Long range dependence in evolving networks

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Outline

- Brief motivation and structure
- Seed detection in dynamic networks
- Change point detection
- Co-evolving networks

In collaboration with Sayan Banerjee, Jain Carmichael, and Zoe Huang
- For each problem area I will describe the motivation of the area in words

  \[\rightarrow\text{Important}\]

- I will describe our specific contributions

  \[\rightarrow\text{Potentially Irrelevant}\]
Seed detection in evolving networks

Our motivation in words:

- Dynamic network started with a single node ("patient zero") or seed graph at time zero.
- Observe network when it is of large size e.g. \( n = 10^6 \) with no temporal information only network topology (adjacency matrix).
- Have a fixed budget say \( K = 30 \).

**Goal:** Output 30 vertices such that with high prob. seed is in the output.
Change Point Detection

U.S. life expectancy

Life expectancy is a calculation of how long a baby born in a given year is expected to live on average.

- Female
- Male
- Total

1918 pandemic
World War II
2020 pandemic

Source: NCHS, National Vital Statistics System: Mortality

Source: Associated Press
- Suppose you have temporal network data.
  - Ex: Adjacency matrix at all or sub-sample of time points
  - Er: Time series observations at each node etc
- Suppose network experiences a shock at some point.
- Can we detect this change point from observations?
- Changes in structural properties of the system?
Network co-evolution: our motivation

- Most real world networks support some particular purpose (e.g. diffusion of information on Twitter)

- Co-evolution: Networks influence individuals; individuals influence networks
- Till date majority of models deal either with
  - Dynamics on a fixed network (e.g. random walk or epidemics on a fixed network).
  - Dynamics "of" a network: Network itself changing in some fashion.
- However kept these two disciplines largely "separate".
  Most network practitioners believe co-evolving networks is the next frontier.

Goal: Understand conjectured phase transitions in one tractable model
Seed detection in evolving networks

Our motivation in words

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- Observe network when it is of large size e.g. $n=10^6$ with no temporal information only network topology (adjacency matrix).

- Have a fixed budget say $K=30$.

- **GOAL**: Output 30 vertices such that with high prob. seed is in the output.
Probabilistic foundations

- Network model: Fix attachment function $f$. Start with single seed vertex.
- At each stage new vertex enters system. Connects to one pre-existing vertex.
- Probability connecting to a vertex $u$ in the system proportional to $f(\text{degree}(u))$.
- $T_n$ = network of size $n$.
Example: $f = 1$ (Random recursive tree)
SIMULATION \( (n = 3000 ?) \)
Example \( f(k) = k \) Preferential attachment
Simulation \( (n = 5000) \)
Setup:

- $G$: space of equivalence classes (up to isomorphisms) of finite unlabelled graphs.
- For finite labelled graph $G$: $G^\circ$ for the isomorphism class of $G$ in $G$.
- Root finding algorithm: Fix $K \geq 1$ and a mapping $H_K$ on $G$ that takes an input finite unlabelled graph $g \in G$ and outputs a subset of $K$ vertices from $g$.

Root finding algorithms

Let $\{G_n : n \geq 0\}$ be a sequence of growing random networks. Fix $0 < \varepsilon < 1$ and $K \geq 1$. A mapping $H_K$ is called a budget $K$ root finding algorithm with error tolerance $\varepsilon$ for the sequence of networks if,

$$\liminf_{n \to \infty} \mathbb{P}(1 \in H_K(G_n^\circ)) \geq 1 - \varepsilon.$$

**Question:** can we choose $K$ independent of $n$? Dependence on $\varepsilon$?
Class of seed detection algorithms

- Centrality based measures
- For each vertex obtain some measure of centrality
  so collection of numbers \( \{ \phi(u) : u = \text{vertex in } \mathbb{Z}_n \} \)
- Example:
  - Degree centrality: \( \phi(u) = \text{degree of } u \)
  - Eigen-vector centrality
  - Centroid or Jordan centrality
ALGORITHM

- Suppose budget \( = K \)
- Output the "top" \( K \) vertices (Could be smallest or largest depending on the measure)
- Say that above has error tolerance \( \varepsilon \) if

\[
\lim_{n \to \infty} P(\text{seed} \in \text{outputed set of } \mathbb{Z}_n) \geq 1 - 3^{-\varepsilon}
\]
Fundamental questions

- For given error tolerance \( \varepsilon \) (e.g. \( \varepsilon = 0.01 \)), can we select \( K \) independent of \( n = \text{size of network} \)?

