A Large Deviation Principle for Block Models

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Let $G(n, p_n)$ be an Erdős-Rényi random graph.
Large deviations on random graphs

- Let $G(n, p_n)$ be an Erdős-Rényi random graph.
- Let $T_n$ denote the number of triangles in $G(n, p_n)$. 

Fix $\delta > 0$.

$P(T_n > (1 + \delta)E[T_n]) = ?$

What is the "structure" of the graph, conditioned on this rare event?

What is responsible for an elevated triangle count?

- More edges spread throughout the graph?
- Some small, dense graphs?
- "localization"
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  - Some small, dense graphs? “localization”
Why is this interesting?

- Setting $(A_{i,j})_{i,j=1}^{n}$ to be the adjacency matrix,

\[ T_n = \sum_{i<j<k} A_{ij} A_{jk} A_{ki} - \text{nonlinear} \]
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  Today’s focus: Large deviations in dense graphs
The Erdős-Rényi case

Key idea: represent an Erdős-Rényi random graph as a graphon [CV’11, LZ’15]

Figure 1: Empirical graphon

Figure 2: A sequence of empirical graphons

Describe large deviations through the language of graphons!

1Images: Forkert 2015
The Erdős-Rényi case

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Figure 1: Empirical graphon

- The region $[0, 1]^2$ is divided into $n \times n$ cells.
- If $(i, j) \in E$, then the $(i, j)$ cell takes value 1.
- If $(i, j) \notin E$, then the $(i, j)$ cell takes value 0.

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![Empirical graphon](image1)

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Graphon topology

Developed by Borgs, Chayes, Lovász, Sos, Szegedy, Vesztergombi, . . .
Graphon topology

A graphon is a measurable function $f : [0, 1]^2 \rightarrow [0, 1]$, satisfying $f(x, y) = f(y, x)$.
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Cut distance: $d_{\square}(f, g) = \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} (f(x, y) - g(x, y)) \, dx \, dy \right|$
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- Cut metric: \( \delta_{\square}(f, g) = \inf_{\phi \in \mathcal{M}} d_{\square}(f, g^\phi) \)
  - \( \mathcal{M} = \{ \phi : [0, 1] \to [0, 1] \text{ : bijective, measure preserving} \} \)
  - \( g^\phi(x, y) = g(\phi(x), \phi(y)) \)
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  - $g^\phi(x, y) = g(\phi(x), \phi(y))$
- Equivalence relation: $f \sim g$ if $\delta_{\Box}(f, g) = 0$.  

\[ \tilde{\mathcal{W}} = \{ \tilde{f} : f \in \mathcal{W} \}, \delta_{\Box}(\tilde{f}, \tilde{g}) = \delta_{\Box}(f, g) \]

Theorem (Lovász & Szegedy (2007)) $(\tilde{\mathcal{W}}, \delta_{\Box})$ is a compact metric space.
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**Theorem (Lovász & Szegedy (2007))**

$(\tilde{\mathcal{W}}, \delta_{\square})$ is a compact metric space.
Homomorphism densities

**Definition (Homomorphism density)**

Fix a subgraph $H$. For $f \in \mathcal{W}$, define

$$t(H, f) = \int_{[0,1]|V(H)|} \prod_{(i,j) \in E(H)} f(x_i, x_j)^{|V(H)|} \prod_{i=1}^{\left|V(H)\right|} dx_i.$$
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Let $f^G$ be the empirical graphon associated with $G$.

$$\frac{6}{n^3} \sum_{i<j<k} A_{ij}A_{jk}A_{ki} = t(\Delta, f^G)$$

Can talk about $t(H, \tilde{f})$ as well!
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Theorem (LS’07,BCLSV’08)

For any fixed graph $H$, $\tilde{f} \mapsto t(H, \tilde{f})$ is continuous under the cut topology.
Consider now the random graph $G(n, p)$ for $p \in (0, 1)$. 

**Definition (Relative entropy)**

Define $I_W: W \to \mathbb{R} \cup \{\infty\}$ as

$$I_W(f) = \frac{1}{2} \int_{[0,1]^2} h_p(f(x,y)) \, dx \, dy,$$

where $h_p(u) =$ \begin{align*}
&u \log u + (1-u) \log (1-u), \quad u \in (0,1), \\
&\infty, \quad \text{otherwise}.
\end{align*}
Consider now the random graph $G(n, p)$ for $p \in (0, 1)$.

