LDPs for the spectral radius of inhomogeneous ERRGs

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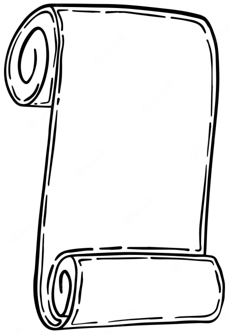
Graph Limits, Nonparametric Models, and Estimation,
26–30 September 2022, Simons Institute, Berkeley, USA.

OUTLINE:

(1) LDP for the inhomogeneous ERRG.

(2) LDP for the spectral radius of the adjacency matrix.

(3) LDP for the spectral radius of the Laplacian matrix.



Parts (2) and (3) are joint work with:

A. Chakrabarty, R.S. Hazra, M. Markering, M. Sfragara

§ LDP FOR THE INHOMOGENEOUS ERRG

1. Let

$$\mathcal{W} = \Big\{h\colon\ [0,1]^2 \to [0,1]\colon\ h(x,y) = h(y,x)\ \forall\, x,y \in [0,1]\Big\}$$

denote the set of graphons. Let $\mathcal{M}$ denote the set of Lebesgue measure-preserving bijective maps $\phi\colon\ [0,1] \mapsto [0,1]$. The cut-distance on $\mathcal{W}$ is defined by

$$d_\square(h_1,h_2) = \sup_{S,T\subset[0,1]} \left| \int_{S\times T} \mathrm{d}x\,\mathrm{d}y \left[h_1(x,y) - h_2(x,y)\right]\right|,$$

and the cut-metric by

$$\delta_\square(h_1,h_2) = \inf_{\phi\in\mathcal{M}} d_\square(h_1,h_2^\phi),$$

where $h_2^\phi(x,y) = h_2(\phi(x),\phi(y))$.

The cut-metric defines an equivalence relation $\sim$ on $\mathcal{W}$ by declaring $h_1 \sim h_2$ if and only if $\delta_\Box(h_1, h_2) = 0$, and leads to the quotient space $\widetilde{\mathcal{W}} = \mathcal{W}/_\sim$. For $h \in \mathcal{W}$, write $\tilde{h}$ to denote the equivalence class of h in $\widetilde{\mathcal{W}}$. The pair $(\widetilde{\mathcal{W}}, \delta_\Box)$ is a compact metric space.

2. Let $r \in \mathcal{W}$ be a measurable reference graphon satisfying

$$\log r,\ \log(1-r) \in L^1.$$

Fix $N \in \mathbb{N}$ and consider the random graph G_N with vertex set $[N] = \{1, \ldots, N\}$ where the pair of vertices $i, j \in [N]$, $i \neq j$, is connected by an edge with probability $r_N(\frac{i}{N}, \frac{j}{N})$, independently of other pairs of vertices, with r_N an $N \times N$ block graphon satisfying

$$\lim_{N\to\infty} \|r_N - r\|_{L^1} = 0,$$

and similarly for $\log r_N$ and $\log(1 - r_N)$.

3. Write $\mathbb{P}_N$ to denote the law of G_N. Use the same symbol for the law on $\mathcal{W}$ induced by the map that associates with the graph G_N its empirical graphon h^{G_N}, defined by

$$h^{G_N}(x,y) = \begin{cases} 1, & \text{if } \lceil Nx \rceil \sim \lceil Ny \rceil, \\ 0, & \text{otherwise}, \end{cases} \qquad (x,y) \in [0,1]^2.$$

Write $\widetilde{\mathbb{P}}_N$ to denote the law of $\widetilde{h}^{G_N}$.