- How does \( K = K(\varepsilon) \) depend on \( \varepsilon \)?

\[
\frac{1}{\varepsilon} \quad ? \quad \frac{1}{\varepsilon \ 100} \quad ? \quad \frac{1}{\varepsilon \ 10000}
\]
Persistence

Fix $K \geq 1$ and a network centrality measure $\Psi$. For a family of network models \{${G_n : n \geq 1}$\} say that this sequence is $(\Psi, K)$ **persistent** if $\exists \ n^* < \infty$ a.s. such that for all $n \geq n^*$ the optimal $K$ vertices $(v_1, \psi(G_n^0), v_2, \psi(G_n^2), \ldots, v_K, \psi(G_n^K))$ remain the same and further the relative ordering amongst these $K$ optimal vertices remains the same.

Example: If degree centrality was persistent this implies, the *identity* of the maximal degree vertex becomes fixed within finite time and no other vertex can overtake the degree of this vertex after this time.

**Such phenomenon once again a hallmark of long range dependence.**
Jordan or centroid centrality*

\[ \phi(w) = \text{size of the largest subtree of a child of } v \]

* Only works for trees. First analyzed by Bubeck - Devroye - Lugosi.
Suppose for some $\overline{C}_f > 0$, $\beta \geq 0$, $f$ satisfies $f_* \leq f(i) \leq \overline{C}_f \cdot i + \beta$ for all $i \geq 1$. Then there exist positive constants $C_1$, $C_2$ such that for any error tolerance $0 < \varepsilon < 1$, the budget requirement satisfies,

$$K_\Psi(\varepsilon) \leq \frac{C_1}{\varepsilon(2\overline{C}_f + \beta)/f_*} \exp\left(\sqrt{C_2 \log 1/\varepsilon}\right).$$

If further the attachment function $f$ is in fact bounded with $f(i) \leq f^*$ for all $i \geq 1$ then one has for any error tolerance $0 < \varepsilon < 1$,

$$K_\Psi(\varepsilon) \leq \frac{C_1}{\varepsilon f^*/f_*} \exp\left(\sqrt{C_2 \log 1/\varepsilon}\right).$$
Centroid centrality necessary bounds

- If \( \exists \; C_f > 0 \) and \( \beta \geq 0 \) such that \( f(i) \geq C_f \cdot i + \beta \) for all \( i \geq 1 \) then \( \exists \) a positive constant \( C'_1 \) such that for any error tolerance \( 0 < \varepsilon < 1 \),

\[
K_\Psi(\varepsilon) \geq \frac{C'_1}{\varepsilon (2C_f + \beta) / f(1)}.
\]

- For general \( f \) one has for any error tolerance \( 0 < \varepsilon < 1 \),

\[
K_\Psi(\varepsilon) \geq \frac{C'_1}{\varepsilon f_*/f(1)}.
\]
Special cases

- **Uniform attachment:** \( f(k) = 1 \)
  \[
  \frac{C_1'}{\varepsilon} \leq K_\psi(\varepsilon) \leq \frac{C_1}{\varepsilon} \exp\left(\sqrt{C_2 \log \frac{1}{\varepsilon}}\right)
  \]

- **Pure Preferential attachment:** \( f(k) = k \)
  \[
  \frac{C_1'}{\varepsilon^2} \leq K_\psi(\varepsilon) \leq \frac{C_1}{\varepsilon^2} \exp\left(\sqrt{C_2 \log \frac{1}{\varepsilon}}\right).
  \]

