Limited Commitment: Mechanism Design meets Information Design

Data-Driven Decision Processes Bootcamp

Laura Doval
Columbia Business School and CEPR

Based on joint work with Vasiliki Skreta
(Static) Mechanism Design:

- Agents have private information: $T_i$ is the set of types of agent $i$ and
  \[ \psi : \Theta \mapsto \Delta(T_1 \times \cdots \times T_N) \]
  describes the information player $i$ has about $\theta$ and the types of other players.
- Payoffs only depend on $A_0$.
- We are given a mapping $\pi : \Theta \mapsto \Delta(A_0)$.
- **Question:** Can we design actions for each player $A_1, \ldots, A_N$ and an outcome function
  \[ f : \times_{i=1}^N A_i \mapsto \Delta(A_0) \] such that $\pi$ is the equilibrium outcome of the game defined by $\langle G, \psi, f \rangle$?

**Example:** Google ad auction design

- $A_0 \subseteq (\{0, 1\} \times \mathbb{R})^N$ and $(q, t) \in A_0$ if, and only if, $0 \leq \sum_{i=1}^N q_i \leq 1$.
- $\Theta = \Theta_1 \times \ldots \Theta_N$; $T_i = \Theta_i$ denotes advertiser i’s value for the slot; $\psi(\cdot | \theta) = \delta_\theta$.
- $\pi$ is the rule that assigns the good to the advertiser w/ highest $\theta_i$. 

**Mechanism Design**

*(Static) Mechanism Design:* (in more standard textbook notation)

- Agents have private information: $\Theta = \times_{i=1}^{N} \Theta_i$ and agent $i$ knows $\theta_i$. That is,

$$\psi: \Theta \mapsto \Delta(\Theta_1 \times \cdots \times \Theta_N)$$

is such that $\psi(\cdot | \theta) = \delta_{\theta}$.

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Given $\pi : \Theta \mapsto \Delta(A_0)$,

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- Mechanism design provides us with a language to do this via the revelation principle.
Theorem (Gibbard, 1973; Myerson, 1979; Dasgupta et al, 1979)

There exists a game that has $\pi$ as an equilibrium outcome if and only if the following game implements $\pi$:

1. **Actions** $M_i = \Theta_i$

2. When players take actions $\theta' = (\theta'_1, \ldots, \theta'_N)$, the outcome is $f(\bar{\theta}) = \pi(\cdot | \theta')$.

Furthermore, it is without loss of generality to assume that the players find it optimal to truthfully report their types.
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The RP does not mean that *all* mechanisms are truthful

- Many real world mechanisms are not truthful (e.g., first price auctions)
- not clear that truthful mechanisms are *better* (e.g., second price auctions) (c.f., Li, 2017, Akbarpour & Li, 2020)
So why the obsession with the revelation principle?

- Truthful mechanisms are a good first cut abstraction,
- It is a recipe for constructing algorithms that implement allocations,
- It transforms an equilibrium problem into a constrained optimization problem.
- From the design perspective, if I cannot find a truthful mechanism that implements my desired rule then no mechanism does.
Mechanism design in the wild

- Sponsored search auctions
- Display advertising
- FCC spectrum auctions
- Kidney exchange
- Healthcare systems
- Recommendation systems
- Routing on the Internet
- Resource allocation in the cloud
- Platform design for a sharing economy
- Energy and electricity markets
- Bitcoin
- Participatory democracy
- Crowdsourcing
Even closer to a data-driven decision process:
- Repeated interactions
- Persistent and/or evolving types

(new) burgeoning area of dynamic mechanism design in Econ, CS, and OR
- internet auctions, government procurement, durable goods, regulation

The designer learns information that is relevant for today but also subsequent periods
- e.g., the optimal reserve price for today may not be optimal tomorrow

what the designer learns today, they can use tomorrow:
- ratchet effect
  - e.g., forward-looking bidders understand that bids today determine reserve prices tomorrow

⇒ additional incentive to shave bids above and beyond the strategic and dynamic interaction
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- Sometimes these rents are large enough that optimal mechanisms do not use the information learned.
  - *e.g.*, **sale of a durable good**
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Period 1

![Diagram](no p? no)
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Period 1

- $\theta > p$
- $p$?

Period 2

- $p_2 < p$?
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![Diagram](image)
Oftentimes, optimal mechanisms are not *sequentially rational*:

- i.e., if we gave the designer the possibility to revise the decision rule given the new learned information, they would have an incentive to do so.

There are many examples with these features

- dynamic (ad) auctions (e.g., Google) (c.f., Kanoria & Nazerzadeh, 2014; Papadimitrou et al, 2014; Balseiro et al, 2022)
- repeated sales (e.g., Lobel & Paes Leme, 2017; Devanur et al, 2019; Immorlica et al, 2017)
- procurement (e.g., Gur et al, 2022)

**Desiderata:** a theory of mechanism design that does not rely so strongly on the assumption that the designer has full commitment
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- Full-term mechanisms with 2-sided renegotiation (e.g., Hart & Tirole, 1988, Dewatripoint, 1989)
- Full-term mechanisms with 1-sided renegotiation (e.g., Baron & Besanko, 1987)
- Long, but not full, term contract with renegotiation (e.g., Rey & Salanie, 1990)
- Cannot commit even to today's mechanism (e.g., Adams & Schwarz, 2007, Vartianen, 2013, Akbarpour & Li, 2020)

Papers in CS & OR that study dynamic lack of commitment focus on this case as well: Papadimitrou et al., 2014; Lobel & Paes Leme, 2017; Devanur et al., 2019; Immorlica et al., 2017; Balseiro et al., 2022; Gur et al., 2022
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Commitment to short-term mechanisms

Setting:

- Uninformed designer interacts with privately & persistently informed agent over time
- Designer can offer short-term mechanisms
- Designer can commit to today’s mechanism, but not to the continuation ones.