The empirical graphon induces a distribution on $(\widetilde{\mathcal{W}}, \delta)$.
Random Graphons

- Consider now the random graph $G(n, p)$ for $p \in (0, 1)$.
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- Naturally induces a sequence of probability measures $\widetilde{\mathbb{P}}_{n,p}$ on $(\widetilde{\mathcal{W}}, \delta_{\square})$. 

Definition (Relative entropy)

Define $I_{\mathcal{W}0}: \mathcal{W} \to \mathbb{R} \cup \{\infty\}$ as

$$I_{W0}(f) = \frac{1}{2} \int_{[0,1]^2} \frac{h_p(f(x,y))}{h_p(u)} du dy,$$

where $h_p(u) = u \log u + (1-u) \log (1-u)$.

LDP for Block Models
Random Graphons

- Consider now the random graph $G(n, p)$ for $p \in (0, 1)$.
- The empirical graphon induces a distribution on $(\hat{W}, \delta_\square)$.
- Naturally induces a sequence of probability measures $\tilde{P}_{n,p}$ on $(\hat{W}, \delta_\square)$!

Derive LDP for graphs in terms of $\tilde{P}_{n,p}$!
Random Graphons

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**Definition (Relative entropy)**

Define $I_{W_0} : \mathcal{W} \rightarrow \mathbb{R} \cup \{\infty\}$ as

$$I_{W_0}(f) = \frac{1}{2} \int_{[0,1]^2} h_p(f(x, y)) \, dx \, dy,$$

where $h_p(u)$ is the usual relative entropy,

$$h_p(u) = u \log \frac{u}{p} + (1 - u) \log \frac{1 - u}{1 - p}.$$
An LDP under cut topology

**Theorem (Chatterjee-Varadhan(2011))**

For any fixed $p \in (0, 1)$, $\{\tilde{P}_{n,p} : n \geq 1\}$ satisfies an LDP with speed $n^2$ and rate function $I_p(\cdot)$. Formally,
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  $$\limsup_{n \to \infty} \frac{1}{n^2} \log \tilde{P}_{n,p}(\tilde{F}) \leq - \inf_{\tilde{h} \in \tilde{F}} I_p(\tilde{h}),$$

- For any open set $\tilde{O} \subset \tilde{\mathcal{W}}$,
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$I_p(\tilde{h}) = \frac{1}{2} \int_{[0,1]^2} I_p(h(x, y)) dxdy.$
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$I_p(\tilde{h}) = \frac{1}{2} \int_{[0,1]^2} I_p(h(x, y)) \, dx \, dy$. 
Typical structure under rare event

Theorem (Chatterjee-Varadhan (2011))

- Let \( \tilde{F} \subset \tilde{W} \) be closed.
Typical structure under rare event

Theorem (Chatterjee-Varadhan (2011))

- Let $\tilde{F} \subset \tilde{\mathcal{W}}$ be closed.
- Let $\tilde{F}^\ast$ be the subset of $\tilde{F}$ where $I_p$ is minimized.
Typical structure under rare event

Theorem (Chatterjee-Varadhan (2011))

- Let $\tilde{F} \subset \tilde{\mathcal{W}}$ be closed.
- Let $\tilde{F}^*$ be the subset of $\tilde{F}$ where $I_p$ is minimized.

Then
- $\tilde{F}^*$ is non-empty and compact.
Typical structure under rare event

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- Let $\tilde{F}^*$ be the subset of $\tilde{F}$ where $I_p$ is minimized.

Then

- $\tilde{F}^*$ is non-empty and compact.
- $\mathbb{P}_{n,p}(\delta_\square(G(n,p), \tilde{F}^*) < \varepsilon | G(n,p) \in \tilde{F}) \geq 1 - \exp(-Cn^2)$ for some $C > 0$.
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If $\widetilde{F}^*$ is a singleton, the conditional distribution is concentrated at a single point!
The upper tail variational problem

\[ \phi(p, t) = \inf \{ I_p(\tilde{f}) : \tilde{f} \in \widetilde{\mathcal{W}}, t(\Delta, \tilde{f}) \geq t \}. \]
The upper tail variational problem

\[ \phi(p, t) = \inf \{ I_p(\tilde{f}) : \tilde{f} \in \tilde{W}, t(\Delta, \tilde{f}) \geq t \} . \]