- **Affine preferential attachment:** \( f(k) = k + \beta \)
  \[
  \frac{C_1'}{\varepsilon^{2+\beta}} \leq K_\psi(\varepsilon) \leq \frac{C_1}{\varepsilon^{2+\beta}} \exp\left(\sqrt{C_2 \log \frac{1}{\varepsilon}}\right).
  \]

- **Sublinear preferential attachment:**
  \[
  \frac{C_1'}{\varepsilon} \leq K_\psi(\varepsilon) \leq \frac{C_1}{\varepsilon^2} \exp\left(\sqrt{C_2 \log \frac{1}{\varepsilon}}\right).
  \]
Disadvantages of Centroid centrality

Essentially need quite precise information of entire network

**Natural question:** How do more local measures like degree centrality perform? Does there exist a *persistent hub* (i.e. maximal degree vertex fixates within finite time)?

**Fake popularity:** Suppose $i$-th vertex enters the system with $m_i$ edges that it attaches to the current existing system (again with popularity of vertices measured via some function $f$). How quickly does $m_i \uparrow \infty$ to break persistence phenomenon?
Assumptions and notation

- \( f_\ast := \inf_{i \geq 0} f(i) > 0 \); further at most linear growth \( f(i) \leq C_i(i) \).
- \( \sum_{i=0}^{\infty} \frac{1}{f(i)} = \infty \).
- \( \Phi_k(x) = \int_0^x \frac{1}{f^k(z)} \, dz \).
- \( \mathcal{K}(t) = \Phi_2 \circ \Phi_1^{-1}(t), \ t \geq 0 \).
- \( d_{\max}(n) := \max_{0 \leq k \leq n} d_k(n) \).
- **Index of the maximal degree:**
  \[ \mathcal{I}_n^* := \inf\{0 \leq i \leq n : d_i(n) \geq d_j(n) \text{ for all } j \leq n\} . \]
Persistence of hubs

**Banerjee + B (2020)**

Under a few technical assumptions on $f$ and $f$ is increasing:

- Suppose $\Phi_2(\infty) < \infty$ (e.g. $f(k) = k^\alpha$ for $\alpha \in (1/2, 1]$) and that $\limsup_{n \to \infty} \frac{\Phi_1(m_n)}{\log s_n} \leq \frac{1}{8C_f}$.

Then a persistent hub emerges almost surely in the random graph sequence.

*Do not need increasing assumption for trees.*
Lack of persistence

Banerjee + B(2020)

Assume $\Phi_2(\infty) = \infty$ (e.g. $f(k) = k^\alpha$ for $\alpha \in (0, 1/2)$) and (we are working in the tree case) and $f(k) \to \infty$ as $k \to \infty$. Then index of maximal degree satisfies:

$$\frac{\log I_n^*}{\mathcal{K} \left( \frac{1}{\lambda^*} \log n \right)} \xrightarrow{P} \frac{\lambda^{*2}}{2}, \text{ as } n \to \infty.$$  

where $\lambda^*$ is the Malthusian rate of growth of the continuous time embedding.

For $f(k) = k^\alpha$ for $\alpha \in (0, 1/2)$,

$$\frac{\log I_n^*}{(\log n)^{\frac{1-2\alpha}{1-\alpha}}} \xrightarrow{P} \frac{(\lambda^*)^{\frac{1}{1-\alpha}}}{2}, \text{ as } n \to \infty.$$  

*Inspired by Morters and Dietrich who proved similar results for a different evolving network model.*
Change Point Detection

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- Suppose you have temporal network data.
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- Suppose network experiences a shock at some point.
- Can we detect this change point from observations?
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Recall: Probabilistic foundations

- **Network model**: Fix attachment function $f$. Start with single seed $v_0$.
- At each stage a new vertex enters the system. Connects to one pre-existing vertex.
- Probability connecting to a vertex $v$ in the system proportional to $f(\text{degree}(v))$.
- $\mathcal{G}_n = \text{network of size } n$
Example \( f(k) = k + d \) Preferential attachment
Known results for $f(k) = k + 2$

- $N_k(n) = \# \text{ of vertices of degree } k \text{ in } \mathbb{Z}_n$

  \[
  \frac{N_k(n)}{n} \rightarrow P_k
  \]

- $P_k \sim \frac{C}{k^{d+3}}$
  
  Degree exponent $= d + 3$

- $\max \text{- degree} = M_n \sim n^{d+2}$
Example of standard change point model

- Fix $\theta \in (0, 1)$.