Examples:
1. Regulation (c.f., Laffont & Tirole, 1988)
2. Procurement
3. Political Economy; e.g., taxation and social insurance,
4. Ad auctions, online shopping

Few papers analyze optimal mechanisms under limited commitment:

- Optimal mechanisms w/ finite horizon, e.g.,
- Infinite Horizon under restrictions, e.g.,
  - iid private information: e.g., Sleet and Yeltekin (2006, 2008), Farhi, Sleet, Yeltekin, and Werning (2012), Golosov and Iovino (2021)
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**Revelation principle for mechanism design with limited commitment**

We characterize a class of mechanisms and strategies that are enough to implement any outcome distribution that can be implemented under limited commitment.
Mechanisms (Myerson ’82, Forges ’85)

- $M$ is a set of input messages,
- $S$ is a set of output messages,
- $\varphi$ assigns to each input message a joint distribution over output messages and allocations
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- $\varphi$ assigns to each input message a joint distribution over output messages and allocations.
Mechanisms (Myerson ’82, Forges ’85)

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Mechanisms

Without loss of generality:

• Communication is direct, i.e., $M = \Theta$.
• Communication is observable: $M$ and $S$ have the same cardinality and $\varphi$ is invertible.
• Equilibrium communication is truthful.
Revelation Principle under commitment

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Limited Commitment 1: Bester & Strausz (ECMA, 2001)

Assume:

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Then, if the principal earns his highest payoff consistent with the agent’s payoff, wlog

- Communication is **direct**, i.e., $M = \Theta$,

However, **Equilibrium/communication is truthful**. (c.f., Papadimitrou et al, 2014)
What we know: Bester & Strausz (2001, 2007)

Limited Commitment 2: Bester & Strausz (JET, 2007)

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Then, without loss of generality:

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Revelation Principle for Limited Commitment

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Direct-Blackwell mechanisms
In a dynamic setting, we need the mechanism to replicate

1. The type-by-type allocation distribution
2. and (at the very least) the type-by-type distribution of beliefs
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1. The type-by-type allocation distribution (mechanism design)
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Restoring the revelation principle: Allocations and information

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\[ \theta \rightarrow M \rightarrow \phi(\cdot|m) \\Rightarrow S \times A \]
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![Diagram](image-url)
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  - Truthtelling + participation + Bayes’ plausibility constraint (designer’s sequential rationality)
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New applications facilitated by the generality of the framework
- no restrictions on the cardinality of $\Theta$
- on the length of the interaction
- extension to Markov settings
- Optimality of posted prices in infinite horizon-binary type durable goods model
- Optimality of coarse product lines (menus) when purchase history leads to price discrimination

Today: Revisit the sale of a durable good w/ a continuum of types and finite horizon
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- Today: Revisit the sale of a durable good w/ a continuum of types and finite horizon
Two final remarks

Two other reasons to care about MDLC in the context of DDDP and AGT:

1. Simplicity
2. Learning
Simple mechanisms

- Limiting the principal’s commitment was also an attempt to justify simple mechanisms,
- ... the idea being that it would force the principal to condition his mechanism on less variables (e.g., non-clairvoyant mechanisms, Balseiro et al, 2022)
- It turns out that the optimal mechanism is not necessarily “simpler”
  - e.g., posted prices may no longer be optimal to sell durable goods in finite horizon settings,
• Platforms use **learning algorithms** to optimize on prices/reserve prices based on historical data (c.f., Kanoria & Nazerzadeh, 2014; Haghtalab, Lykouris, Nietert, & Wei, 2022)
Learning mechanisms

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- And yet, these algorithms will do “the best” with the information collected so far according to some objective function
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  - the algorithm takes the role of the “sequentially rational principal”.

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• Our result provides a way of representing these **Bayesian** algorithms and the outcomes that can arise from the strategic interaction with a forward looking agent.
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Our result provides a way of representing these *Bayesian* algorithms and the outcomes that can arise from the strategic interaction with a forward looking agent.

- The analyst is forced to jointly describe the way information is stored and how it is used to determine the allocation.
Sale of a durable good: binary types and two periods
- A seller and a buyer interact over two periods.

\[ U(\cdot, \theta) = 2 \sum_{t=1}^{T} \delta^{t-1}(q_t \theta - x_t) \]

\[ W(\cdot, \theta) = 2 \sum_{t=1}^{T} \delta^{t-1} x_t \]

where \( \delta \in (0,1) \) is a common discount factor.
- A seller and a buyer interact over two periods.
- The seller owns one unit of a durable good and assigns value 0 to it.
Sale of a durable good

- A seller and a buyer interact over two periods.
- The seller owns one unit of a durable good and assigns value 0 to it.
- The buyer has private information indexed by \( \theta \in \Theta \equiv \{\theta_L, \theta_H\} \) and \( \mu_0 = \Pr(\theta = \theta_H) \)
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- An allocation is a pair \((q, x) \in \{0, 1\} \times \mathbb{R}\),
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- An allocation is a pair $(q, x) \in \{0, 1\} \times \mathbb{R}$,
  - $q$ indicates whether the good is sold ($q = 1$) or not ($q = 0$), and
  - $x$ is a payment from the buyer to the seller.
- If the good is sold in the first period, the game ends.
- If the final allocation is $\{(q_t, x_t)\}_{t \in \{1, 2\}}$, buyer and seller’s payoffs are
  $$U(\cdot, \theta) = 2 \sum_{t=1}^{\delta_t - 1} \delta_t (q_t \theta - x_t)$$
  and
  $$W(\cdot, \theta) = 2 \sum_{t=1}^{\delta_t - 1} \delta_t x_t$$
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- If the final allocation is $\{(q_t, x_t)\}_{t \in \{1, 2\}}$, buyer and seller’s payoffs are

$$U(\cdot, \theta) = \sum_{t=1}^{2} \delta^{t-1} (q_t \theta - x_t) \quad \text{and} \quad W(\cdot, \theta) = \sum_{t=1}^{2} \delta^{t-1} x_t$$

where $\delta \in (0, 1)$ is a common discount factor.
Timing: At the beginning of each period $t \in \{1, 2\}$
Sale of a durable good