The following LDP is an extension of the celebrated LDP for homogeneous ERRG derived by Chatterjee, Varadhan 2011

THEOREM 1 Dhara & Sen 2021, Markering 2022

The sequence $(\widetilde{\mathbb{P}}_N)_{N\in\mathbb{N}}$ *satisfies the LDP on* $(\widetilde{\mathcal{W}}, \delta_\square)$ *with rate* $\binom{N}{2}$ *and with rate function* $J_r\colon \widetilde{\mathcal{W}} \to \mathbb{R}$ *given by*

$$J_r(\widetilde{h}) = \inf_{\phi\in\mathcal{M}} I_r(h^\phi),$$

where h *is any representative of* $\widetilde{h}$ *and*

$$I_r(h) = \int_{[0,1]^2} \mathrm{d}x\,\mathrm{d}y\,\mathcal{R}\Big(h(x,y)\mid r(x,y)\Big), \quad h \in \mathcal{W},$$

with

$$\mathcal{R}(a\mid b) = a\log\tfrac{a}{b} + (1-a)\log\tfrac{1-a}{1-b}.$$

§ GRAPHON OPERATORS

With $h \in \mathcal{W}$ we associate a graphon operator acting on $L^2([0,1])$, defined as the linear integral operator

$$(\mathcal{T}_h u)(x) = \int_{[0,1]} \mathrm{d}y \, h(x,y) u(y), \qquad x \in [0,1].$$

The operator norm of $\mathcal{T}_h$ is defined as

$$\|\mathcal{T}_h\| = \sup_{\substack{u \in L^2([0,1]) \\ \|u\|_2 = 1}} \|\mathcal{T}_h u\|_2,$$

Given a graphon $h \in \mathcal{W}$, we have $\|\mathcal{T}_h\| \leq \|h\|_2$. Therefore, any graphon sequence converging in the L^2-norm is also converging in the operator norm.

SOME BASIC FACTS:

For any $h \in \mathcal{W}$:

(i) $\mathcal{T}_h$ is self-adjoint, bounded and continuous.

(ii) The maximal eigenvalue and associated eigenfunction of $\mathcal{T}_h$ are strictly positive.

(iii) The maximal eigenvalue of $\mathcal{T}_h$ equals the operator norm $\|\mathcal{T}_h\|$.

§ LDP FOR THE ADJACENCY MATRIX

Chakarabarty, Hazra, den Hollander, Sfragara 2021

Let λ_N be the maximal eigenvalue of the adjacency matrix A_N of G_N. Write $\mathbb{P}_N^*$ to denote the law of λ_N/N.

THEOREM 2

The sequence $(\mathbb{P}_N^*)_{N\in\mathbb{N}}$ *satisfies the LDP on* $\mathbb{R}$ *with rate* $\binom{N}{2}$ *and with rate function*

$$\psi_r(\beta) = \inf_{\substack{\widetilde{h}\in\widetilde{\mathcal{W}} \\ \|\mathcal{T}_{\widetilde{h}}\|=\beta}} J_r(\widetilde{h}) = \inf_{\substack{h\in\mathcal{W} \\ \|\mathcal{T}_h\|=\beta}} I_r(h), \qquad \beta\in\mathbb{R}.$$

Note that $\lambda_N/N = \|\mathcal{T}_{h^{G_N}}\|$, because $\|\mathcal{T}_{\tilde{h}}\| = \|\mathcal{T}_{h^\phi}\|$ for all $\phi \in \mathcal{M}$. Since $\widetilde{h} \mapsto \|\mathcal{T}_{\tilde{h}}\|$ is bounded and continuous on $\widetilde{\mathcal{W}}$, the claim follows from Theorem 1 via the contraction principle.

Put

$$C_r = \|\mathcal{T}_r\|.$$

When $\beta = C_r$, the graphon h that minimizes $I_r(h)$ such that $\|\mathcal{T}_h\| = C_r$ equals r almost everywhere, for which $I_r(r) = 0$ and no large deviation occurs.