- For $t \in [1, n\theta]$, network uses attachment function
  \[ f(k) = k + \alpha \]

- For $t \in [n\theta + 1, n]$, network uses
  \[ g(k) = k + \beta \]
Any guesses on the degree exponent?

Recall under no change

\[ f(n) = n^2 + 2 \]

degree exponent \( = n^2 + 3 \)

\[ g(n) = n^2 + \beta \]

degree exponent \( = \beta^2 + 3 \)
Irrespective of how small $\delta$ is (e.g. $\delta = 0.01$ or $\delta = 0.000000001$), the initializer function wins!  

Always
standard change point model

- Fix $\tau \in (0, 1)$.
- For $t \in [1, n\tau]$, network uses attachment function $f(k) = \text{general function}$
- For $t \in [n\tau+1, n]$, network uses $g(k) = \text{general function}$
Fix \( t \in [0, 1] \). Let \( N_k(nt) = \# \) vertices of degree \( k \) in \( T_{nt} \).

**Theorem [Banerjee, B, Carmichael]**

Under conditions on \( f \) and \( g \) \( f \) explicit probability mass functions \( \xi (p_k(t))_{k \geq 1} : t \in [0, 1] \) such that

\[
\sup_{t \in [0, 1]} \left| \frac{N_k(nt)}{nt} - p_k(t) \right| \xrightarrow{t \to 0} 0
\]
Theorem [Banerjee, B., Carmichael]

Under above technical conditions on \( f \) & \( g \), irrespective of how small \( x \) is \( f \) always wins!

- So if degree exponent with \( f \) and no change point is \( \delta \) \& it is the model with change point.
Change point estimator: For each \( t \in (0, 1) \) compare the degree distribution \( \left( \frac{N_k^t(Nt)}{nt} \right)_{k \geq 1} \) with the degree distribution when network is of size \( \frac{N}{\ln n} \) (recall change point at \( \frac{n}{\ln n} \)).

And become alarmed the first time there seems to be a big change in degree distn.
Change point estimator

\[ \frac{h_n}{n} \quad \gamma \quad t \quad d_n(t) \]

Nonparametric change point estimator

Fix any two sequences \( h_n \to \infty, b_n \to \infty: \frac{\log h_n}{\log n} \to 0, \frac{\log b_n}{\log n} \to 0. \) Define

\[ \hat{\tau}_n = \inf \left\{ t \geq \frac{1}{h_n} : \sum_{k=0}^{\infty} 2^{-k} \left| \frac{D_n(k, T_{[nt]}^\theta)}{nt} - \frac{D_n(k, T_{[n/nh_n]}^\theta)}{n/h_n} \right| > \frac{1}{b_n} \right\}. \]

Then \( \hat{\tau}_n \xrightarrow{p} \gamma. \)
Lots of open problems

**Simulations**

Figure: $n = 2 \times 10^6$, $\gamma = 0.5$, $f_0(i) = i + 2$, $f_1(i) = \sqrt{i + 2}$, $h_n = \log \log n$, $b_n = n^{1/\log \log n}$

$$d_n(m) := \sum_{k=0}^{\infty} 2^{-k} \left| \frac{D_n(k, T_{m}^\theta)}{m} - \frac{D_n(k, T_{[n/h_n]}^\theta)}{n/h_n} \right|, \quad \frac{n}{\log \log n} < m \leq n.$$
The big bang model: What if the change happened very early in the system?

![Big Bang: Getty images](image)

Fix functions $f_0, f_1 : \{0, 1, 2, \ldots \} \to \mathbb{R}_+$ and $\gamma \in (0, 1)$. Let $\theta = (f_0, f_1, \gamma)$.