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- Seller offers mechanism
- Buyer accepts mechanism
- Buyer rejects mechanism

When rejected, trade stops and the following occurs:

$$ (q, x) = (0, 0) \rightarrow t + 1 $$
Timing: At the beginning of each period $t \in \{1, 2\}$

- Seller offers mechanism
- Buyer accepts
- Buyer participates
- Buyer rejects

$$(q, x) = (0, 0) \quad \rightarrow \quad t + 1$$
Timing: At the beginning of each period $t \in \{1, 2\}$

- The seller offers a durable good.
- A buyer participates in the mechanism.
- If the buyer accepts the offer, the good is sold.
- If the buyer rejects the offer, no trade occurs.

Mathematically, this is represented as:

$$(q, x) = (0, 0) \rightarrow t + 1$$
**Sale of a durable good**

**Timing:** At the beginning of each period $t \in \{1, 2\}$
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The seller offers a mechanism to the buyer. The buyer can either accept or reject the offer.

- **Accepts**: The buyer participates in the allocation. The outcome is $(q, x) = (0, 0)$, and the allocation proceeds to the next period $t + 1$.

- **Rejects**: No trade occurs, and the allocation process stops at $t + 1$. 

The timing diagram illustrates the sequence of events:

1. Seller offers a mechanism.
2. Buyer accepts or rejects.
3. If accepted, allocation proceeds; if rejected, no trade.
4. End of period $t$.
5. Move to period $t + 1$. 

This structure captures the timing of decisions and outcomes in a durable good sale process.
Sale of a durable good

- Final period: seller has full commitment. (Standard) Revelation principle applies.
- Let $\mu_2$ denote the seller’s belief that $\theta = \theta_H$. 
Sale of a durable good

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- Let $\mu_2$ denote the seller’s belief that $\theta = \theta_H$.
- The optimal mechanism is as follows:

$$\bar{\mu} \equiv \frac{\theta_L}{\theta_H}$$
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\begin{align*}
\text{sell at } \theta_L & \\
\mu \equiv \frac{\theta_L}{\theta_H} & \\
\mu_2 &
\end{align*}
\]
Sale of a durable good

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$$\mu \equiv \frac{\theta_L}{\theta_H}$$

- Why $\mu_2$? Whenever the seller sells to both types, he leaves rents $\mu_2 \Delta \theta$ to $\theta_H$.

$$\theta_L = \mu_2(\theta_H - \Delta \theta) + (1 - \mu_2) \theta_L = \mu_2 \theta_H + (1 - \mu_2)(\theta_L - \mu_2 \Delta \theta) = \mu_2 \theta_H + (1 - \mu_2) \hat{\theta}_L(\mu_2)$$

When $\mu_2 = \mu$, then $\hat{\theta}_L(\mu_2) = 0$. 
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sell at $\theta_L$  \hspace{2cm} sell at $\theta_H$

$$\mu_2$$
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\begin{align*}
\mu_2 &\equiv \frac{\theta_L}{\theta_H} \\
\mu &\equiv \frac{\theta_L}{\theta_H} \\
\text{sell at } \theta_L &\approx \text{sell at } \theta_H
\end{align*}
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\theta_L &= \mu_2(\theta_H - \Delta \theta) + (1 - \mu_2)\theta_L = \mu_2\theta_H + (1 - \mu_2)(\theta_L - \frac{\mu_2}{1 - \mu_2}\Delta \theta) \\
&= \mu_2\theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_2)
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\begin{align*}
\theta_L &= \mu_2(\theta_H - \Delta \theta) + (1 - \mu_2)\theta_L = \mu_2\theta_H + (1 - \mu_2)(\theta_L - \frac{\mu_2}{1 - \mu_2}\Delta \theta) \\
&= \mu_2\theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_2)
\end{align*}
\]
Sale of a durable good

• Final period: seller has full commitment. (Standard) Revelation principle applies.
• Let $\mu_2$ denote the seller’s belief that $\theta = \theta_H$.
• The optimal mechanism is as follows:

![Diagram showing optimal mechanism]

- Why $\bar{\mu}$? Whenever the seller sells to both types, he leaves rents $\mu_2 \Delta \theta$ to $\theta_H$.

$$\theta_L = \mu_2(\theta_H - \Delta \theta) + (1 - \mu_2)\theta_L = \mu_2 \theta_H + (1 - \mu_2)(\theta_L - \frac{\mu_2}{1 - \mu_2} \Delta \theta)$$

$$= \mu_2 \theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_2)$$

When $\mu_2 = \bar{\mu}$, then $\hat{\theta}_L(\mu_2) = 0$. 
Wrapping up:

\[ R_2(\mu_2) = \begin{cases} 
\theta_L & \text{if } \mu_2 \leq \bar{\mu} \\
\mu_2\theta_H & \text{if } \mu_2 > \bar{\mu} 
\end{cases} = \begin{cases} 
\mu_2\theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_2) & \text{if } \mu_2 \leq \bar{\mu} \\
\mu_2\theta_H & \text{if } \mu_2 > \bar{\mu} 
\end{cases} \]

Seller's payoff in period 2
Sale of a durable good

• Recall $\mu_1$ is the prior probability that $\theta = \theta_H$. 