THEOREM 3

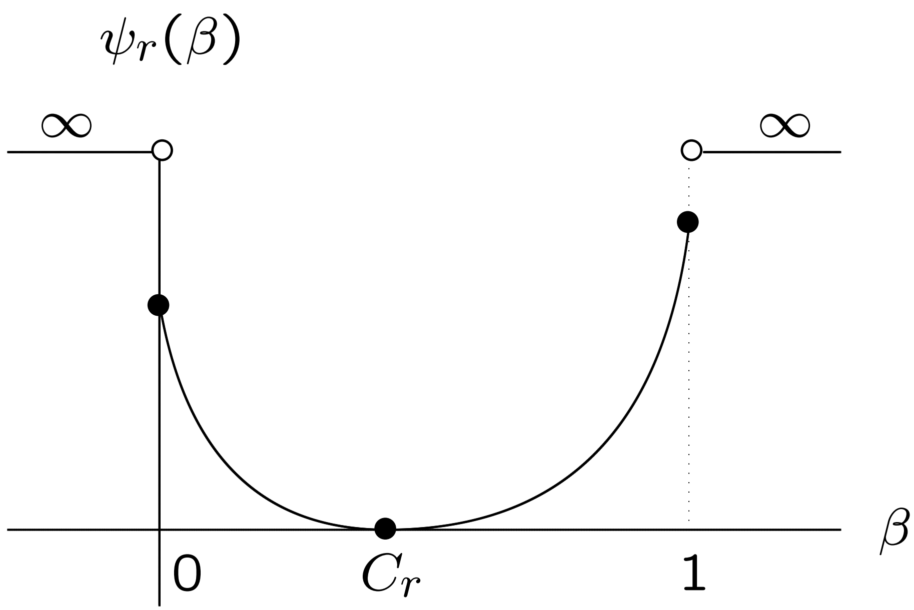
(i) ψ_r is continuous and unimodal on $[0,1]$, with a unique zero at C_r.

(ii) ψ_r is strictly decreasing on $[0, C_r]$ and strictly increasing on $[C_r, 1]$.

(iii) For every $\beta \in [0,1]$, the set of minimisers in the first variational formula for $\psi_r(\beta)$ is non-empty and compact in $\widetilde{\mathcal{W}}$.

The LDP says that

$$\mathbb{P}^*_N(\lambda_N/N \approx \beta) \approx \exp\left[-\binom{N}{2}\psi_r(\beta)\right].$$



Graph of $\beta \mapsto \psi_r(\beta)$.

§ SCALING NEAR THE BOTTOM

If the reference graphon r is of rank 1, i.e.,

$$r(x,y) = \nu(x)\,\nu(y), \quad (x,y) \in [0,1]^2,$$

for some $\nu\colon [0,1] \to [0,1]$ that is bounded away from 0 and 1, then we are able to say more.

Define

$$m_k = \int_{[0,1]} \nu^k, \qquad k \in \mathbb{N}.$$

It is easily shown that $C_r = m_2$ when r is rank 1. Put

$$B_r = \int_{[0,1]^2} r^3(1-r) = m_3^2 - m_4^2.$$

THEOREM 4

(i) If r *is rank* 1*, then*

$$\psi_r(\beta) \sim K_r\,(\beta - C_r)^2, \qquad \beta \to C_r,$$

with

$$K_r = \frac{(C_r)^2}{2B_r} = \frac{m_2^2}{2(m_3^2 - m_4^2)}.$$

(ii) If r *is rank* 1*, then any* $h_\beta \in \mathcal{W}$ *that minimises the second variational formula for the rate function* ψ_r *satisfies*

$$\lim_{\beta \to C_r} (\beta - C_r)^{-1} \| h_\beta - r - (\beta - C_r)\Delta \|_2 = 0,$$

with

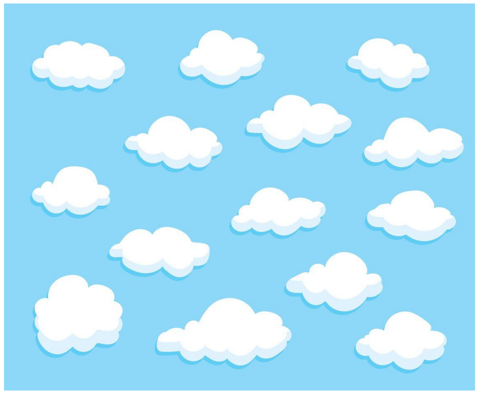
$$\Delta(x,y) = \frac{C_r}{B_r}\, r(x,y)^2 [1 - r(x,y)], \qquad (x,y) \in [0,1]^2.$$

§ REMARKS

1. It remains open whether ψ_r is convex or not. We do not expect ψ_r to be analytic, because bifurcations may occur in the set of minimisers of ψ_r as β is varied.