**Model**

- **Time** $1 \leq m \leq n^\gamma$ Vertices perform attachment with probability proportional to $f_0(out - deg)$.
- **Time** $n^\gamma < m \leq n$ Vertices perform attachment with probability probability proportional to $f_1(out - deg)$.
Result 1

- Here change point at $n^\gamma$ (e.g. $\sqrt{n}$).
- Here

$$\frac{N_n(k)}{n} \xrightarrow{p} p_k^1$$

namely the degree distribution of the model run purely with attachment function $f_1$.

So what changes?

1. **Uniform $\leadsto$ Linear**: $f_0 \equiv 1$ whilst $f_1(k) = k + 1 + \alpha$ for fixed $\alpha > 0$. Then for $\omega_n \uparrow \infty$,

$$\frac{\frac{1-\gamma}{2+\alpha} \log n}{\omega_n} \ll M_n(1) \ll n^{\frac{1-\gamma}{2+\alpha}} (\log n)^2.$$

2. **Linear $\leadsto$ Uniform**: $f_0(k) = k + 1 + \alpha$ whilst $f_1(\cdot) \equiv 1$.

$$\frac{n^{\frac{\gamma}{2+\alpha}} \log n}{\omega_n} \ll M_n(1) \ll n^{\frac{\gamma}{2+\alpha}} (\log n)^2.$$

3. **Linear $\leadsto$ Linear**: $f_0(k) = k + 1 + \alpha$ whilst $f_1(k) = k + 1 + \beta$ where $\alpha \neq \beta$. Then $M_n(1)/n^{\eta(\alpha, \beta)}$ is tight where

$$\eta(\alpha, \beta) := \frac{\gamma(2 + \beta) + (1 - \gamma)(2 + \alpha)}{(2 + \alpha)(2 + \beta)}.$$

(5)
Motivation

- Most real-world networks support some particular purpose (e.g., diffusion of information on Twitter)
- Co-evolution: Network influences individuals and vice versa
Motivation 2: More sophisticated models for PA

Motivations

Despite PA being heavily used, number of limitations

Assumes global knowledge of network. Each new vertex needs complete knowledge of network.

In principle attractiveness should not depend ONLY on degree but potentially on "attenuated" neighborhood features.

Example: Page rank score attachment scheme.
Defn [Page rank Scores] Fix "damping factor" $c$. For directed graph $G = (V,E)$, page rank score $(\Pi_\infty: v \in V)$ is the stationary distribution of a random walk that at each step

- with prob $c$ does usual random walk using outgoing edges
- with prob $1-c$ jumps to a randomly selected vertex uniformly at random

[Diagram]

Mr. McKenty
Thus \((Tvw)\) satisfies linear system of equations

\[
Tvw = \frac{1-c}{n} + c \sum_{u \in N^{-}(v)} \frac{d^+(u)}{\alpha(d^+(u))}
\]

Special case

Directed tree, directions to the root

Can check for \(v \neq \text{root}\)

\[
Tvw = \frac{1-c}{n} \sum_{k=0}^{8} c^k L_k(v)
\]
\( L_k(v) = \# \) of individuals at level \( k \) below \( v \)

Thus, if one does Preferential attachment using Page rank scores, then one does attachment using more global attractiveness function.
Motivation which might be contradictory to the previous motivations: Local exploration based attachment schemes

- Might want network evolution schemes where vertices decide to attach to a previous vertex after exploring the network "web-surfing" for some time.
Co-evolutionary network model \((P)\)

\[ T_1 = \begin{array}{ccc} & v_0 & \vdots \\ v_0 & & v_1 \end{array} \]

1. Having constructed \(T_n\)
2. At time \(n+1\) a new vertex \(v_{n+1}\) enters system
3. Selects vertex \(v_n\) at random in \(T_n\)
4. Selects \# of "exploration steps to root" variable \(Z_{n+1} \sim P\)
5. Goes up that many steps and attaches, stopping at root if need be.