$\mu_1$ is the prior probability that $\theta = \theta_H$. 

$t = 2$
• Recall $\mu_1$ is the prior probability that $\theta = \theta_H$.

• A mechanism is a tuple

\[
M \xrightarrow{\varphi(\cdot|m)} S \times A
\]

- $M$ is the set of input messages
- $S$ is the set of output messages
- $\varphi : M \mapsto \Delta(S \times A)$
• Recall $\mu_1$ is the prior probability that $\theta = \theta_H$.

• A mechanism is a tuple

\[ M \xrightarrow{\varphi(\cdot|m)} S \times A \]

• $M$ is the set of input messages
• $S$ is the set of output messages
• $\varphi : M \mapsto \Delta(S \times A)$ (finite support) – without loss with finitely many types
• Recall \( \mu_1 \) is the prior probability that \( \theta = \theta_H \).

• A mechanism is a tuple

\[
\text{Buyer} \quad \xrightarrow{\text{sends a message}} \quad M \quad \xrightarrow{\varphi(\cdot|m)} \quad S \times A
\]

• \( M \) is the set of input messages
• \( S \) is the set of output messages
• \( \varphi : M \mapsto \Delta(S \times A) \) (finite support)– without loss with finitely many types
• Recall $\mu_1$ is the prior probability that $\theta = \theta_H$.
• A mechanism is a tuple

\begin{align*}
\text{Buyer} & \xrightarrow{\text{sends a message}} M \xrightarrow{\varphi(\cdot|m)} S \times A \\
\text{Seller}
\end{align*}

• $M$ is the set of input messages
• $S$ is the set of output messages
• $\varphi : M \mapsto \Delta(S \times A)$ (finite support)— without loss with finitely many types
Some simplifications:

\[ M_1 = \theta \]

\[ \phi_1(S_1, a_1 | m_1) \]

\[ \phi_1(S_3, a_3 | m_1) \]

\[ \phi_1(S_1, a_2 | m_2) \]

\[ \phi_1(S_3, a_3 | m_2) \]

\[ \theta_1 - \theta_{a_1} \phi_1(S_2, a_2 | m_1) + (1 - \theta_{a_1}) \phi_1(S_2, a_3 | m_2) \]

Bester and Strausz (JET, 2007)
Some simplifications:

- $M_1 = ?$
Some simplifications:

- \( M_1 = ? \)

\[ \varphi_1(s_1, a_1 | m_1) \rightarrow s_1, a_1 \]

\[ \varphi_1(s_3, a_3 | m_1) \rightarrow s_3, a_3 \]

\[ \varphi_1(s_1, a_2 | m_2) \rightarrow s_1, a_1 \]

\[ \varphi_1(s_3, a_3 | m_2) \rightarrow s_3, a_3 \]
Some simplifications:

- $M_1 = ?$

\[
\begin{align*}
\varphi_1(s_1, a_1 | m_1) &\rightarrow s_1, a_1 \\
\varphi_1(s_2, a_2 | m_1) &\rightarrow s_2, a_2 \\
p_\theta(s_1, a_1 | m_1) &\rightarrow s_1, a_1 \\
p_\theta(s_3, a_3 | m_1) &\rightarrow s_3, a_3 \\
p_\theta(s_2, a_2 | m_2) &\rightarrow s_2, a_2 \\
\varphi_1(s_3, a_3 | m_2) &\rightarrow s_3, a_3 \\
\varphi_1(s_2, a_2 | m_2) &\rightarrow s_2, a_2 \\
\varphi_1(s_3, a_3 | m_2) &\rightarrow s_3, a_3 \\
\end{align*}
\]
Some simplifications:

- $M_1 = ?$

\[
\begin{align*}
M_1 &= \theta_m s_1 a_1 s_2 a_2 s_3 a_3 m_2 \\
M_2 &= \theta_m s_1 a_1 s_2 a_2 s_3 a_3 m_2
\end{align*}
\]
Some simplifications:

- \( M_1 =? \)

\[
\begin{align*}
\varphi_1(s_1, a_1 | m_1) & \rightarrow s_1, a_1 \\
\varphi_1(s_3, a_3 | m_1) & \rightarrow s_3, a_3 \\
p_\theta & \rightarrow s_2, a_2 \\
1 - p_\theta & \rightarrow s_1, a_1 \\
\varphi_1(s_1, a_1 | m_2) & \rightarrow s_1, a_1 \\
\varphi_1(s_3, a_3 | m_2) & \rightarrow s_3, a_3 \\
\end{align*}
\]
Some simplifications:

- \( M_1 = \Theta \)

\( \theta 
\begin{align*}
&\varphi_1(s_1, a_1|m_1) \rightarrow s_1, a_1 \\
&\varphi_1(s_3, a_3|m_1) \rightarrow s_3, a_3 \\
&1 - p_\theta \varphi_1(s_1, a_1|m_2) \rightarrow s_1, a_1 \\
&\varphi_1(s_3, a_3|m_2) \rightarrow s_3, a_3
\end{align*}

\( \phi^*(s_1, a_1|\theta) \rightarrow s_1, a_1 \\
\phi^*(s_2, a_2|\theta) \rightarrow s_2, a_2 \\
\phi^*(s_3, a_3|\theta) \rightarrow s_3, a_3 \)
Some simplifications:

- $M_1 = \Theta$
Some simplifications:

- $M_1 = \Theta$

- Bester and Strausz (JET, 2007)
$S_1 = ?$
Output messages

\( S_1 = ? \)

\[ \varphi_1(s_1, a_1|\theta) \]

\[ \varphi_1(s_3, a_3|\theta) \]

\[ \theta \]

\[ s_1, a_1 \]

\[ s_2, a_2 \]

\[ s_3, a_3 \]
Output messages

\[ S_1 =? \]

\[ \varphi_1(s_1, a_1|\theta) \]

\[ \theta \]

\[ \varphi_1(s_3, a_3|\theta) \]

\[ s_1, a_1 \rightarrow \mu, a_1 \]

\[ s_2, a_2 \rightarrow \mu, a_2 \]

\[ s_3, a_3 \rightarrow \mu', a_3 \]
Output messages

$S_1 = ?$

$\theta$

$\varphi_1(s_1, a_1 | \theta) \rightarrow s_1, a_1 \rightarrow \mu, a_1, p_2(\mu)$

$\varphi_1(s_3, a_3 | \theta) \rightarrow s_3, a_3 \rightarrow \mu', a_3, p_2(\mu')$

$\theta$

$\varphi_1(s_2, a_2 | \theta) \rightarrow s_2, a_2 \rightarrow \mu, a_2, p_2(\mu)$

$\text{period 1}$
Output messages

\[ S_1 = ? \]

\[ \varphi_1(\mu, a_1 | \theta) \]

\[ \theta \]

\[ \varphi_1(\mu', a_3 | \theta) \]

\[ \mu', a_3 \rightarrow \mu', a_3, p_2(\mu') \]

\[ \mu, a_1 \rightarrow \mu, a_1, p_2(\mu) \]

\[ \mu, a_2 \rightarrow \mu, a_2, p_2(\mu) \]

\[ \text{period 1} \]
Output messages

\[ S_1 = \Delta(\Theta) \]
Separating information and allocation design

\[ \varphi_1(\mu, a_1 | \theta) \]

\[ \theta \]

\[ \varphi_1(\mu', a_3 | \theta) \]

\[ \mu, a_1 \rightarrow \mu, a_1, p_2(\mu) \]

\[ \mu, a_2 \rightarrow \mu, a_2, p_2(\mu) \]

\[ \mu', a_3 \rightarrow \mu', a_3, p_2(\mu') \]
Separating information and allocation design

\[ \varphi_1(\mu, a_1 | \theta) \]

\[ \varphi_1(\mu', a_3 | \theta) \]

\[ \theta \rightarrow \mu, a_1 \rightarrow \mu, a_1, p_2(\mu) \]

\[ \theta \rightarrow \mu, a_2 \rightarrow \mu, a_2, p_2(\mu) \]

\[ \theta \rightarrow \mu', a_3 \rightarrow \mu', a_3, p_2(\mu') \]
Separating information and allocation design

\[ \begin{align*}
\theta & \quad \mu, a_1 \quad \rightarrow \quad \mu, a_1, p_2(\mu) \\
\phi_1(\mu, a_1 | \theta) & \\
\theta & \quad \mu, a_2 \quad \rightarrow \quad \mu, a_2, p_2(\mu) \\
\phi_1(\mu', a_3 | \theta) & \\
\mu', a_3 & \quad \rightarrow \quad \mu', a_3, p_2(\mu') \\
\end{align*} \]

- \[ \Pr_{\phi}(\mu | \theta) = \phi_1(\mu, a_1 | \theta) + \phi_1(\mu, a_2 | \theta) \]
Separating information and allocation design

$$\varphi_1(\mu, a_1 | \theta)$$

$$\varphi_1(\mu', a_3 | \theta)$$

$$\theta \rightarrow \mu, a_1 \rightarrow \mu, a_1, p_2(\mu)$$

$$\theta \rightarrow \mu, a_2 \rightarrow \mu, a_2, p_2(\mu)$$

$$\theta \rightarrow \mu', a_3 \rightarrow \mu', a_3, p_2(\mu')$$

$$\theta \rightarrow \mu' \rightarrow \mu'$$

- $$\Pr_\varphi(\mu | \theta) = \varphi_1(\mu, a_1 | \theta) + \varphi_1(\mu, a_2 | \theta)$$
Separating information and allocation design

\[ \varphi_1(\mu, a_1|\theta) \]

\[ \varphi_1(\mu', a_3|\theta) \]

\[ \mu, a_1 \rightarrow \mu, a_1, p_2(\mu) \]

\[ \mu, a_2 \rightarrow \mu, a_2, p_2(\mu) \]

\[ \mu', a_3 \rightarrow \mu', a_3, p_2(\mu') \]

\[ \text{Pr}_{\varphi}(\mu|\theta) = \varphi_1(\mu, a_1|\theta) + \varphi_1(\mu, a_2|\theta) \]

\[ \text{Pr}_{\varphi}(a_1|\theta, \mu) = \varphi_1(\mu, a_1|\theta)/\text{Pr}_{\varphi}(\mu|\theta) \]
Separating information and allocation design

- \( \text{Pr}_\varphi(\mu|\theta) = \varphi_1(\mu, a_1|\theta) + \varphi_1(\mu, a_2|\theta) \)

- \( \text{Pr}_\varphi(a_1|\theta, \mu) = \varphi_1(\mu, a_1|\theta) / \text{Pr}_\varphi(\mu|\theta) \)
Separating information and allocation design

- \( \Pr_{\varphi}(\mu|\theta) = \varphi_1(\mu, a_1|\theta) + \varphi_1(\mu, a_2|\theta) \)
- \( \Pr_{\varphi}(a_1|\theta, \mu) = \varphi_1(\mu, a_1|\theta)/\Pr_{\varphi}(\mu|\theta) \)
Separating information and allocation design

\[ \phi_1(s_1, a_1 | \theta) \]

\[ \phi_1(s_3, a_3 | \theta) \]

