N + \eta_{j,k}$, the individual j jumps from the I to the R compartment.
- As a result, the total force of infection at time t and location x_k is

$$\mathfrak{F}_k^N(t) = \sum_{j=1}^{I_k^N(0)} \lambda_{-j,k}(t - \tau_{-j,k}) + \sum_{j=1}^{A_k^N(t)} \lambda_{j,k}(t - \tau_{j,k}),$$

where $A^N(t)$ is the number of initially susceptible who have been infected on the time interval $[0, t]$. We also define $\mathfrak{F}^N(t,x) = \sum_{k=1}^K \mathfrak{F}_k^N(t) 1_{\mathrm{I}_k^N}(x)$.

Varying infectivity

- In each location x_k, there are two types of individuals : those who are infected at time $t = 0$, there are $I_k^N(0)$ of those, who have been infected at times $\tau_{-j,k}^N < 0$, $1 \le j \le I_k^N(0)$, and those who are susceptible at time $t = 0$, and may get infected (i.e. jump from S to I) at times $\tau_{1,k}^N < \tau_{2,k}^N < \cdots$.
- Let $\{\lambda_{j,k},\ j \in \mathbb{Z}\backslash\{0\},\ 1 \le k \le K\}$ be an i.i.d. collection of random elements of $\mathrm{D}(\mathbb{R};\mathbb{R}_+)$, which satisfy $\lambda(t) = 0$ for $t < 0$. Let $\eta_{j,k} = \sup\{t > 0, \lambda_{j,k}(t) > 0\}$. At time $\tau_{j,k}^N + \eta_{j,k}$, the individual j jumps from the I to the R compartment.
- As a result, the total force of infection at time t and location x_k is

$$\mathfrak{F}_k^N(t) = \sum_{j=1}^{I_k^N(0)} \lambda_{-j,k}(t - \tau_{-j,k}) + \sum_{j=1}^{A_k^N(t)} \lambda_{j,k}(t - \tau_{j,k}),$$

where $A^N(t)$ is the number of initially susceptible who have been infected on the time interval $[0, t]$. We also define $\mathfrak{F}^N(t, x) = \sum_{k=1}^K \mathfrak{F}_k^N(t) 1_{\mathrm{I}_k^N}(x)$.

Varying infectivity

- In each location x_k, there are two types of individuals : those who are infected at time $t = 0$, there are $I_k^N(0)$ of those, who have been infected at times $\tau_{-j,k}^N < 0$, $1 \le j \le I_k^N(0)$, and those who are susceptible at time $t = 0$, and may get infected (i.e. jump from S to I) at times $\tau_{1,k}^N < \tau_{2,k}^N < \cdots$.
- Let $\{\lambda_{j,k},\ j \in \mathbb{Z}\backslash\{0\},\ 1 \le k \le K\}$ be an i.i.d. collection of random elements of $\mathrm{D}(\mathbb{R};\mathbb{R}_+)$, which satisfy $\lambda(t) = 0$ for $t < 0$. Let $\eta_{j,k} = \sup\{t > 0, \lambda_{j,k}(t) > 0\}$. At time $\tau_{j,k}^N + \eta_{j,k}$, the individual j jumps from the I to the R compartment.
- As a result, the total force of infection at time t and location x_k is

$$\mathfrak{F}_k^N(t) = \sum_{j=1}^{I_k^N(0)} \lambda_{-j,k}(t - \tau_{-j,k}) + \sum_{j=1}^{A_k^N(t)} \lambda_{j,k}(t - \tau_{j,k}),$$

where $A^N(t)$ is the number of initially susceptible who have been infected on the time interval $[0, t]$. We also define $\mathfrak{F}^N(t, x) = \sum_{k=1}^K \mathfrak{F}_k^N(t) 1_{\mathrm{I}_k^N}(x)$.

