Common graphs with large chromatic number

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Joint work with D. Král’ and F. Wei.
Ramsey multiplicities / Common graphs

\[ R(3) = 6 \Rightarrow \text{any RED/BLUE col of } E(K_n) \text{ contains } \approx \frac{n^3}{20} \text{ mono-\Delta} \]
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Goodman’s bound: \[ 6\binom{n}{3} - 6 \left[ \frac{n}{2} \cdot \left( \frac{n-1}{2} \right)^2 \right] \geq \frac{n(n-1)(n-5)}{4} \text{ mono-\Delta} \]
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In general: \( \forall \) R/B col of \( E(K_n) \) has \( \geq \frac{|Aut(H)|}{R(H)|V(H)|} \cdot n|V(H)| \) mono-\( H \)
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\[ H \text{ Sidorenko} \equiv t(H, W) \geq t(H, p)^{e(H)} = p^{e(H)} \quad \text{where } p = \int W \]
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Conjecture (Sidorenko '91): \( H \text{ is bipartite } \implies H \text{ is Sidorenko?} \)
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compare \( p^{e(H)} \) to \( \min t(H, W) \) only over \( W \) that are close to constant \( p \)

(close in subgraph counts \equiv \text{cut-distance})
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Fox-Wei ('17): $H$ locally Sidorenko $\iff$ the girth of $H$ is even
Common graphs $H$ with $\chi(H) > 3$

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Our main result: $\forall k : \exists$ common graph $H_k$ with $\chi(H_k) = k$
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$\forall F : \text{girth}(F) \geq 42 \ \exists N_0 \ s.t. \ \forall m \geq n \geq N_0 \ \text{and} \ \ell \approx 2n \rightarrow \text{common}$
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$\forall F : \text{girth}(F) \geq 42 \ \exists N_0 \text{ s.t. } \forall m \geq n \geq N_0$ and $\ell \approx 2n \rightarrow$ common

Proof idea: if $(W, 1 - W)$ is FAR from the constant $1/2$, then
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either find sparse spot $S$ in (say) red $\rightarrow$ induct on $S$ in blue
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When $W$ (and also $1 - W$) is CLOSE to the constant $1/2$, then
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Our main result: $\forall k : \exists$ common graph $H_k$ with $\chi(H_k) = k$

$\forall F : \text{girth}(F) \geq 42 \exists N_0 \text{ s.t. } \forall m \geq n \geq N_0 \text{ and } \ell \approx 2n \rightarrow \text{common}$

Proof idea: if $(W, 1 - W)$ is FAR from the constant $1/2$, then

either find sparse spot $S$ in (say) red $\longrightarrow$ induct on $S$ in blue

or $\forall \nu \in [0, 1]$ is in $\delta_0 > 0$ of $H$’s & use boost of $\#K_{m,n}$ by non-random

When $W$ (and also $1 - W$) is CLOSE to the constant $1/2$, then

- Girth of $H$ is four so Fox-Wei local-Sidorenko does apply…
Common graphs $H$ with $\chi(H) > 3$

Jagger-Šťovíček-Thomason ('96): Is there any? What about the 5-wheel?
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When $W$ (and also $1 - W$) is CLOSE to the constant $1/2$, then
- Girth of $H$ is four so Fox-Wei local-Sidorenko does apply. . .
- . . .however, we need control on $m, n, \ell \rightarrow$ restrict only to $H_k$
Conclusion (1/2)

Our main result: \( \forall k : \exists \) common graph \( H_k \) with \( \chi(H_k) = k \)
Conclusion (1/2)

Our main result: \( \forall k : \exists \) common graph \( H_k \) with \( \chi(H_k) = k \)

\[
\begin{array}{c}
K_{m,n} \hspace{2cm} (\ell - 1)\text{-edge path} \hspace{2cm} girth \geq 42 \\
\hline
F \hspace{2cm} 12\text{-edge path} \hspace{2cm} C_4
\end{array}
\]