- \( \Pr_\varphi(\mu | \theta) = \phi_1(\mu, a_1 | \theta) + \phi_1(\mu, a_2 | \theta) \)

- \( \Pr_\varphi(a_1 | \theta, \mu) = \phi_1(\mu, a_1 | \theta) / \Pr_\varphi(\mu | \theta) \)
Separating information and allocation design

- $\Pr_\varphi(\mu|\theta) = \varphi_1(\mu, a_1|\theta) + \varphi_1(\mu, a_2|\theta)$
- $\Pr_\varphi(a_1|\theta, \mu) = \varphi_1(\mu, a_1|\theta)/\Pr_\varphi(\mu|\theta)$
Separating information and allocation design

- Separate the design of the information from that of the allocation
- $\beta$ is the mechanism's disclosure rule and $\alpha$ is the mechanism's allocation rule.
One last simplification

Quasilinearity + separation between allocation and information:

- No need to randomize on transfers: \( x(\mu_2) \) is the (expected) payment when output message is \( \mu_2 \)
- \( q(\mu_2) \) is the probability of selling the good when output message is \( \mu_2 \)
Thus, the seller’s optimal outcome solves:

$$\max_{\text{mechanisms}} \text{ Revenue}$$

where $M_1 = \Theta, S_1 = \Delta(\Theta), \varphi = \beta \otimes \alpha$ subject to

- Participation
- Truth telling
- Consistency between beliefs and output messages.
Thus, the seller’s optimal outcome solves:
Thus, the seller’s optimal outcome solves:

$$[x(\mu_2) + (1 - q(\mu_2))\delta R_2(\mu_2)]$$
Thus, the seller’s optimal outcome solves:

\[
\left( \sum_{\theta \in \Theta} \mu_1(\theta) \beta(\mu_2 | \theta) \right) \left[ x(\mu_2) + (1 - q(\mu_2)) \delta R_2(\mu_2) \right]
\]
Thus, the seller’s optimal outcome solves:

$$\max_{\beta,q,x} \sum_{\mu_2 \in \Delta(\Theta)} \left( \sum_{\theta \in \Theta} \mu_1(\theta) \beta(\mu_2 | \theta) \right) \left[ x(\mu_2) + (1 - q(\mu_2)) \delta R_2(\mu_2) \right]$$
Thus, the seller’s optimal outcome solves:

\[ R_1(\mu_1) \equiv \max_{\beta,q,x} \sum_{\mu_2 \in \Delta(\Theta)} \left( \sum_{\theta \in \Theta} \mu_1(\theta) \beta(\mu_2|\theta) \right) [x(\mu_2) + (1 - q(\mu_2))\delta R_2(\mu_2)], \]
Thus, the seller’s optimal outcome solves:

\[ R_1(\mu_1) \equiv \max_{\beta, q, x} \sum_{\mu_2 \in \Delta(\Theta)} \left( \sum_{\theta \in \Theta} \mu_1(\theta) \beta(\mu_2|\theta) \right) \left[ x(\mu_2) + (1 - q(\mu_2)) \delta R_2(\mu_2) \right], \]

subject to for all \( \theta \in \{\theta_L, \theta_H\} \):
Thus, the seller’s optimal outcome solves:

\[ R_1(\mu_1) \equiv \max_{\beta, q, x} \sum_{\mu_2 \in \Delta(\Theta)} \left( \sum_{\theta \in \Theta} \mu_1(\theta) \beta(\mu_2|\theta) \right) \left[ x(\mu_2) + (1 - q(\mu_2))\delta R_2(\mu_2) \right], \]

subject to for all \( \theta \in \{\theta_L, \theta_H\} \):

\[ \text{Participation}_\theta: \sum_{\mu_2 \in \Delta(\Theta)} \beta(\mu_2|\theta)(\theta q(\mu_2) - x(\mu_2) + (1 - q(\mu_2))\delta u^*(\mu_2, \theta)) \geq 0 \]
Thus, the seller’s optimal outcome solves:

\[
R_1(\mu_1) \equiv \max_{\beta, q, x} \sum_{\mu_2 \in \Delta(\Theta)} \left( \sum_{\theta \in \Theta} \mu_1(\theta) \beta(\mu_2 | \theta) \right) \left[ x(\mu_2) + (1 - q(\mu_2)) \delta R_2(\mu_2) \right],
\]

subject to for all \( \theta \in \{\theta_L, \theta_H\} \):

**Participation**\(_{\theta} \): \[ \sum_{\mu_2 \in \Delta(\Theta)} \beta(\mu_2 | \theta)(\theta q(\mu_2) - x(\mu_2) + (1 - q(\mu_2)) \delta u^*(\mu_2, \theta)) \geq 0 \]

**Truth telling**\(_{\theta, \theta'} \): \[ \sum_{\mu_2 \in \Delta(\Theta)} (\beta(\mu_2 | \theta) - \beta(\mu_2 | \theta'))(\theta q(\mu_2) - x(\mu_2) + (1 - q(\mu_2)) \delta u^*(\mu_2, \theta)) \geq 0 \]
Thus, the seller's optimal outcome solves:

\[ R_1(\mu_1) \equiv \max_{\beta,q,x} \sum_{\mu_2 \in \Delta(\Theta)} \left( \sum_{\theta \in \Theta} \mu_1(\theta)\beta(\mu_2|\theta) \right) \left[ x(\mu_2) + (1 - q(\mu_2))\delta R_2(\mu_2) \right], \]

subject to for all \( \theta \in \{\theta_L, \theta_H\} \):

- **Participation** \( \theta \):  
  \[ \sum_{\mu_2 \in \Delta(\Theta)} \beta(\mu_2|\theta)(\theta q(\mu_2) - x(\mu_2) + (1 - q(\mu_2))\delta u^*(\mu_2, \theta)) \geq 0 \]