The propagation of the epidemic

- $\{P_k,\ 1 \le k \le K\}$ denoting mutually independent standard Poisson processes, we let

$$A_k^N(t) = P_k\left(\int_0^t \Upsilon_k^N(s)ds\right), \quad \text{where}$$
$$\Upsilon_k^N(t) = \frac{S_k^N(t)}{B_k^N}\frac{1}{K}\sum_{k'=1}^{K} \beta_{k,k'}^N \mathfrak{F}_{k'}^N(t)\,.$$

- Finally we define the number of infected indiv. at location x_k and time t, who have have been infected for a duration less than $\mathfrak{a}$:

$$\mathfrak{I}_k^N(t,\mathfrak{a}) = \sum_{j=1}^{I_k^N(0)} 1_{\tau^N_{-j,k}+\eta_{-j,k}>t} 1_{-\tau^N_{-j,k}\le(\mathfrak{a}-t)^+} + \sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t)} 1_{\tau^N_{j,k}+\eta_{j,k}>t}$$

Also $\mathfrak{I}^N(t,\mathfrak{a},x) = \sum_{k=1}^{K} \mathfrak{I}_k^N(t,\mathfrak{a}) 1_{\mathrm{I}_k^N}(x)$.

The propagation of the epidemic

- $\{P_k,\ 1 \le k \le K\}$ denoting mutually independent standard Poisson processes, we let

$$A_k^N(t) = P_k\left(\int_0^t \Upsilon_k^N(s)ds\right), \quad \text{where}$$
$$\Upsilon_k^N(t) = \frac{S_k^N(t)}{B_k^N}\frac{1}{K}\sum_{k'=1}^{K}\beta_{k,k'}^N\mathfrak{F}_{k'}^N(t)\,.$$

- Finally we define the number of infected indiv. at location x_k and time t, who have have been infected for a duration less than $\mathfrak{a}$:

$$\mathfrak{I}_k^N(t,\mathfrak{a}){=}\sum_{j=1}^{I_k^N(0)} 1_{\tau_{-j,k}^N+\eta_{-j,k}>t}1_{-\tau_{-j,k}^N\le(\mathfrak{a}-t)^+}+\sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t)} 1_{\tau_{j,k}^N+\eta_{j,k}>t}$$

Also $\mathfrak{I}^N(t,\mathfrak{a},x) = \sum_{k=1}^K \mathfrak{I}_k^N(t,\mathfrak{a})1_{\mathtt{I}_k^N}(x)$.

The stochastic finite population model

- We have

$$S^N(t,x) = S^N(0,x) - A^N(t,x),$$

$$I^N(t,x) = \sum_{j=1}^{I^N(0,x)} 1_{\tau^N_{-j,k}+\eta_{-j,k}>t} + \sum_{j=1}^{A^N(t,x)} 1_{\tau^N_{j,k}+\eta_{j,k}>t}, \text{ if } x \in \mathtt{I}_k$$

$$R^N(t,x) = R^N(0,x) + \sum_{j=1}^{I^N(0,x)} 1_{\tau^N_{-j,k}+\eta_{-j,k}\le t} + \sum_{j=1}^{A^N(t,x)} 1_{\tau^N_{j,k}+\eta_{j,k}\le t}\,.$$

- We let $\bar{X}^N(t,x) := \frac{X^N(t,x)}{B^N(x)}$, where $B^N(x) = \sum_{k=1}^{K} B^N_k 1_{\mathtt{I}^N_k}(x)$.
- We wish to take the liit as $N \to \infty$ in $(\bar{S}^N(t,x), \bar{\mathfrak{F}}^N(t,x), \bar{I}^N(t,x), \bar{R}^N(t,x))$ and also in $\bar{\mathfrak{I}}^N(t,\mathfrak{a},x)$.