- If minimizer is constant - Erdős-Rényi with higher edge density. (symmetry)
The upper tail variational problem

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- If minimizer is constant - Erdős-Rényi with higher edge density. (symmetry)

- If minimizer non-constant - what happens? (symmetry-breaking)
The Symmetry/Symmetry-breaking transition

- $G \sim G(n, p)$, conditioned on an elevated triangle count
The Symmetry/Symmetry-breaking transition

- $G \sim G(n, p)$, conditioned on an elevated triangle count
- $r$: the edge probability for which the elevated triangle count is typical
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Figure 3: The upper tail phase diagram for triangles. [Lubetzky-Zhao (2015)]
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- Blue region: symmetric regime $\rightarrow$ mimic $G(n, r)$
The Symmetry/Symmetry-breaking transition

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![Figure 3: The upper tail phase diagram for triangles. [Lubetzky-Zhao (2015)]](image)

- Blue region: *symmetric regime* $\rightarrow$ mimic $G(n, r)$
- White region: *non-symmetric regime* $\rightarrow$ distribution does not match $G(n, r)$
What is left to know?

- Phase diagram for non-regular graphs $H$?
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A lot remains unknown!
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A lot remains unknown!

Our focus: Large deviations beyond the Erdős-Rényi case
Random graphs with inhomogeneities or constraints are common.

(a) The $G(n,m)$ model. [Dembo-Lubetzky (2018)]
(b) Random regular graphs.
(c) Block models.
Random graphs with inhomogeneities or constraints are common.

(a) The $G(n, m)$ model. [Dembo-Lubetzky (2018)]
(b) Random regular graphs.
(c) Block models.

Large deviations in this context is of natural interest!

Expect new phenomena . . .
Construct a graphon with $k$ blocks of equal length.
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Edge probabilities are specified by \( (p_{ab})_{1 \leq a,b \leq k} \), where \( p_{ab} = p_{ba} \).
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**Base graphon** $W_0$ takes value $p_{ab}$ on the $(a, b)$ block.
Block Models

- Construct a graphon with $k$ blocks of equal length.
- Edge probabilities are specified by $(p_{ab})_{1 \leq a, b \leq k}$, where $p_{ab} = p_{ba}$.
- **Base graphon** $W_0$ takes value $p_{ab}$ on the $(a, b)$ block.
- Our random graph has $kn$ vertices, with $n$ vertices associated to each block.

Add edges independently with probability $p \left\lceil \frac{i}{n} \right\rceil \left\lceil \frac{j}{n} \right\rceil$.

In other words, if Vertex $i$ is in block $a$ and Vertex $j$ is in block $b$, then connect $i$ and $j$ with probability $p_{ab}$.

Note: repeated $p_{ab}$ are allowed; we can accommodate rational-length blocks.

Sampled graph $\leftrightarrow$ Empirical graphon

Distribution over graphs $\leftrightarrow \tilde{P}_{n,W_0}$, the induced law on $(\tilde{W},\delta)$.

Derive LDP for graphs in terms of $\tilde{P}_{n,W_0}$!
Construct a graphon with \( k \) blocks of equal length.

Edge probabilities are specified by \((p_{ab})_{1 \leq a,b \leq k}\), where \( p_{ab} = p_{ba} \).

Base graphon \( W_0 \) takes value \( p_{ab} \) on the \((a, b)\) block.

Our random graph has \( kn \) vertices, with \( n \) vertices associated to each block.

Add edges independently with probability \( p_{[i/n][j/n]} \).

In other words, if \( \text{Vertex } i \) is in block \( a \) and \( \text{Vertex } j \) is in block \( b \), then connect \( i \) and \( j \) with probability \( p_{ab} \).

Note: repeated \( p_{ab} \) are allowed; we can accommodate rational-length blocks.

Sampled graph \( \leftrightarrow \) Empirical graphon

Distribution over graphs \( \leftrightarrow \tilde{P}_{n,W_0} \), the induced law on \( (\tilde{W},\delta_{\square}) \).

