# Analytic Approach to Quasirandomness

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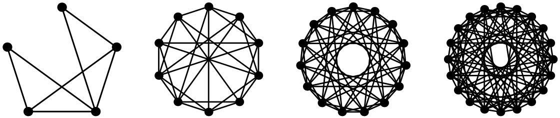
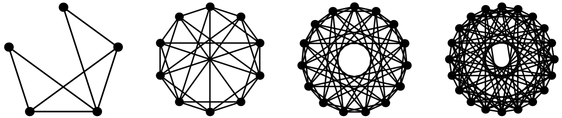
# Classical results

- quasirandom graph $\approx$ Erdős-Rényi graph $G_{n,p}$
  not a property of a single graph but a sequence



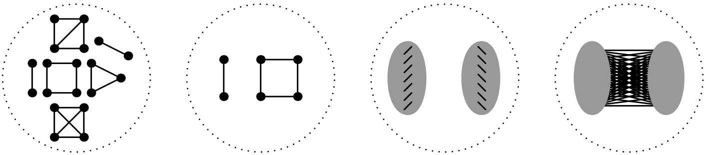
# Classical results

- quasirandom graph $\approx$ Erdős-Rényi graph $G_{n,p}$
  not a property of a single graph but a sequence
- Rödl, Thomason, Chung, Graham and Wilson (1980's)
- $d(H, G)$ = induced density of $H$ in $G$
  $t(H, G)$ = homomorphic density of $H$ in $G$
- $G_1, G_2, \ldots$ is quasirandom if $d(H, G_i) \to \mathbb{E}\, d(H, G_{n,p})$
  equivalently, if $t(H, G_i) \to \mathbb{E}\, t(H, G_{n,p})$



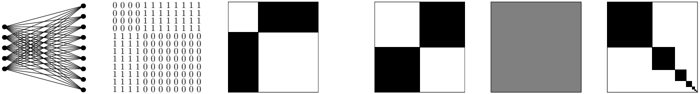
# Equivalent characterizations

- $G_1, G_2, \ldots$ is quasirandom if $d(H, G_i) \to \mathbb{E}\, d(H, G_{n,p})$
  $\Leftrightarrow$ $t(H, G_i) \to \mathbb{E}\, t(H, G_{n,p})$
  $\Leftrightarrow$ $t(K_2, G_i) \to p$ and $t(C_4, G_i) \to p^4$
  $\Leftrightarrow$ every $n$-vertex subset induces $\approx pn^2/2$ edges
  $\Leftrightarrow$ number of edges between $A$ and $B$ is $\approx p\,|A|\,|B|$
  $\Leftrightarrow$ spectrum of the adjacency matrix is $\{pn, o(n), \ldots, \}$



# Graph limit view

- a sequence $G_i$ is convergent if $t(H, G_i)$ converges
  quasirandom $\Leftrightarrow$ $t(H, G_i) \to \mathbb{E}\, t(H, G_{n,p})$
- graphon analytic representation of the limit
  $W : [0,1]^2 \to [0,1]$, a "continuous" adjacency matrix
  regularity decompositions, martingale convergence
- possible to define $t(H, W)$ for every graph $H$



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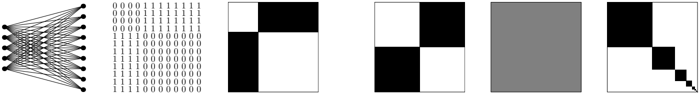
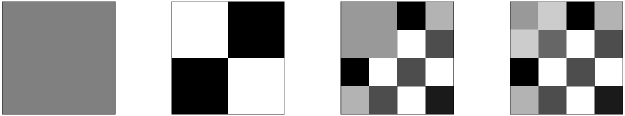
- graphon analytic representation of the limit
  $W : [0, 1]^2 \to [0, 1]$, a "continuous" adjacency matrix
  density $t(H, W)$ of a graph $H$ in $W$
- a sequence $G_i$ is quasirandom iff $W = 1/2$ a.e.
  $t(K_2, W) = p$ and $t(C_4, W) = p^4 \Leftrightarrow W = p$
- this implies that $t(C_4, W) \geq t(K_2, W)^4$ for every $W$



# Step graphons

- Theorem (Lovász, Sós, 2008)
  $K$-step graphon characterized by $O(K^K)$-vertex graphs



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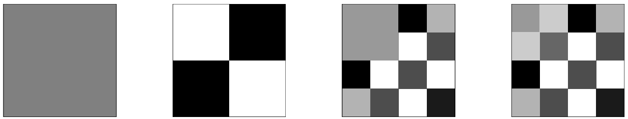