The stochastic finite population model

- We have

$$S^N(t,x) = S^N(0,x) - A^N(t,x),$$

$$I^N(t,x) = \sum_{j=1}^{I^N(0,x)} 1_{\tau^N_{-j,k}+\eta_{-j,k}>t} + \sum_{j=1}^{A^N(t,x)} 1_{\tau^N_{j,k}+\eta_{j,k}>t}, \text{ if } x \in \mathtt{I}_k$$

$$R^N(t,x) = R^N(0,x) + \sum_{j=1}^{I^N(0,x)} 1_{\tau^N_{-j,k}+\eta_{-j,k}\le t} + \sum_{j=1}^{A^N(t,x)} 1_{\tau^N_{j,k}+\eta_{j,k}\le t}\,.$$

- We let $\bar{X}^N(t,x) := \frac{X^N(t,x)}{B^N(x)}$, where $B^N(x) = \sum_{k=1}^K B^N_k 1_{\mathtt{I}^N_k}(x)$.
- We wish to take the liit as $N \to \infty$ in $(\bar{S}^N(t,x), \bar{\mathfrak{F}}^N(t,x), \bar{I}^N(t,x), \bar{R}^N(t,x))$ and also in $\bar{\mathfrak{I}}^N(t,\mathfrak{a},x)$.

The stochastic finite population model

- We have

$$S^N(t,x) = S^N(0,x) - A^N(t,x),$$

$$I^N(t,x) = \sum_{j=1}^{I^N(0,x)} 1_{\tau^N_{-j,k}+\eta_{-j,k}>t} + \sum_{j=1}^{A^N(t,x)} 1_{\tau^N_{j,k}+\eta_{j,k}>t}, \text{ if } x \in \mathtt{I}_k$$

$$R^N(t,x) = R^N(0,x) + \sum_{j=1}^{I^N(0,x)} 1_{\tau^N_{-j,k}+\eta_{-j,k}\le t} + \sum_{j=1}^{A^N(t,x)} 1_{\tau^N_{j,k}+\eta_{j,k}\le t}\,.$$

- We let $\bar{X}^N(t,x) := \frac{X^N(t,x)}{B^N(x)}$, where $B^N(x) = \sum_{k=1}^K B_k^N 1_{\mathtt{I}_k^N}(x)$.
- We wish to take the liit as $N \to \infty$ in $(\bar{S}^N(t,x), \bar{\mathfrak{F}}^N(t,x), \bar{I}^N(t,x), \bar{R}^N(t,x))$ and also in $\bar{\mathfrak{I}}^N(t,\mathfrak{a},x)$.

Assumptions

- $K^N \to \infty$ as $N \to \infty$, $\inf_k B_k^N \to \infty$, and $\sup_{N,k,k'} \frac{B_k^N}{B_{k'}^N} < \infty$.
- We assume that the initial conditions converge towards appropriate limits, namely for all $\mathfrak{a} > 0$, as $N \to \infty$,

$$\|\bar{S}^N(0,\cdot)-\bar{S}(0,\cdot)\|_1+\|\bar{\mathfrak{I}}^N(0,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}(0,\mathfrak{a},\cdot)\|_1+\|\bar{R}^N(0,\cdot)-\bar{R}(0,\cdot)\|_1 \to 0$$

- $\sup_{N,k,k'} \beta_{k,k'}^N < \infty$ and $\beta^N(x,x') \to \beta(x,x')$ in $L^1([0,1]^2)$, where

$$\beta^N(x,x') = \sum_{k,k'=1}^{K} \frac{B_{k'}^N}{B_k^N} \beta_{k,k'}^N \mathbb{1}_{\mathtt{I}_k^N}(x) \mathbb{1}_{\mathtt{I}_{k'}^N}(x') .$$

- $\lambda(t) \le \lambda^*$ a.s. for all $t \ge 0$, λ has at most a given finite number of jumps and is uniformly continuous between its jumps. Notation : $\bar{\lambda}(t) = \mathbb{E}[\lambda(t)]$.