2. Note that the scaling corrections are not rank 1. It remains open to determine what happens near C_r when r is not of rank 1. Higher rank can be included, but at the cost of more technicalities.

3. The inverse curvature $1/K_r$ equals the variance in the central limit theorem derived in Chakrabarty, Chakraborty, Hazra 2020. This is in line with the standard folklore of large deviation theory.



§ LDP FOR THE LAPLACIAN MATRIX

Hazra, den Hollander, Markering, in progress

The Laplacian matrix is defined as

$$L_N = D_N - A_N,$$

where D_N is the diagonal matrix whose elements are the degrees of the vertices.

The LDP for the spectral radius of L_N poses new challenges, because L_N is a more delicate object than A_N.

The upward and the downward large deviations live on different scales, and the norm of the Laplacian operator lacks certain continuities properties that hold for the norm of the adjacency operator.

GRAPHON OPERATORS:

Define the degree function for a graphon $h \in \mathcal{W}$ as

$$d_h(x) = \int_{[0,1]} \mathrm{d}y\, h(x,y), \qquad x \in [0,1].$$

The degree operator is defined by

$$(\mathcal{D}_h u)(x) = d_h(x) u(x), \qquad x \in [0,1].$$

The Laplacian operator is defined by

$$(\mathcal{L}_h u)(x) = \int_{[0,1]} \mathrm{d}y\, h(x,y)[u(x) - u(y)], \qquad x \in [0,1].$$

Note that

$$\mathcal{L}_h = \mathcal{D}_h - \mathcal{T}_h.$$

The operator $\mathcal{L}_h$ is not as well-behaved as the operator $\mathcal{T}_h$. In fact, even if a sequence of graphons $(h_n)_{n\in\mathbb{N}}$ converges in the cut-distance to a graphon h, then the eigenvalues and eigenvectors of $\mathcal{L}_{h_n}$ may not converge to those of $\mathcal{L}_h$.

If h^{G_N} is the empirical graphon of the graph G_N, then $N\|\mathcal{T}_h\|$ is the largest eigenvalue of G_N, and $N\|\mathcal{D}_h\|$ is the maximum degree of G_N. In fact, for any graphon h the operator norm of $\mathcal{D}_h$ equals

$$\|\mathcal{D}_h\| = \|d_h\|_\infty,$$

Let

$$\|\mathcal{L}_h\| = \sup_{\substack{u\in L^2([0,1])\\ \|u\|_2=1}} \|\mathcal{L}_h u\|_2$$

be the operator norm of $\mathcal{L}_h$, which equals the maximal eigenvalue of $\mathcal{L}_h$.

SOME BASIC FACTS:

(i) $\mathcal{L}_h$ is bounded and $h \mapsto \|\mathcal{L}_h\|$ is lower semi-continuous in the cut-distance.

(ii) $\|L_N\|/N = \|\mathcal{L}_{h^G}\|$ for all N.

> Note that $h \mapsto \|\mathcal{L}_h\|$ is not continuous in the cut-distance. For example, consider the sequence of graphons $(h_N)_{N\in\mathbb{N}}$ such that h_N is the empirical graphon of the N-star graph. Then $h_N \downarrow 0$ as $N \to \infty$ in the cut-distance, but $\|\mathcal{L}_{h_N}\| = 1$ for all $N \in \mathbb{N}$.

Define

$$\widehat{C}_r = \|\mathcal{L}_r\|.$$

§ DOWNWARD LDP

Let $\widehat{\lambda}_N$ be the maximal eigenvalue of the Laplacian matrix L_N of G_N. Write $\mathbb{P}_N^*$ to denote the law of $\widehat{\lambda}_N/N$.