- **Truth telling** \( \theta, \theta' \):
  \[ \sum_{\mu_2 \in \Delta(\Theta)} (\beta(\mu_2|\theta) - \beta(\mu_2|\theta'))(\theta q(\mu_2) - x(\mu_2) + (1 - q(\mu_2))\delta u^*(\mu_2, \theta)) \geq 0 \]

- **Consistency** \( \mu_2 \):
  \[ \mu_2(\theta_H) \left[ \sum_{\theta} \mu_1(\theta)\beta(\mu_2|\theta) \right] = \mu_1(\theta_H)\beta(\mu_2|\theta_H) \]
At the optimum, the following hold:

- The seller extracts all surplus from low-valuation buyer ($\Rightarrow$ participation binds for $\theta_L$)
- High-valuation buyer is indifferent between reporting $\theta_H$ and $\theta_L$ ($\Rightarrow$ Truthtelling binds for $\theta_H$)
At the optimum, the following hold:

- The seller extracts all surplus from low-valuation buyer (⇒ participation binds for $\theta_L$)
- High-valuation buyer is indifferent between reporting $\theta_H$ and $\theta_L$ (⇒ Truth telling binds for $\theta_H$)

Hence, we can rewrite the problem as:
At the optimum, the following hold:

- The seller extracts all surplus from low-valuation buyer (⇒ participation binds for $\theta_L$)
- High-valuation buyer is indifferent between reporting $\theta_H$ and $\theta_L$ (⇒ Truth telling binds for $\theta_H$)

Hence, we can rewrite the problem as:

$$q(\mu_2)$$
At the optimum, the following hold:

- The seller extracts all surplus from low-valuation buyer (⇒ participation binds for \( \theta_L \) )
- High-valuation buyer is indifferent between reporting \( \theta_H \) and \( \theta_L \) (⇒ Truthtelling binds for \( \theta_H \) )

Hence, we can rewrite the problem as:

\[
q(\mu_2)(\mu_2\theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_1))
\]
At the optimum, the following hold:

- The seller extracts all surplus from low-valuation buyer (⇒ participation binds for $\theta_L$)
- High-valuation buyer is indifferent between reporting $\theta_H$ and $\theta_L$ (⇒ Truthfulness binds for $\theta_H$)

Hence, we can rewrite the problem as:

$$q(\mu_2)(\mu_2 \theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_1)) + (1 - q(\mu_2))$$
At the optimum, the following hold:

- The seller extracts all surplus from low-valuation buyer (⇒ participation binds for $\theta_L$)
- High-valuation buyer is indifferent between reporting $\theta_H$ and $\theta_L$ (⇒ Truth-telling binds for $\theta_H$)

Hence, we can rewrite the problem as:

$$q(\mu_2)(\mu_2\theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_1)) + (1 - q(\mu_2))\delta R(\mu_2; \mu_1)$$
At the optimum, the following hold:

- The seller extracts all surplus from low-valuation buyer ($\Rightarrow$ participation binds for $\theta_L$)
- High-valuation buyer is indifferent between reporting $\theta_H$ and $\theta_L$ ($\Rightarrow$ Truth telling binds for $\theta_H$)

Hence, we can rewrite the problem as:

$$\left(\sum_{\theta \in \Theta} \mu_1(\theta) \beta(\mu_2 | \theta)\right) \left[ q(\mu_2) (\mu_2 \theta_H + (1 - \mu_2) \hat{\theta}_L(\mu_1)) + (1 - q(\mu_2)) \delta R(\mu_2; \mu_1)\right]$$
At the optimum, the following hold:

• The seller extracts all surplus from low-valuation buyer (⇒ participation binds for $\theta_L$)
• High-valuation buyer is indifferent between reporting $\theta_H$ and $\theta_L$ (⇒ Truth-telling binds for $\theta_H$

Hence, we can rewrite the problem as:

$$\tau(\mu_2) \left[ q(\mu_2)(\mu_2 \theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_1)) + (1 - q(\mu_2))\delta R(\mu_2; \mu_1) \right]$$
Seller optimal outcome

At the optimum, the following hold:

- The seller extracts all surplus from low-valuation buyer (⇒ participation binds for $\theta_L$)
- High-valuation buyer is indifferent between reporting $\theta_H$ and $\theta_L$ (⇒ Truth-telling binds for $\theta_H$)

Hence, we can rewrite the problem as:

$$R_1(\mu_1) = \max_{\tau, q} \sum_{\mu_2 \in \Delta(\Theta)} \tau(\mu_2) \left[ q(\mu_2)(\mu_2 \theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_1)) + (1 - q(\mu_2))\delta R(\mu_2; \mu_1) \right]$$
At the optimum, the following hold:

- The seller extracts all surplus from low-valuation buyer (⇒ participation binds for $\theta_L$)
- High-valuation buyer is indifferent between reporting $\theta_H$ and $\theta_L$ (⇒ Truth-telling binds for $\theta_H$)

Hence, we can rewrite the problem as:

$$R_1(\mu_1) = \max_{\tau,q} \sum_{\mu_2 \in \Delta(\Theta)} \tau(\mu_2) \left[ q(\mu_2)(\mu_2 \theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_1)) + (1 - q(\mu_2))\delta R(\mu_2; \mu_1) \right]$$

subject to

$$\sum_{\mu_2 \in \Delta(\Theta)} \tau(\mu_2)\mu_2(\theta_H) = \mu_1(\theta_H)$$
At the optimum, the following hold:

- The seller extracts all surplus from low-valuation buyer (⇒ participation binds for $\theta_L$)
- High-valuation buyer is indifferent between reporting $\theta_H$ and $\theta_L$ (⇒ Truth-telling binds for $\theta_H$)

Hence, we can rewrite the problem as:

$$R_1(\mu_1) = \max_{\tau, q} \sum_{\mu_2 \in \Delta(\Theta)} \tau(\mu_2) \left[ q(\mu_2)(\mu_2 \theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_1)) + (1 - q(\mu_2))\delta R(\mu_2; \mu_1) \right]$$

subject to

$$\sum_{\mu_2 \in \Delta(\Theta)} \tau(\mu_2) \mu_2(\theta_H) = \mu_1(\theta_H)$$
Sale of a durable good: $t = 1$

$$\delta R_2(\mu_2; \mu_1) = \begin{cases} 
\delta(\mu_2\theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_1)) & \text{if } \mu_2 < \bar{\mu} \\
\delta\mu_2\theta_H & \text{if } \mu_2 > \bar{\mu}
\end{cases}$$
Sale of a durable good: $t = 1$

\[ \delta R_2(\mu_2; \mu_1) = \begin{cases} 
\delta(\mu_2 \theta_H + (1 - \mu_2) \hat{\theta}_L(\mu_1)) & \text{if } \mu_2 < \bar{\mu} \\
\delta \mu_2 \theta_H & \text{if } \mu_2 > \bar{\mu}
\end{cases} \]
Sale of a durable good: $t = 1$

$$\delta R_2(\mu_2; \mu_1) = \begin{cases} 
\delta (\mu_2 \theta_H + (1 - \mu_2) \hat{\theta}_L(\mu_1)) & \text{if } \mu_2 < \bar{\mu} \\
\delta \mu_2 \theta_H & \text{if } \mu_2 > \bar{\mu}
\end{cases}$$
Sale of a durable good: $t = 1$

$$
\delta R_2(\mu_2; \mu_1) = \begin{cases} 
\delta (\mu_2 \theta_H + (1 - \mu_2) \hat{\theta}_L(\mu_1)) & \text{if } \mu_2 < \bar{\mu} \\
\delta \mu_2 \theta_H & \text{if } \mu_2 > \bar{\mu}
\end{cases}
$$
The "Bayesian persuasion" feel is a consequence of $S \simeq \Delta(\Theta) \Rightarrow$ Constrained Information Design

$$\max_{\tau, q} \sum_{\mu_2 \in \Delta(\Theta)} \tau(\mu_2) \left[ q(\mu_2)(\mu_2 \theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_1)) + (1 - q(\mu_2))\delta R_2(\mu_2; \mu_1) \right]$$
 Seller optimal outcome

\[
\max_{\tau,q} \sum_{\mu_2 \in \Delta(\Theta)} \tau(\mu_2) \left[ 0 \times (\mu_2 \hat{\theta}_H + (1 - \mu_2) \hat{\theta}_L(\mu_1)) + 1 \times \delta R_2(\mu_2; \mu_1) \right]
\]
Seller optimal outcome

\[
\max_{\tau,q} \sum_{\mu_2 \in \Delta(\Theta)} \tau(\mu_2) \left[ 1 \times (\mu_2 \theta_H + (1 - \mu_2)\hat{\theta}_L(\mu_1)) + 0 \times \delta R_2(\mu_2; \mu_1) \right]
\]

The "Bayesian persuasion" feel is a consequence of $S \approx \Delta(\Theta) \Rightarrow$ Constrained Information Design
The "Bayesian persuasion" feel is a consequence of

\[
\max_{\tau} \sum_{\mu_2 \in \Delta(\Theta)} \tau(\mu_2) \max\{\mu_2 \theta_H + (1 - \mu_2) \hat{\theta}_L(\mu_1), \delta R(\mu_2; \mu_1)\}
\]
The "Bayesian persuasion" feel is a consequence of $\mathcal{S} \simeq \Delta(\Theta) \Rightarrow$ Constrained Information Design.

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The "Bayesian persuasion" feel is a consequence of \( S \approx \Delta(\Theta) \Rightarrow \) Constrained Information Design.
• Seller splits $\mu_1$ between $\mu_2 = \bar{\mu}$ and $\mu_2 = 1$.
• He sells when $\mu_2 = 1$ ($q(1) = 1$) and delays when $\mu_2 = \bar{\mu}$ ($q(\bar{\mu}) = 0$).
• Posted price of $\theta_H$ in both periods.
Seller optimal outcome

- Seller splits $\mu_1$ between $\mu_2 = \bar{\mu}$ and $\mu_2 = 1$
- He sells when $\mu_2 = 1$ ($q(1) = 1$) and delays when $\mu_2 = \bar{\mu}$ ($q(\bar{\mu}) = 0$)
- Posted price of $\theta_H$ in both periods.

The “Bayesian persuasion” feel is a consequence of $S \simeq \Delta(\Theta)$
⇒ Constrained Information Design
**Economic trade-off:** tailor the allocation to the agent’s report vs. learning about the agent’s type.

- No such trade-off when there is commitment: acquired information can always be “forgotten.”
- The seller slows down learning:
  - Similar to Kanoria & Nazerzadeh, 2014; Abernethy et al., 2019; Haghtalab, Lykouris, Nietert, & Wei, 2022
Open questions
This is a problem that had been open in Economics for 30 years. There’s much to do!

1. Most glaring: multiple agents (the existing counterexamples do not survive with our mechanisms)
   - How to aggregate the information from the multiple agents? (e.g., Halpern & Teague, 2006)

2. More practical: How to implement direct-Blackwell mechanisms?
   - Multiple (infinite?) rounds of indirect observable communication?
   - Cryptographic commitments? (e.g., Ferreira & Weinberg, 2020)