Assumptions

- $K^N \to \infty$ as $N \to \infty$, $\inf_k B_k^N \to \infty$, and $\sup_{N,k,k'} \frac{B_k^N}{B_{k'}^N} < \infty$.
- We assume that the initial conditions converge towards appropriate limits, namely for all $\mathfrak{a} > 0$, as $N \to \infty$,

$$\|\bar{S}^N(0,\cdot)-\bar{S}(0,\cdot)\|_1+\|\bar{\mathfrak{I}}^N(0,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}(0,\mathfrak{a},\cdot)\|_1+\|\bar{R}^N(0,\cdot)-\bar{R}(0,\cdot)\|_1 \to 0$$

- $\sup_{N,k,k'} \beta_{k,k'}^N < \infty$ and $\beta^N(x,x') \to \beta(x,x')$ in $L^1([0,1]^2)$, where

$$\beta^N(x,x') = \sum_{k,k'=1}^{K} \frac{B_{k'}^N}{B_k^N} \beta_{k,k'}^N \mathbb{1}_{\mathtt{I}_k^N}(x) \mathbb{1}_{\mathtt{I}_{k'}^N}(x').$$

- $\lambda(t) \le \lambda^*$ a.s. for all $t \ge 0$, λ has at most a given finite number of jumps and is uniformly continuous between its jumps. Notation : $\bar{\lambda}(t) = \mathbb{E}[\lambda(t)]$.

Assumptions

- $K^N \to \infty$ as $N \to \infty$, $\inf_k B_k^N \to \infty$, and $\sup_{N,k,k'} \frac{B_k^N}{B_{k'}^N} < \infty$.
- We assume that the initial conditions converge towards appropriate limits, namely for all $\mathfrak{a} > 0$, as $N \to \infty$,

$$\|\bar{S}^N(0,\cdot)-\bar{S}(0,\cdot)\|_1+\|\bar{\mathfrak{I}}^N(0,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}(0,\mathfrak{a},\cdot)\|_1+\|\bar{R}^N(0,\cdot)-\bar{R}(0,\cdot)\|_1 \to 0$$

- $\sup_{N,k,k'} \beta_{k,k'}^N < \infty$ and $\beta^N(x,x') \to \beta(x,x')$ in $L^1([0,1]^2)$, where

$$\beta^N(x,x') = \sum_{k,k'=1}^{K} \frac{B_{k'}^N}{B_k^N} \beta_{k,k'}^N 1_{\mathtt{I}_k^N}(x) 1_{\mathtt{I}_{k'}^N}(x') \,.$$

- $\lambda(t) \le \lambda^*$ a.s. for all $t \ge 0$, λ has at most a given finite number of jumps and is uniformly continuous between its jumps. Notation : $\bar{\lambda}(t) = \mathbb{E}[\lambda(t)]$.

Assumptions

- $K^N \to \infty$ as $N \to \infty$, $\inf_k B_k^N \to \infty$, and $\sup_{N,k,k'} \frac{B_k^N}{B_{k'}^N} < \infty$.
- We assume that the initial conditions converge towards appropriate limits, namely for all $\mathfrak{a} > 0$, as $N \to \infty$,

$$\|\bar{S}^N(0,\cdot)-\bar{S}(0,\cdot)\|_1+\|\bar{\mathfrak{I}}^N(0,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}(0,\mathfrak{a},\cdot)\|_1+\|\bar{R}^N(0,\cdot)-\bar{R}(0,\cdot)\|_1 \to 0$$

- $\sup_{N,k,k'} \beta_{k,k'}^N < \infty$ and $\beta^N(x,x') \to \beta(x,x')$ in $L^1([0,1]^2)$, where

$$\beta^N(x,x') = \sum_{k,k'=1}^{K} \frac{B_{k'}^N}{B_k^N} \beta_{k,k'}^N 1_{\mathtt{I}_k^N}(x) 1_{\mathtt{I}_{k'}^N}(x') \,.$$

- $\lambda(t) \le \lambda^*$ a.s. for all $t \ge 0$, λ has at most a given finite number of jumps and is uniformly continuous between its jumps. Notation : $\bar{\lambda}(t) = \mathbb{E}[\lambda(t)]$.