Derive LDP for graphs in terms of \( \tilde{P}_{n,W_0} \).
Block Models

- Construct a graphon with $k$ blocks of equal length.
- Edge probabilities are specified by $(p_{ab})_{1 \leq a,b \leq k}$, where $p_{ab} = p_{ba}$.
- **Base graphon** $W_0$ takes value $p_{ab}$ on the $(a, b)$ block.
- Our random graph has $kn$ vertices, with $n$ vertices associated to each block.
- Add edges independently with probability $p_{[i/n][j/n]}$.
- In other words, if
  - Vertex $i$ is in block $a$
  - Vertex $j$ is in block $b$,
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**Sampled graph ↔ Empirical graphon**

Distribution over graphs ↔ $\tilde{P}_{n,W_0}$, the induced law on $(\tilde{W}, \delta_{\square})$

Derive LDP for graphs in terms of $\tilde{P}_{n,W_0}$!
A subtlety when working with block models

- Some blocks can be equal to 0 or 1
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- Leads to issues when calculating relative entropy of a graphon $f$ compared to $W_0$. 

\[ p_{ab} \in \{0, 1\} \quad \text{and} \quad f(x, y) \neq p_{a,b} \Rightarrow h_{p_{a,b}}(f(x, y)) = \infty \]

Issue does not arise in Erdős-Rényi context

Solution: Restrict support appropriately for LDP.

$\Omega = \{(x, y) : W_0(x, y) \in (0, 1)\}$

$W_\Omega = \{f \in W : f = W_0 \lambda - a.s. \text{ on } \Omega \}$

Graphons that "agree" with $W_0$

$\tilde{W}_\Omega = \{\tilde{f} \in \tilde{W} : \delta_\square(f, g) = 0 \text{ for some } g \in W_\Omega\}$

Equivalence classes that "agree" with $W_0$

$\tilde{W}_\Omega$ closed, $\tilde{P}_{n,W_0}$ supported on $\tilde{W}_\Omega$. 

LDP for Block Models
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Equivalence classes that “agree” with $W_0$
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Equivalence classes that “agree” with \( W_0 \)

\[
\widetilde{\mathcal{W}}_\Omega \text{ closed, } \tilde{P}_{n,W_0} \text{ supported on } \widetilde{\mathcal{W}}_\Omega.
\]
Recall the rate function from the Erdős-Rényi setting:

\[ I_{W_0}(f) = \frac{1}{2} \int_{[0,1]^2} h_p(f(x,y)) \, dx \, dy, \]
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First guess: \[ I_{W_0}(f) = \frac{1}{2} \int_{[0,1]^2} h_{W_0(x,y)}(f(x,y)) \, dx \, dy. \]
The rate function

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- Not well-defined on \((\tilde{\mathcal{W}}, \delta_{\square})\).
- The rate function should be lower semi-continuous.

Our rate function:

\[
J_{W_0}(\tilde{f}) = \begin{cases} 
\sup_{\eta > 0} \inf_{h \in B(\tilde{f}, \eta)} I_{W_0}(h) & \text{if } \tilde{f} \in \tilde{\mathcal{W}}_\Omega, \\
\infty & \text{o.w.}
\end{cases}
\]
Theorem (BCGPS '20+)

The sequence $\tilde{P}_{kn,W_0}$ satisfies an LDP with speed $n^2$ and rate function $J_{W_0}$.
LDP for dense block models

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  \[ J_{W_0}(\tilde{f}) = \inf_{h : \delta_\square(h, \tilde{f}) = 0} I_{W_0}(h). \]

- Markering '22 showed that the same rate function applies when \( \log(W_0), \log(1 - W_0) \in L^1([0, 1]^2) \).
Theorem (BCGPS’20+)  

Fix $H$. Set $t_{\text{max}} = \max_{\tilde{f} \in \tilde{W}} t(H, \tilde{f})$. For $t < t_{\text{max}}$ define

$$
\phi(W_0, t) = \inf \{ J_{W_0}(\tilde{f}) : t(H, \tilde{f}) \geq t \}.
$$
Application to homomorphism densities

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(i) For $t < t_{\text{max}}$, \( \lim_{n \to \infty} \frac{1}{(kn)^2} \log \mathbb{P}_{kn,W_0}(t(H, G_{kn}) \geq t) = -\phi(W_0, t). \)
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(ii) Fix $t < t_{\text{max}}$.