\( H \) is \( \ell \)-common: \( \forall (W_1, \ldots, W_\ell) \text{ s.t. } \sum W_i = 1: \sum t(H, W_i) \geq \ell^{1-e(H)} \)
Conclusion (1/2)

Our main result: $\forall k : \exists$ common graph $H_k$ with $\chi(H_k) = k$

$H$ is $\ell$-common: $\forall (W_1, \ldots, W_\ell)$ s.t. $\sum W_i = 1$: $\sum t(H, W_i) \geq \ell^{1-e(H)}$

- $H$ not $\ell$-common $\implies$ $H$ not $(\ell + 1)$-common
- if $\chi(H) \geq 3$, then $\exists \ell_0$ s.t. $H$ is not $\ell_0$-common
Conclusion (1/2)

Our main result: \( \forall k : \exists \text{ common graph } H_k \text{ with } \chi(H_k) = k \)

\[ K_{m,n} \rightarrow (\ell - 1)\text{-edge path} \rightarrow F \rightarrow 12\text{-edge path} \rightarrow C_4 \]

girth \( \geq 42 \)

\( H \) is \( \ell \)-common: \( \forall (W_1, \ldots, W_\ell) \text{ s.t. } \sum W_i = 1: \sum t(H, W_i) \geq \ell^{1 - e(H)} \)

- \( H \) not \( \ell \)-common \( \implies H \) not \((\ell + 1)\)-common
- \( H \) not \( \ell \)-common \( \implies H \) not \( \ell_0 \)-common
- \( H \) is Sidorenko \( \implies H \) is \( \ell \)-common for every \( \ell \)
Conclusion (1/2)

Our main result: \( \forall k : \exists \) common graph \( H_k \) with \( \chi(H_k) = k \)

\[ K_{m,n} \quad \text{(\( \ell - 1 \)-edge path)} \quad F \quad \text{(girth } \geq 42) \quad C_4 \quad \text{(12-edge path)} \]

\( H \) is \( \ell \)-common: \( \forall (W_1, \ldots, W_\ell) \text{ s.t. } \sum W_i = 1: \sum t(H, W_i) \geq \ell^{1-e(H)} \)

\( \includegraphics[width=0.5\textwidth]{graphic.png} \)

- \( H \) not \( \ell \)-common \( \implies H \) not \( (\ell + 1) \)-common
- if \( \chi(H) \geq 3 \), then \( \exists \ell_0 \) s.t. \( H \) is not \( \ell_0 \)-common

Thm: \( H \) is Sidorenko \( \iff H \) is \( \ell \)-common for every \( \ell \)

Král’, Noel, Norin, V., Wei (’22) (independently on Sidorenko conj.)
Conclusion (1/2)

Our main result: \( \forall k : \exists \) common graph \( H_k \) with \( \chi(H_k) = k \)

\[ K_{m,n} \quad \text{--} \quad \text{girth } \geq 42 \quad \text{--} \quad F \quad \text{--} \quad C_4 \]

\( (\ell - 1) \)-edge path

12-edge path

\( H \) is \( \ell \)-common: \( \forall (W_1, \ldots, W_\ell) \) s.t. \( \sum W_i = 1: \sum t(H, W_i) \geq \ell^{1-e(H)} \)

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Theorem (KNNVW ’22): \( \forall \ell : \exists \) \( \ell \)-common \( H_\ell \) with \( \chi(H_\ell) = 3 \)
Conclusion (1/2)

Our main result: \( \forall k : \exists \text{ common graph } H_k \text{ with } \chi(H_k) = k \)

\[ K_{m,n} \quad \cdots \quad (\ell - 1)\text{-edge path} \quad F \quad \text{girth } \geq 42 \quad C_4 \quad \text{12-edge path} \]

\( H \) is \( \ell \)-common: \( \forall (W_1, \ldots, W_\ell) \text{ s.t. } \sum W_i = 1: \sum t(H, W_i) \geq \ell^{1 - e(H)} \)