The limiting quantities

- Consider the system (with $F(t)=\mathbb{P}(\eta_{1,1}\le t)$, $F^c(t)=1-F(t)$)

$$\bar{S}(t,x)=\bar{S}(0,x)-\int_0^t \bar{\Upsilon}(s,x)ds,$$

$$\bar{\mathfrak{F}}(t,x)=\int_0^\infty \bar{\lambda}(\mathfrak{a}+t)\bar{\mathfrak{I}}(0,d\mathfrak{a},x)+\int_0^t \bar{\lambda}(t-s)\bar{\Upsilon}(s,x)ds,$$

$$\bar{\mathfrak{I}}(t,\mathfrak{a},x)=\int_0^{(\mathfrak{a}-t)^+}\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\bar{\mathfrak{I}}(0,d\mathfrak{a}',x)+\int_{(t-\mathfrak{a})^+}^t F^c(t-s)\bar{\Upsilon}(s,x)ds$$

$$\bar{R}(t,x)=\bar{R}(0,x)+\int_0^\infty\left(1-\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\right)\bar{\mathfrak{I}}(0,d\mathfrak{a}',x)+\int_0^t F(t-s)\bar{\Upsilon}(s,x)ds$$

- where $\bar{\Upsilon}(t,x)=\bar{S}(t,x)\int_0^1\beta(x,x')\bar{\mathfrak{F}}(t,x')dx'$.

The limiting quantities

- Consider the system (with $F(t) = \mathbb{P}(\eta_{1,1} \le t)$, $F^c(t) = 1 - F(t)$)

$$\bar{S}(t,x) = \bar{S}(0,x) - \int_0^t \bar{\Upsilon}(s,x)ds,$$

$$\bar{\mathfrak{F}}(t,x) = \int_0^\infty \bar{\lambda}(\mathfrak{a}+t)\bar{\mathfrak{I}}(0,d\mathfrak{a},x) + \int_0^t \bar{\lambda}(t-s)\bar{\Upsilon}(s,x)ds,$$

$$\bar{\mathfrak{I}}(t,\mathfrak{a},x) = \int_0^{(\mathfrak{a}-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\bar{\mathfrak{I}}(0,d\mathfrak{a}',x) + \int_{(t-\mathfrak{a})^+}^t F^c(t-s)\bar{\Upsilon}(s,x)ds$$

$$\bar{R}(t,x) = \bar{R}(0,x) + \int_0^\infty \left(1 - \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\right)\bar{\mathfrak{I}}(0,d\mathfrak{a}',x) + \int_0^t F(t-s)\bar{\Upsilon}(s,x)ds$$

- where $\bar{\Upsilon}(t,x) = \bar{S}(t,x)\int_0^1 \beta(x,x')\bar{\mathfrak{F}}(t,x')dx'$.

The convergence result

- We have

Theorem

$$\|\bar{S}^N(t,\cdot) - \bar{S}(t,\cdot)\|_1 \to 0,\ \|\bar{\mathfrak{F}}^N(t,\cdot) - \bar{\mathfrak{F}}(t,\cdot)\|_1 \to 0,\ \|\bar{R}^N(t,\cdot) - \bar{R}(t,\cdot)\|_1 \to$$
$$\|\bar{\mathfrak{I}}^N(t,\mathfrak{a},\cdot) - \bar{\mathfrak{I}}^N(t,\mathfrak{a},\cdot)\|_1 \to 0$$

in probability as $N \to \infty$, locall uniformly in t and $\mathfrak{a}$, where the limits are given by the unique solution to the above set of integral equations.

- In fact the pair $(\bar{S}, \bar{\mathfrak{F}})$ is the unique solution of an integral equation, and the other quantities are then expressed in terms of that solution.
- The result follows from the proof that as $N \to \infty$, locally uniformly in t,

$$\|\bar{S}^N(t,\cdot) - \bar{S}(t,\cdot)\|_1 \to 0, \quad \|\bar{\mathfrak{F}}^N(t,\cdot) - \bar{\mathfrak{F}}(t,\cdot)\|_1 \to 0$$

The convergence result

- We have

Theorem

$$\|\bar{S}^N(t,\cdot) - \bar{S}(t,\cdot)\|_1 \to 0,\ \|\bar{\mathfrak{F}}^N(t,\cdot) - \bar{\mathfrak{F}}(t,\cdot)\|_1 \to 0,\ \|\bar{R}^N(t,\cdot) - \bar{R}(t,\cdot)\|_1 \to$$
$$\|\bar{\mathfrak{I}}^N(t,\mathfrak{a},\cdot) - \bar{\mathfrak{I}}^N(t,\mathfrak{a},\cdot)\|_1 \to 0$$

in probability as $N \to \infty$, locall uniformly in t and $\mathfrak{a}$, where the limits are given by the unique solution to the above set of integral equations.