THEOREM 5

Suppose that $\lim_{N\to\infty} \|r_N - r\|_\infty = 0$*. Then the sequence* $(\mathbb{P}_N^*)_{N\in\mathbb{N}}$ *satisfies the downward LDP on* $\mathbb{R}$ *with rate* $\binom{N}{2}$ *and with rate function*

$$\psi_r^-(\beta) = \inf_{\substack{\widetilde{h}\in\widetilde{\mathcal{W}} \\ \|\mathcal{L}_{\widetilde{h}}\|\leq\beta}} J_r(\widetilde{h}) = \inf_{\substack{h\in\mathcal{W} \\ \|\mathcal{L}_h\|\leq\beta}} I_r(h).$$

The second equality uses that $\|\mathcal{L}_r\| = \|\mathcal{L}_{r^\phi}\|$ for any $\phi \in \mathcal{M}$. Since the maximal eigenvalue is invariant under relabelling of the vertices, we need not worry about the equivalence classes.

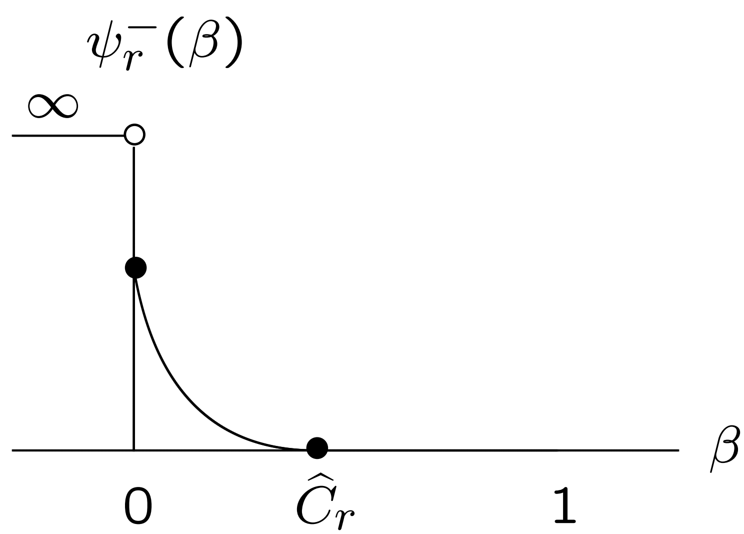
The downward LDP says that

$$\mathbb{P}^*_N(\widehat{\lambda}_N/N \leq \beta) \approx \exp\left[-\binom{N}{2}\psi_r^-(\beta)\right].$$

THEOREM 6

(i) ψ_r^- *is right-continuous and strictly decreasing on* $[0,\widehat{C}_r]$*, with* $\psi_r^-(0) > 0$ *and* $\psi_r^-(\widehat{C}_r) = 0$*.*

(ii) For every $\beta \in [0,\widehat{C}_r]$*, the set of minimisers in the first variational formula for* $\psi_r^-(\beta)$ *is non-empty and compact.*



Graph of $\beta \mapsto \psi_r^-(\beta)$.

Under the additional assumption that

$$(x, y) \mapsto r(x, y) \text{ is positive definite,}$$

which is weaker than rank 1, we are able to say more. Put

$$J_r(x, \beta) = \int_{[0,1]} \mathrm{d}y \, \mathcal{R}\Big(\widehat{r}(x, y) \mid r(x, y)\Big), \qquad x \in [0, 1],$$

where

$$\widehat{r}(x, y) = \frac{\mathrm{e}^{\theta(x)} r(x, y)}{\mathrm{e}^{\theta(x)} r(x, y) + [1 - r(x, y)]}$$

with $\theta(x)$, $x \in [0, 1]$, chosen such that

$$\int_{[0,1]} \mathrm{d}y \, \widehat{r}(x, y) = \beta, \qquad x \in [0, 1].$$

Recall that $d_r(x) = \int_{[0,1]} \mathrm{d}y \, r(x, y)$. It is easily shown that $\widehat{C}_r = \|d_r\|_\infty$ when r is positive definite.