- Let $\tilde{F}^*$ be the subset of $\{ \tilde{f} : t(H, \tilde{f}) \geq t \}$ where $J_{W_0}$ is minimized.
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- $\tilde{F}^\ast$ is non-empty and compact.
Application to homomorphism densities

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Fix $H$. Set $t_{\text{max}} = \max_{\tilde{f} \in \tilde{\mathcal{W}}} t(H, \tilde{f})$. For $t < t_{\text{max}}$ define

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- $\tilde{F}^*$ is non-empty and compact.
- $\mathbb{P}_{kn, W_0}(\delta(\mathbb{F}_{kn}^*, \tilde{F}^*) < \varepsilon | t(H, G_{kn}) \geq t) \geq 1 - \exp(-Cn^2)$ for some $C > 0$.  

Application to homomorphism densities

**Theorem (BCGPS’20+)**

*Fix $H$. Set $t_{\text{max}} = \max_{\tilde{f} \in \tilde{W}} t(H, \tilde{f})$. For $t < t_{\text{max}}$ define*

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\phi(W_0, t) = \inf \{ J_{W_0}(\tilde{f}) : t(H, \tilde{f}) \geq t \}.
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(i) *For $t < t_{\text{max}}$, $\lim_{n \to \infty} \frac{1}{(kn)^2} \log \mathbb{P}_{kn, W_0}(t(H, G_{kn}) \geq t) = -\phi(W_0, t)$.*

(ii) *Fix $t < t_{\text{max}}$.*

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- $\mathbb{P}_{kn, W_0}(\delta(\tilde{G}_{kn}, \tilde{F}^*) < \varepsilon | t(H, G_{kn}) \geq t) \geq 1 - \exp(-Cn^2)$ for some $C > 0$.
- *If $\tilde{F}^*$ is a singleton, the conditional distribution is concentrated at a single point.*
A new notion of symmetry

Question

Do the minimizers of \( \min \left\{ J_{W_0}(\tilde{f}) : \tilde{f} \in \tilde{W}, t(H, \tilde{f}) \geq t \right\} \) have the same block structure as \( W_0 \)?
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Theorem (BCGPS'20+)
Fix a d-regular graph \( H \). Set \( t_{\text{max}} = \max_{\tilde{f} \in \tilde{W}} t(H, \tilde{f}) \).

\[ \begin{align*}
\text{Fix a d-regular graph } H. \text{ Set } t_{\text{max}} &= \max_{\tilde{f} \in \tilde{W}} t(H, \tilde{f}).
\end{align*} \]
A new notion of symmetry

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**Theorem (BCGPS’20+)**

Fix a \( d \)-regular graph \( H \). Set \( t_{\text{max}} = \max_{\tilde{f} \in \tilde{W}} t(H, \tilde{f}) \).

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(ii) There exists \( \eta > 0 \) such that if \( t \in [(1 - \eta) t_{\text{max}}, t_{\text{max}}] \) the minimizer is unique and symmetric.
In the symmetry-breaking regime, the optimizing graphon has a different structure from the base graphon.
Symmetry-breaking

- In the symmetry-breaking regime, the optimizing graphon has a different structure from the base graphon.
- Can establish symmetry-breaking for certain $W_0$. 

Know the specific symmetry/symmetry-breaking boundary for Erdős-Rényi bipartite graphs.

In these examples, this establishes a “re-entrant phase transition.”
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Know the specific symmetry/symmetry-breaking boundary for Erdős-Rényi bipartite graphs.
Proof ideas: LDP

- Prior work [CV ’11] relies on the fact that an Erdős-Rényi random graph remains invariant in law under vertex permutation.
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1. Apply Szemerédi’s Regularity Lemma:
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2. “Method of types”-style argument
Proof ideas: LDP

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Two-step approach:

1. Apply Szemeredi’s Regularity Lemma:
   - Construct a Szemeredi net of block graphons
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2. “Method of types”-style argument
   - Each vertex is a member of some block (“type”)
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   - Each vertex is a member of some block ("type")
   - Its type influences how likely it forms edges with vertices of other types.
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Two-step approach:

1. Apply Szemerédi’s Regularity Lemma:
   - Construct a Szemerédi net of block graphons
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   - It suffices to characterize the limiting probability of each open ball.