- In fact the pair $(\bar{S}, \bar{\mathfrak{F}})$ is the unique solution of an integral equation, and the other quantities are then expressed in terms of that solution.
- The result follows from the proof that as $N \to \infty$, locally uniformly in t,

$$\|\bar{S}^N(t,\cdot) - \bar{S}(t,\cdot)\|_1 \to 0, \quad \|\bar{\mathfrak{F}}^N(t,\cdot) - \bar{\mathfrak{F}}(t,\cdot)\|_1 \to 0$$

The convergence result

- We have

Theorem

$$\|\bar{S}^N(t,\cdot)-\bar{S}(t,\cdot)\|_1 \to 0,\ \|\bar{\mathfrak{F}}^N(t,\cdot)-\bar{\mathfrak{F}}(t,\cdot)\|_1 \to 0,\ \|\bar{R}^N(t,\cdot)-\bar{R}(t,\cdot)\|_1 \to$$
$$\|\bar{\mathfrak{I}}^N(t,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}^N(t,\mathfrak{a},\cdot)\|_1 \to 0$$

in probability as $N \to \infty$, locall uniformly in t and $\mathfrak{a}$, where the limits are given by the unique solution to the above set of integral equations.

- In fact the pair $(\bar{S},\bar{\mathfrak{F}})$ is the unique solution of an integral equation, and the other quantities are then expressed in terms of that solution.
- The result follows from the proof that as $N \to \infty$, locally uniformly in t,

$$\|\bar{S}^N(t,\cdot)-\bar{S}(t,\cdot)\|_1 \to 0, \quad \|\bar{\mathfrak{F}}^N(t,\cdot)-\bar{\mathfrak{F}}(t,\cdot)\|_1 \to 0$$

Main idea of the proof

- We have

$$[\bar S^N - \bar S](t,x) = [\bar S^N - \bar S](0,x) - \int_0^t [\bar\Upsilon^N - \bar\Upsilon](s,x)ds + M_A^N(t,x),$$

$$[\bar{\mathfrak{F}}^N - \bar{\mathfrak{F}}](t,x) = [\bar{\mathfrak{F}}_0^N - \bar{\mathfrak{F}}_0](t,x) + \int_0^t \bar\lambda(t-s)[\bar\Upsilon^N - \bar\Upsilon](s,x)ds + \mathcal{E}_{\mathfrak{F}}^N(t,x)\,.$$

- An important step is to show that M_A^N, $\bar{\mathfrak{F}}_0^N - \bar{\mathfrak{F}}_0$ and $\mathcal{E}_{\mathfrak{F}}^N$ tend to 0 in probability, in $L^1([0,1])$, locally uniformly in t.
- Moreover

$$[\bar\Upsilon^N - \bar\Upsilon](t,x) = \bar S^N(t,x)\int_0^1 \beta^N(x,x')\bar{\mathfrak{F}}^N(t,x')dx' - \bar S(t,x)\int_0^1 \beta(x,x')\bar{\mathfrak{F}}(t,x')dx'\,.$$

Thanks to a priori estimates on $\sup_{N,t\le T,x}\left\{\bar S^N(t,x) + \bar{\mathfrak{F}}^N(t,x)\right\}$, and $\sup_{t\le T,x}\left\{\bar S(t,x) + \bar{\mathfrak{F}}(t,x)\right\}$, we deduce the wished convergence by standard inequalities and Gronwall's Lemma.