\( \blacklozenge \) \( H \) not \( \ell \)-common \( \implies \ H \) not \( (\ell + 1) \)-common

\( \blacklozenge \) if \( \chi(H) \geq 3, \text{ then } \exists \ell_0 \text{ s.t. } H \) is not \( \ell_0 \)-common

Thm: \( H \) is Sidorenko \( \iff \) \( H \) is \( \ell \)-common for every \( \ell \)

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Our main result\(^+\): \( \forall k, \ell : \exists \ell \text{-common } H_{k,\ell} \text{ with } \chi(H_{k,\ell}) = k \)
Conclusion (1/2)

Our main result: \( \forall k : \exists \) common graph \( H_k \) with \( \chi(H_k) = k \)

\[ \begin{array}{c}
\begin{array}{c}
K_{m,n} \\
\vdots \\
C_4
\end{array}
\end{array} \]

\( girth \geq 42 \)

\( (\ell - 1)\)-edge path

\( F \)

\( 12\)-edge path

\[ \begin{array}{c}
\begin{array}{c}
H \text{ is } \ell\text{-common: } \forall (W_1, \ldots, W_\ell) \text{ s.t. } \sum W_i = 1: \sum t(H, W_i) \geq \ell^{1 - e(H)} \\
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\[ \begin{array}{c}
\begin{array}{c}
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Thm: \( H \) is Sidorenko \( \iff \) \( H \) is \( \ell\)-common for every \( \ell \)

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Our main result\(^+\): \( \forall k, \ell : \exists \) \( \ell\)-common \( H_{k,\ell} \) with \( \chi(H_{k,\ell}) = k \)

Is the graph \( K_{5,5} - C_{10} \) common?
Conclusion (2/2)

Our main result\(^+\): \(\forall k, \ell : \exists \ell\)-common \(H_{k,\ell}\) with \(\chi(H_{k,\ell}) = k\)

Theorem (Ko-Lee): \(\exists \ell\)-common \(H_{k,\ell, m}\) with \(\chi = k\), connectivity \(m\)

Open problem: \(\exists\) high-\(\chi\) & high-girth & high-connectivity common \(H\)?
Conclusion (2/2)

Our main result$: \forall k, \ell : \exists \ell\text{-common } H_{k,\ell} \text{ with } \chi(H_{k,\ell}) = k$

\[K_{m,n} \xrightarrow{\text{(}$\ell - 1\text{)}\text{-edge path}} F \xrightarrow{\text{girth } \geq 42} \xrightarrow{\text{12-edge path}} C_4\]

Theorem (Ko-Lee): $\exists \ell\text{-common } H_{k,\ell,m} \text{ with } \chi = k, \text{ connectivity } m$
Conclusion (2/2)

Our main result: \( \forall k, \ell : \exists \ell\text{-common } H_{k,\ell} \text{ with } \chi(H_{k,\ell}) = k \)

Theorem (Ko-Lee): \( \exists \ell\text{-common } H_{k,\ell,m} \text{ with } \chi = k, \text{ connectivity } m \)

Open problem: \( \exists \text{ high-}\chi \text{ & high-girth & high-connectivity common } H? \)
Conclusion (2/2)

Our main result: \( \forall k, \ell : \exists \ell\text{-common } H_{k,\ell} \text{ with } \chi(H_{k,\ell}) = k \)

\[ K_{m,n} \longrightarrow (\ell - 1)\text{-edge path} \longrightarrow F \longrightarrow 12\text{-edge path} \longrightarrow C_4 \]

Theorem (Ko-Lee): \( \exists \ell\text{-common } H_{k,\ell,m} \text{ with } \chi = k, \text{ connectivity } m \)

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Our main result$^+$: $\forall k, \ell : \exists \ell$-common $H_{k,\ell}$ with $\chi(H_{k,\ell}) = k$

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