THEOREM 7

If r *is positive-definite, then*

$$\psi_r^-(\beta) = 2 \int_{S_r(\beta)} \mathrm{d}x \, J_r(x,\beta), \qquad \beta \in [0, \widehat{C}_r],$$

where

$$S_r(\beta) = \{x \in [0,1] \colon \, d_r(x) \geq \beta\} \, .$$

THEOREM 8

If r *is positive-definite, then*

$$\psi_r^-(\beta) \asymp (\widehat{C}_r - \beta)^2 \, |S_r(\beta)|, \qquad \beta \uparrow \widehat{C}_r.$$

Note that $\lim_{\beta \uparrow \widehat{C}_r} |S_r(\beta)| = 0$ because $d_r(x) \leq \widehat{C}_r$ for all $x \in [0,1]$. Hence the decay is faster than quadratic. Several scenarios are possible, depending on how d_r scales near its maximum.

§ UPWARD LDP

THEOREM 9

The sequence $(\mathbb{P}^*_N)_{N\in\mathbb{N}}$ *satisfies the upward LDP on* $\mathbb{R}$ *with rate* N *and with rate function*

$$\psi_r^+(\beta) = \inf_{x\in[0,1]} J_r(x,\beta).$$

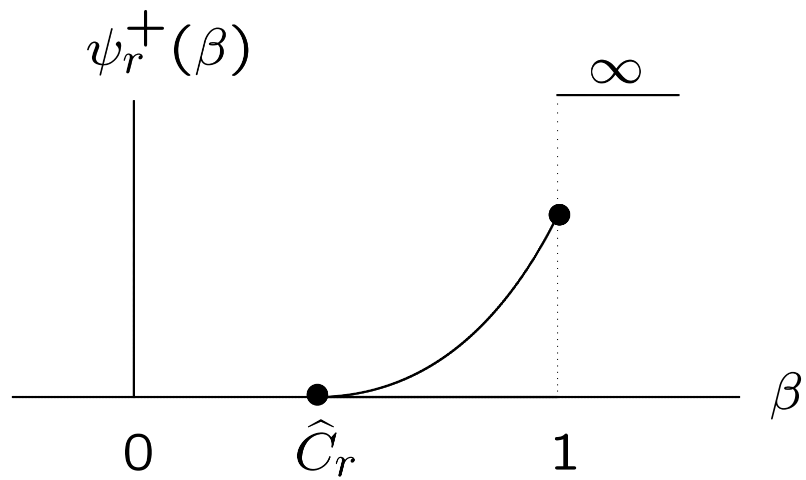
THEOREM 10

If r *is positive-definite, then*

$$\widehat{\psi}_r(\beta) \asymp (\beta - \widehat{C}_r)^2, \qquad \beta \downarrow \widehat{C}_r.$$

The upward LDP says that

$$\mathbb{P}^*_N(\widehat{\lambda}_N/N \geq \beta) \approx \exp\left[-N\psi_r^+(\beta)\right].$$



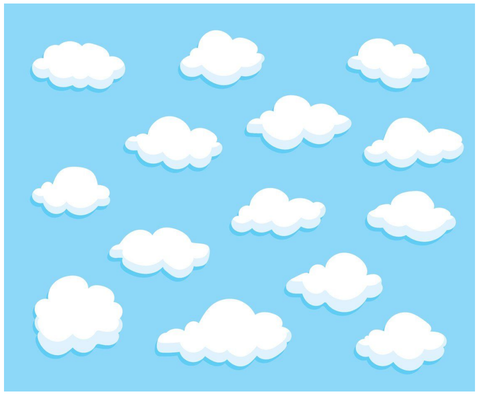
Graph of $\beta \mapsto \psi_r^+(\beta)$.

§ REMARKS

1. The fine properties of ψ_r^- and ψ_r^+ remain elusive.

2. It remains open to determine what happens near $\widehat{C}_r$ when r is not positive-definite.

3. No central limit theorem is known for $\widehat{\lambda}_N/N$.



Challenge: What can be said about the second largest eigenvalue of A_N and L_N?


Thank you!