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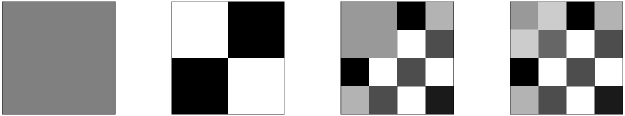

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- Theorem (Grzesik, K., Pikhurko, 2022+)
  degrees of parts different $\Rightarrow \max\{2K + 1, 4\}$ vertices

# Tournaments

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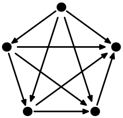
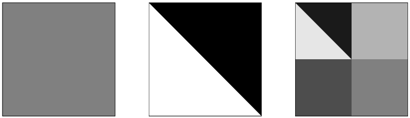
- tournamenton: $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) + W(y,x) = 1$





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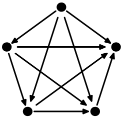
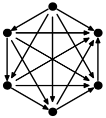
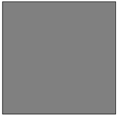
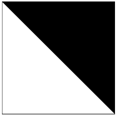
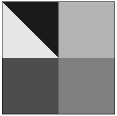
  $t(T,W) = 2^{-\binom{|T|}{2}}$ iff $W \equiv 1/2$

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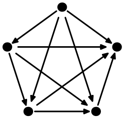
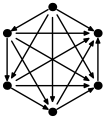
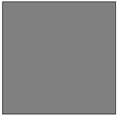
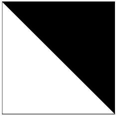
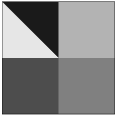
- Every transitive tournament with $k \geq 4$ vertices.

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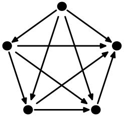
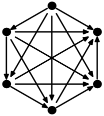
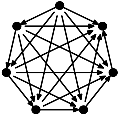
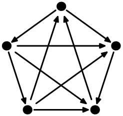

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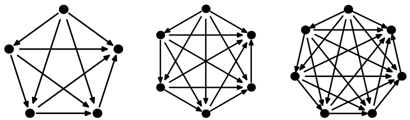

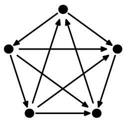

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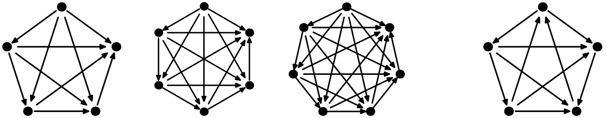

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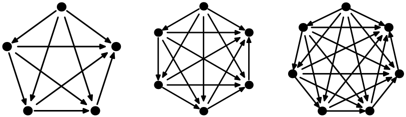
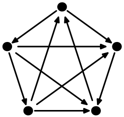
- no $\geq$ 7-vertex (Bucić, Long, Shapira, Sudakov, 2019+)
- no additional tournament (Hancock, Kabela, K., Martins, Parente, Skerman, Volec, 2019+)





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  $c(k) = 1$ if and only if $k$ is not divisible by four
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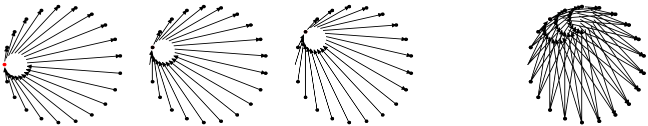

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- Theorem (Grzesik, K., Lovász Jr., Volec, 2020+)
  $c(k) = 1 \Leftrightarrow k$ not divisible by four
  $C_k$ is quasirandom-forcing if $k = 2 \mod 4$
  $1 + 2 \cdot (2/\pi)^k \leq c(k) \leq 1 + (2/\pi + o(1))^k$ if $4|k$
  $c(8) = 332/315$

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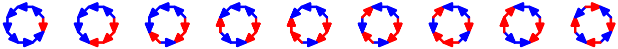
- Theorem (Grzesik, Il'kovič, Kielak, K., 2022+)
  Full characterization of orientations upto length 12.
  No orientation of an odd cycle is quasirandom-forcing.