2. “Method of types”-style argument
   - Each vertex is a member of some block ("type")
   - Its type influences how likely it forms edges with vertices of other types.
   - Compare base graphon to empirical graphon according to alignment of types
Proof Ideas: Symmetric Regime

Definition

Let $p \in (0, 1)$ and $d \geq 2$. We define $\psi_p : [0, 1] \to \mathbb{R}$ as

$$
\psi_p(x) = h_p(x^{1/d}),
$$

and let $\hat{\psi}_p(x)$ denote the convex minorant of $\psi_p(x)$.

Figure 4: Illustration of the function $x \mapsto h_p(x^{1/\gamma})$ and its convex minorant (Lubetzky–Zhao 2015))
Proof Ideas: Symmetric Regime

Figure 5: A graphon $f = (f_{ij})_{i, j \in [m]}$
Proof Ideas: Symmetric Regime

Figure 5: A graphon $f = (f_{ij})_{i,j \in [m]}$

Let $\|g\|_d = \left( \int_{[0,1]^2} g(x,y)^d \, dx \, dy \right)^{\frac{1}{d}}$. 
The Convex Minorant Condition

Definition

- Let $W_0 = (p_{ij})_{i,j \in [m]}$ be the base graphon
- Let $f = (f_{ij})_{i,j \in [m]}$
The Convex Minorant Condition

**Definition**

- Let \( W_0 = (p_{ij})_{i,j \in [m]} \) be the base graphon
- Let \( f = (f_{ij})_{i,j \in [m]} \)

We say \( f \) satisfies the \( \varepsilon \)-neighborhood minorant condition if for all \((i, j)\) such that \( p_{ij} \in (0, 1)\)

\[
x \in (\|f_{ij}\|_d - \varepsilon, \|f_{ij}\|_d + \varepsilon) \cap [0, 1] \implies \psi_{p_{ij}}(x) = \hat{\psi}_{p_{ij}}(x).
\]
The Convex Minorant Condition

Definition

- Let \( W_0 = (p_{ij})_{i,j \in [m]} \) be the base graphon
- Let \( f = (f_{ij})_{i,j \in [m]} \)

We say \( f \) satisfies the \( \varepsilon \)-neighborhood minorant condition if for all \((i, j)\) such that \( p_{ij} \in (0, 1) \)

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x \in (\|f_{ij}\|_d - \varepsilon, \|f_{ij}\|_d + \varepsilon) \cap [0, 1] \implies \psi_{p_{ij}}(x) = \hat{\psi}_{p_{ij}}(x).
\]

Figure 6: The function \( x \mapsto h_p(x^{1/\gamma}) \) and its convex minorant [LZ '15]
Key Lemma for Symmetric Regime

Lemma

Let $W_0 = (p_{ij})_{i,j \in [m]}$ be the base graphon
Key Lemma for Symmetric Regime

Lemma

- Let $W_0 = (p_{ij})_{i,j \in [m]}$ be the base graphon
- Suppose $\tilde{f}$ is a minimizer of the variational problem for $\tau = t(H, \cdot)$
Lemma

- Let $W_0 = (p_{ij})_{i,j \in [m]}$ be the base graphon
- Suppose $\tilde{f}$ is a minimizer of the variational problem for $\tau = t(H, \cdot)$
- Suppose there exists a sequence of graphons $f_n \in \mathcal{W}_\Omega$ such that
Lemma

- Let $W_0 = (p_{ij})_{i,j \in [m]}$ be the base graphon
- Suppose $\tilde{f}$ is a minimizer of the variational problem for $\tau = t(H, \cdot)$
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Build a non-symmetric graphon such that:

\[ \gamma \quad 0 \quad \alpha_2 \quad \alpha_3 \]

\[ \alpha_1 \quad r \quad r_1 \quad r \quad \alpha_1 \]

\[ r_1 \quad r \quad r_2 \quad \alpha_4 \]

\[ r \quad r_2 \quad 1 - \gamma \quad 0 \]

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Proof Ideas: Non-Symmetric Regime

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- The constraint (e.g. homomorphism density) is satisfied
- The relative entropy is strictly lower than what the symmetric solution attains.
Subsequent Developments

- Dupuis, Medvedev’20—inhomogeneous LDP (proof using weak convergence methods)
- Chakraborty, Hazra, den Hollander, Sfragara ’20 (variational problem for spectral radius)
- Braunsteins, den Hollander, Mandjes’20 (sample path large deviations)
- Grebik, Pikhurko ’21 (irrational block lengths)
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Thank you!