\[ P = \text{pmf} = \{ P_0, P_1, \ldots \} \quad P(Z=i) = P_i, \quad i \geq 0 \]
Example

\[ C(2) = v_1, \quad Z_2 = 0 \]
\[ C(3) = v_2, \quad Z_3 = 4 \]
\[ C(4) = v_2, \quad Z_4 = 1 \]
\[ C(5) = v_0, \quad Z_5 = 2 \]
\[ C(6) = v_2, \quad Z_1 = 1 \]

\[ v_1 \]
\[ v_2 \]
\[ v_0 \]

Diagram:

\[ v_4 \rightarrow v_3 \rightarrow v_2 \]
\[ v_4 \rightarrow v_1 \rightarrow v_0 \]
\[ v_4 \rightarrow v_2 \rightarrow v_1 \]
Special cases

1. $p_0 = 1 \rightarrow$ Random recursive tree (Uniform Attachment)

2. $p_0 = p, \; p_1 = 1 - p \rightarrow$ Preferential attachment
   $f(k) = k + \frac{(1 - 2p)}{p}$

3. $p_0 = p, \; p_1 = p(1-p), \; p_2 = p(1-p)^2, \ldots$
   "Page rank model"

Theorem [Chebolu + Melsted 200x]

3. Page rank attachment scheme with $1 - c = p$
Theorems [Chebolu+ Melsted] Phase transition!

- ♂ $\mathbb{P} \leq \frac{1}{2}$
  \[ E(\text{degree of root}) = \tilde{\Theta}(c_n) \]

- ♂ $\mathbb{P} > \frac{1}{2}$
  \[ E(\text{degree of root}) = \tilde{\Theta}(n^{4p_{2l}}) \]

* $\tilde{\Theta} \Leftrightarrow O(n \log^{k_r}(c_n)) \quad r \in \mathbb{Z}$
Theorem(s) [Banerjee, SB, Huang]

Let $Z \sim P$

1. Assume $E(Z) < \infty$. Then the sequence of trees $\{T_n\}_{n \geq 1}$ converge in the local weak convergence sense to a limiting infinite $\sin$-tree $\sin$: tree with single infinite path to $\infty$.
For example, for every fixed $k \geq 0$

$$N_k(n) = \# \text{ of vertices with } k \text{ children}$$

Then

$$\frac{N_k(n)}{n} \xrightarrow{a.s.} p_k \rightarrow \sum_{k \geq 0} p_k = \text{PMF}$$

$\implies E(Z) \leq 1$

$\implies E(Z) > 1$

Intuition for mass escaping to $\infty$: Condensation
Let $\mathbb{E}[\xi \geq 1] = \sum_{\xi=1}^{\infty} P \cdot f(s) = pgf = \sum_{k=0}^{\infty} P_k s^k$

Let $S_n = S_0 + \sum_{k=1}^{n} (\xi_i - 1) = \text{random walk started at } S_0$

Assumptions

1. $p_0 \in (0, 1)$, $p_0 + p_1 < 1$ [else if $p_0 + p_1 = 1 \iff \text{Preferential attachment regime}]

2. $f(s)$ is analytic at $s = 1$
Assumptions \Rightarrow by work of [Daley 69] with a few more technical assumptions*

\[ \Rightarrow \text{if we let } T_0 = \inf \{ n \geq 1 : S_n = 0 \} \text{ then} \]

\[ \begin{align*}
\text{If } E(\xi) \neq 1 \text{ then} \\
&P. (n < T_0 < \infty) \approx e^{-n \log R} \\
f&or R = R(p) > 1 \\
&\text{i.e. } S_0 = 1
\end{align*} \]

\[ \begin{align*}
\text{If } E(\xi) = 1 \text{ then } R = 1
\end{align*} \]

* Aperiodicity + analyticity of pgf at \( s = 1 \).
Results [non-root degree]

Fix $k > 0$ and consider $D_{2k}(n) = \deg \varphi_k$ at time $n$

Then $\forall \delta > 0$

$$\frac{D_{2k}(n)}{n^{\frac{1}{k}} - \delta} \to \infty$$

$$\frac{D_{2k}(n)}{n^{1/\varphi}(\log n)^{1+\delta}} \to 0$$
If $E(2) \leq 1$

Let $D$ be a random variable with limiting degree distribution.