Yes: 

No: 

# Quasirandom permutations

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- Question (Graham)
  Does there exist $k_0$ such that quasirandomness
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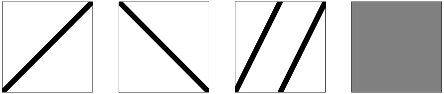
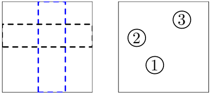
# Permutation limits

- a convergent sequence is represented by a permuton
  probability measure $\mu$ on $[0,1]^2$ with unit marginals
  Hoppen, Kohayakawa, Moreira, Ráth and Sampaio
  similar ideas in work of Presutti and Stromquist
- $\mu$-random permutation
  choose $n$ random points, $x$- and $y$-coordinates
  $(0.2, 0.6), (0.4, 0.3), (0.7, 0.8) \quad \rightarrow \quad 213$





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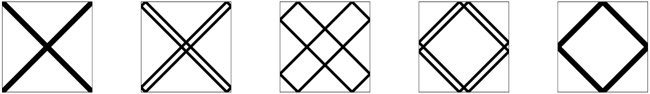
- $k_0 = 3$ is not sufficient: $d(123, .)$ ranges from $1/4$ to $1/8$



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- Theorem (Crudele, Dukes, Noel, 2022++)
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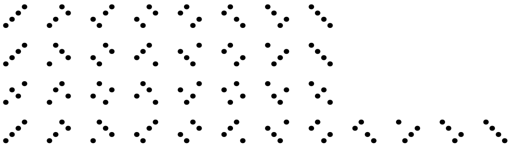
- Theorem (Kurečka, 2022)
  At least 4 permutations (regardless of orders) needed.

# Sum forcing

- Do we need that $d(\pi, \mu) = 1/24$ for all $\pi \in S_4$?

- Theorem (Chan, K., Noel, Pehova, Sharifzadeh, Volec)
  characterization of sets $T \subseteq S_4$ such that
  $\mu$ is uniform $\Leftrightarrow \sum_{\pi \in T} d(\pi, \mu) = |T|/24$

- $T \subseteq S_4$ is quasirandom-forcing iff $T$ is



or symmetric/complementary to one of these four sets

# Latin squares

- Latin square
  each row/column contain all numbers $1, \ldots, n$

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & \color{red}{1} & \color{red}{4} & 5 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & \color{red}{3} & \color{red}{5} & 1 & 4 \\ 5 & 4 & 2 & 3 & 1 \end{matrix} \quad \rightarrow \quad \begin{matrix} \color{red}{1} & \color{red}{3} \\ \color{red}{2} & \color{red}{4} \end{matrix}$$

# Latin squares

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  each row/column contain all numbers $1, \ldots, n$
- pattern density: choose rows and columns

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 3 & \color{red}{1} & \color{red}{4} & 5 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & \color{red}{3} & \color{red}{5} & 1 & 4 \\ 5 & 4 & 2 & 3 & 1 \end{array} \rightarrow \begin{array}{cc} \color{red}{1} & \color{red}{3} \\ \color{red}{2} & \color{red}{4} \end{array}$$

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- Latin square
  each row/column contain all numbers $1, \ldots, n$
- pattern density: choose rows and columns
- limit theory by Garbe, Hancock, Hladký, Sharifzadeh
  sampling is tricky (existence of designs)

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 5 & 1 & 4 \\ 5 & 4 & 2 & 3 & 1 \end{matrix} \quad \rightarrow \quad \begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix}$$

# Latin squares

- Conjecture (Garbe, Hancock, Hladký, Sharifzadeh)
  quasirandomness $\Leftrightarrow$ density of $k \times \ell$ pattern is $1/(k\ell)!$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | | | |
| 3 | 1 | 4 | 5 | 2 | | 1 | 3 |
| 4 | 5 | 1 | 2 | 3 | $\to$ | | |
| 2 | 3 | 5 | 1 | 4 | | 2 | 4 |
| 5 | 4 | 2 | 3 | 1 | | | |

# Latin squares

- Conjecture (Garbe, Hancock, Hladký, Sharifzadeh)
  quasirandomness $\Leftrightarrow$ density of $k \times \ell$ pattern is $1/(k\ell)!$

- Theorem (Cooper, K., Lamaison, Mohr, 2022)
  quasirandomness $\Leftrightarrow$ density of $2 \times 3$ pattern is $1/720$

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 5 & 1 & 4 \\ 5 & 4 & 2 & 3 & 1 \end{matrix} \quad \to \quad \begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix}$$

# Latin squares

- Conjecture (Garbe, Hancock, Hladký, Sharifzadeh)
  quasirandomness $\Leftrightarrow$ density of $k \times \ell$ pattern is $1/(k\ell)!$
- Theorem (Cooper, K., Lamaison, Mohr, 2022)
  quasirandomness $\Leftrightarrow$ density of $2 \times 3$ pattern is $1/720$
- $2 \times 3$ cannot be replaced with $1 \times \ell$ or $2 \times 2$

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 5 & 1 & 4 \\ 5 & 4 & 2 & 3 & 1 \end{matrix} \quad \rightarrow \quad \begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix}$$

Thank you for your attention!