Main idea of the proof

- We have

$$[\bar{S}^N - \bar{S}](t,x) = [\bar{S}^N - \bar{S}](0,x) - \int_0^t [\bar{\Upsilon}^N - \bar{\Upsilon}](s,x)ds + M_A^N(t,x),$$

$$[\bar{\mathfrak{F}}^N - \bar{\mathfrak{F}}](t,x) = [\bar{\mathfrak{F}}_0^N - \bar{\mathfrak{F}}_0](t,x) + \int_0^t \bar{\lambda}(t-s)[\bar{\Upsilon}^N - \bar{\Upsilon}](s,x)ds + \mathcal{E}_{\mathfrak{F}}^N(t,x)\,.$$

- An important step is to show that M_A^N, $\bar{\mathfrak{F}}_0^N - \bar{\mathfrak{F}}_0$ and $\mathcal{E}_{\mathfrak{F}}^N$ tend to 0 in probability, in $L^1([0,1])$, locally uniformly in t.
- Moreover

$$[\bar{\Upsilon}^N - \bar{\Upsilon}](t,x) = \bar{S}^N(t,x)\int_0^1 \beta^N(x,x')\bar{\mathfrak{F}}^N(t,x')dx' - \bar{S}(t,x)\int_0^1 \beta(x,x')\bar{\mathfrak{F}}(t,x')dx'\,.$$

Thanks to a priori estimates on $\sup_{N,t\le T,x}\left\{\bar{S}^N(t,x)+\bar{\mathfrak{F}}^N(t,x)\right\}$, and $\sup_{t\le T,x}\left\{\bar{S}(t,x)+\bar{\mathfrak{F}}(t,x)\right\}$, we deduce the wished convergence by standard inequalities and Gronwall's Lemma.

Main idea of the proof

- We have

$$[\bar{S}^N - \bar{S}](t,x) = [\bar{S}^N - \bar{S}](0,x) - \int_0^t [\bar{\Upsilon}^N - \bar{\Upsilon}](s,x)ds + M_A^N(t,x),$$

$$[\bar{\mathfrak{F}}^N - \bar{\mathfrak{F}}](t,x) = [\bar{\mathfrak{F}}_0^N - \bar{\mathfrak{F}}_0](t,x) + \int_0^t \bar{\lambda}(t-s)[\bar{\Upsilon}^N - \bar{\Upsilon}](s,x)ds + \mathcal{E}_{\mathfrak{F}}^N(t,x)\,.$$

- An important step is to show that M_A^N, $\bar{\mathfrak{F}}_0^N - \bar{\mathfrak{F}}_0$ and $\mathcal{E}_{\mathfrak{F}}^N$ tend to 0 in probability, in $L^1([0,1])$, locally uniformly in t.
- Moreover

$$[\bar{\Upsilon}^N - \bar{\Upsilon}](t,x) = \bar{S}^N(t,x)\int_0^1 \beta^N(x,x')\bar{\mathfrak{F}}^N(t,x')dx' - \bar{S}(t,x)\int_0^1 \beta(x,x')\bar{\mathfrak{F}}(t,x')dx'\,.$$

Thanks to a priori estimates on $\sup_{N,t\le T,x}\left\{\bar{S}^N(t,x)+\bar{\mathfrak{F}}^N(t,x)\right\}$, and $\sup_{t\le T,x}\left\{\bar{S}(t,x)+\bar{\mathfrak{F}}(t,x)\right\}$, we deduce the wished convergence by standard inequalities and Gronwall's Lemma.

PDE models 1

We suppose that F is absolutely continuous, with the density $f(t)$. We define the associated hazard function : $\mu(\mathfrak{a}) = \frac{f(\mathfrak{a})}{F^c(\mathfrak{a})}$. We have