$$\lim_{k \to \infty} \frac{\log P(D \geq k)}{\log k} = -R$$

Intuitively, $P(D \geq k) \approx \frac{C}{k^R}$

(For $E(2) > 1$, get upper and lower bounds for degree exponent.)
CONDENSATION

- Assume $E(\varepsilon) > 1$
- Few more technical Conditions

Then $D_{290}(n) \xrightarrow{n\to\infty} \text{limit random variable} > 0$

E.g. Random surfer model $p < \frac{1}{2}$ above is true!
No phase transition in Height

\[ x_0 = \inf_{SE(0,1)} \frac{f(r)}{S \log(1/s)} \]

\[ \frac{Hn}{\log n} \xrightarrow{P} x_0 \]
Connection between Random Walks + Trees

- What is the first step in studying such models?
- [Chebolu + Melded idea for Page rank driven model]
- Fix a vertex $u \neq$ root
- Fix a vertex $t+1$ at time $\geq u$
- What is $P(t+1 \text{ attaches to } u \mid \text{info till time } t)$?
Let $L_i(t; u) = \# of vertices at distance $i$ below vertex $u$ at time $t$. 

$L_0(t; u) = 1$
$L_1(t; u) = 3$
$L_2(t; u) = 4$
$L_3(t; u) = 3$
\[ P(t+1 \rightarrow u \mid \text{information till time } t) = \frac{L_0(t; u)}{t} p_0 + \frac{L_1(t; u)}{t} p_1 + \frac{L_2(t; u)}{t} p_2 + \ldots \]

Check: This leads to evolution equation for \( \{ L_k(t; u) : k \geq 0 \} \) as
\[
P(C_t(t+1) \sim L_{R}(t+1) \sim | L_{R}(t))
\]
\[
= \frac{p_0}{t} L_{R-1}(t) + \frac{p_1}{t} L_{R}(t) + \ldots
\]
\[
= \left[ A \cdot \frac{L_{R}(t)}{t} \right]_R
\]

where

\[
A = \begin{pmatrix}
0 & 0 & 0 & \ldots \\
p_0 & p_1 & p_2 & \ldots \\
0 & p_0 & p_1 & \ldots \\
0 & \ldots & p_0 & \ldots
\end{pmatrix}
\]

\rightarrow \text{Easier to do things in continuous time. Continuous time version of what is happening below a vertex $u \in R$.}
Let $T$ denote the space of rooted, directed, labelled trees. Let $T^*(\cdot)$ be the continuous time process of growing trees started with $T^*(0) = \{v_0\}$, where $v_0$ is the root of the tree. The vertices in $T^*(\cdot)$ are labelled $v_0, v_1, v_2, \ldots$ in order of appearance. $T^*(\cdot)$ is generated by the following procedure:

Each vertex reproduces at rate 1. When vertex $v$ reproduces, a random variable $Z$ following the law $F$ is sampled independently.

- If $Z \leq \text{dist}(v_0, v)$, then a new vertex $\tilde{v}$ is attached to the unique vertex $u$ lying on the path between $v$ and $v_0$ that satisfies $\text{dist}(v, u) = Z$ via a directed edge from $\tilde{v}$ to $u$.

- If $Z > \text{dist}(v_0, v)$, nothing occurs.

Thus the probability of a new vertex being born to a current vertex is uniform distance.
Two at first disjoint objects

Random walk: \( S_n = S_0 + \sum_{l=1}^{n} (2l-1) \)

\( T_R = \inf \{ n \geq 0 : S_n = 0 \mid S_0 = k^2 \} \)

Branching process: Let \( L_k(t) = \# \text{ of vertices in generation} \)

\( k \) in the tree process described on previous page

Lemma: \( E(L_k(t)) = \sum_{i=0}^{\infty} \frac{t^i}{i!} P(T_k=i) \)
Thank you for your attention!

Any questions?