Proposition

Assume that for each $x \in [0,1]$, $\mathfrak{a} \mapsto \bar{\mathfrak{I}}(0,\mathfrak{a},x)$ is absolutely continuous, and let $\bar{\mathfrak{i}}(0,\mathfrak{a},x) = \bar{\mathfrak{I}}_{\mathfrak{a}}(0,\mathfrak{a},x)$. Then for all $t, \mathfrak{a} > 0$, a.e. $x \in [0,1]$, $\mathfrak{a} \mapsto \bar{\mathfrak{I}}(t,\mathfrak{a},x)$ is absolutely continuous, and $\bar{\mathfrak{i}}(t,\mathfrak{a},x) := \bar{\mathfrak{I}}_{\mathfrak{a}}(t,\mathfrak{a},x)$ satisfies

$$\frac{\partial \bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial t} + \frac{\partial \bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial \mathfrak{a}} = -\mu(\mathfrak{a})\bar{\mathfrak{i}}(t,\mathfrak{a},x),$$

with the initial condition $\bar{\mathfrak{i}}(0,\mathfrak{a},x) = \bar{\mathfrak{I}}_{\mathfrak{a}}(0,\mathfrak{a},x)$ and the boundary condition

$$\bar{\mathfrak{i}}(t,0,x) = \bar{S}(t,x)\int_0^1 \beta(x,x')\int_0^{t+\bar{\mathfrak{a}}} \frac{\bar{\lambda}(\mathfrak{a}')}{\frac{F^c(\mathfrak{a}')}{F^c(\mathfrak{a}'-t)}}\bar{\mathfrak{i}}(t,\mathfrak{a}',x')d\mathfrak{a}'dx' \,.$$

Note that $\mathfrak{I}(0,\infty,x) = \mathfrak{I}(0,\bar{\mathfrak{a}},x)$.

PDE models 2

Moreover, we have $\frac{\partial \bar{S}(t,x)}{\partial t} = -\bar{\mathfrak{i}}(t,0,x)$, the above PDE has a unique solution given as follows :

$$\bar{\mathfrak{i}}(t,\mathfrak{a},x) = 1_{\mathfrak{a}\geq t}\frac{F^c(\mathfrak{a})}{F^c(\mathfrak{a}-t)}\bar{\mathfrak{i}}(0,\mathfrak{a}-t,x) + 1_{\mathfrak{a}<t}F^c(\mathfrak{a})\bar{\mathfrak{i}}(t-\mathfrak{a},0,x),$$

where the boundary function is the unique solution of the integral equation

$$\bar{\mathfrak{i}}(t,0,x) = \left(\bar{S}(0,x) - \int_0^t \bar{\mathfrak{i}}(s,0,x)ds\right) \times \int_0^1 \beta(x,x')\left(\int_0^\infty \bar{\lambda}(\mathfrak{a}+t)\bar{\mathfrak{i}}(0,\mathfrak{a},x')d\mathfrak{a} + \int_0^t \bar{\lambda}(t-s)\bar{\mathfrak{i}}(s,0,x')ds\right)dx' .$$

References 1

- G. Pang, É.P., Functional limit theorems for non-Markovian epidemic models, *Annals of Applied Probability* **32**, pp. 1615–1665, 2022.
- R. Forien, G. Pang, É.P., Estimating the state of the Covid-19 epidemic in France using a non-Markovian model, *Royal Society Open Science* **8** : 202327, 2021.
- R. Forien, G. Pang, É.P., Epidemic models with varying infectivity, *SIAM J. Applied Math.* **81**, pp. 1893-1930, 2021.
- R. Forien, G. Pang, É.P., Multipatch multigroup epidemic model with varying infectivity, *Probability, Uncertainty and Quantitative Risk*, to appear, Dec 2022.

References 2

- O. Diekmann, Thresholds and travelling waves for the geographical spread of infection, *J. of Mathematical Biology* **6**, pp. 58–73, 1979.
- D.G. Kendall, Mathematical models of the spread of infection *Mathematics and computer science in Biology and Medicine*, pp. 213–225, 1965.
- W.O. Kermack and A.G. McKendrick, A Contribution to the Mathematical Theory of Epidemics, *Proc. Royal Soc. London, Series A*, **115**, 700–721, 1927.
- H.R. Thieme A model for the spatial spread of an epidemic, *J. of Mathematical Biology* **4**, pp. 337–351, 1977.

THANK YOU FOR YOUR ATTENTION !