Schelling Segregation

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Based on joint works with Christina Brandt, Gautam Kamath, Robert D. Kleinberg, Brendan Lucier, and M. Zadomighaddam
Theory: How do individual decisions impact population-level phenomena?
MODEL:

one-dimensional

two-dimensional
INDIVIDUAL PREFERENCES:

$n$ nodes, two types

neighborhood = $2w + 1$ nearest neighbors, happy if at least $\tau = 0.5$ neighbors of like-type

Pr[\( \bullet = \bullet \)] = Pr[\( \bullet = \bigcirc \)] = $\frac{1}{2}$

model parameters:

- tolerance $\tau = 0.5$
- window size $w$
- society size $n$
INDIVIDUAL BEHAVIOR:

Each day, two randomly selected individuals swap nodes if both unhappy oppositely-colored.
MEASURE OF SEGREGATION:

A run is a maximal sequence of like-colored individuals.
Behavior may converge to a variety of states.
HISTORY:

Simulations: For $n = 70$ and $w = 4$, the average segregation was 12.

- **societal impact**: shifted discourse about segregation which until then had been attributed to discrimination.
- **theoretical impact**: became archetypical example of global emergent structure from simple local rules.

[Schelling’69]
HISTORY:

Approximations predict total segregation

- Stochastically stable states: In a perturbation, states minimize the number of bi-chromatic edges. [Young'01], [Zhang'04]
- Spin systems: Monochromatic with high probability as temperature approaches 0. [Bhakta, Miracle, and Randall’14]

But approximations have exponential mixing time; predictions contradict simulations.
TODAY:

The dynamics converge after poly\((n)\) steps with \(O(w)\) segregation.

The distribution of run lengths is such that for all \(\lambda > 0\), the probability a randomly selected node is in a run of length \(> \lambda w\) is bounded above by \(c^\lambda\) for some constant \(c < 1\).

[Brandt, Immorlica, Kamath, Kleinberg, 2012]
KEY STRUCTURE:

Defn. A **firewall** is a sequence of $w + 1$ consecutive individuals of the same type.

Claim: Firewalls stable with respect to dynamics.
Corollary: Segregation at most distance bt firewalls.
BIRTH OF FIREWALLS:

Simulation with $n = 10000$, $w = 10$. 

Time $t = 0$
BIRTH OF FIREWALLS:

time $t = 40$

Simulation with $n = 10000, w = 10$. 
BIRTH OF FIREWALLS:

time $t = 80$

Simulation with $n = 10000, w = 10$. 
BIRTH OF FIREWALLS:

time $t = 120$

Simulation with $n = 10000$, $w = 10$. 
BIRTH OF FIREWALLS:

time \ t = 160

Simulation with \ n = 10000, \ w = 10.
EMERGENT STRUCTURE:

Simulation with $n = 10000$, $w = 10$.

Time $t = 260$ (final)
1. Define *firewall incubators*, frequent at initialization.
2. Show firewall incubators are likely to become firewalls.

**TECHNIQUE:**

A ● firewall incubator, rich in ●-type nodes.

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Definition. The bias of a node at time $t$ is the sum of the signs of sites in its neighborhood.
FIREWALL INCUBATORS:

Definition. A firewall incubator is a sequence of 3 sufficiently long and sufficiently biased blocks.

Diagram: A firewall incubator is represented by a sequence of symbols, indicating the structure and characteristics of the blocks involved. The diagram illustrates the concept visually, with various symbols and lines representing the different components of the incubator.
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**TECHNIQUE:**

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THE BIRTH OF A FIREWALL:

To show this incubator turns into a blue firewall:

1. analyze random order of flips instead of swaps
2. argue due to bias all good events happen before too many bad events with probability $O(1/w)$. 
FLIPS VERSUS SWAPS:

Swaps are random order if there is
# unhappy red = # unhappy blue.
DEFINE STATE VARIABLES:

- \( \sigma \in \{-1, +1\}^{w+1} \) is a labeling of a neighborhood
- \( X_\sigma(t) = \# \) of nodes with neighborhood labeling \( \sigma \)
DEFINE STATE VARIABLES:

ring of size $n$

tainted nodes (initially, $w + 1$)

\[ \frac{n}{L(w)} \] rings of size $L(w)$
DEFINE STATE VARIABLES:

• $\sigma \in \{-1, +1\}^{L(w)}$ is a labeling of a subring
• $X_{\sigma}(t) =$ number of nodes $i$ such that subring containing $i$ has label $\sigma$ at time $t$ (clockwise starting from $i$)
DEFINE STATE VARIABLES:

• \( \sigma \in \{-1, +1\}^{L(w)} \): labeling of a subring
• \( X_\sigma(t) \): label clockwise from \( i \) is \( \sigma \) at time \( t \)

Question. \# unhappy red – \# unhappy blue?

\[ \Delta(\vec{X}) = \sum_{\sigma} u(\sigma) X_\sigma(t) \]

where \( u(\sigma) = +1 \) if starts with unhappy red, and \( u(\sigma) = -1 \) if starts with unhappy blue
CALCULATE EXPECTATION:

For all $j, \sigma', \sigma''$, let $a_{\sigma}(j, \sigma', \sigma'') = 1$ if swapping node 1 of $\sigma'$ with node $j$ of $\sigma''$ creates labeling $\sigma$ (and correspondingly for $-1$).

\[
E\left[ X_{\sigma}(t + 1) - X_{\sigma}(t) \middle| G_t \right] = \sum_{j, \sigma', \sigma''} 2a(j, \sigma', \sigma'') \left( \frac{X_{\sigma'}(t)}{n} \right) \left( \frac{X_{\sigma''}(t)}{n} \right) + O \left( \frac{L(w)}{n} \right)
\]
CHECK WORMALD CONDITIONS:

\[
E[ X_\sigma(t + 1) - X_\sigma(t) | G_t ] \\
= \sum_{j, \sigma', \sigma''} 2a(j, \sigma', \sigma'') \left( \frac{X_{\sigma'}(t)}{n} \right) \left( \frac{X_{\sigma''}(t)}{n} \right) + O \left( \frac{L(w)}{n} \right)
\]

- Bounded? \(|X_\sigma(t + 1) - X_\sigma(t)| \leq 2L(w)\)
- Lipschitz? \(f(y, \{x_\sigma\})\) bounded quadratic
SOLVING DIFF EQ:

\[
\frac{dx_\sigma(y)}{dy} = \sum 2a(...)(y)x_\sigma'(y)x_\sigma''(y)
\]

We actually only care about balance:

\[
\Delta(\tilde{\chi}) = \sum u(\sigma)x_\sigma
\]

where \(u(\sigma) = +1\) if first node is unhappy blue, and \(u(\sigma) = -1\) otherwise.
AVOIDING DIFF EQS:

Flipping labels. Let $\bar{\sigma}$ be “flip” of labeling $\sigma$ and $\iota(\hat{x})$ be vector whose $\sigma^{th}$ component is $x_{\bar{\sigma}}$.

1. vector $\iota(\hat{x})$ is permutation of state vector $\hat{x}$

2. fixed point set $\{\hat{x} | \iota(\hat{x}) = \hat{x}\}$ is
   • balanced as $x_{\sigma} = x_{\bar{\sigma}}$ and $u(\sigma) = -u(\bar{\sigma})$
   • invariant under diff eq by symmetry

3. initial state close to fixed point set w.h.p.
   and all close points have high balance
CONCLUDING BALANCE:

**Theorem.** For all sufficiently large $n$, with high probability the # of unhappy red and blue are approximately balanced for sufficiently long.

**Proof Sketch.** This is true of differential equation and therefore true of discrete process for as long as diff eq tracks discrete process closely.
Theorem. Average segregation is $O(w^2)$.

Proof. A block of length $O(w)$ contains an incubator with constant probability. This incubator turns into a firewall with probability $\Omega(1/w)$.

Linear result: define a stronger incubator.
EPILOGUE:

On preaching tolerance...

$n$ nodes, two types

happy if at least $\tau = 0.5$

neighbors of like-type

Segregation exponential in $w$ for $\tau = 0.5 - \epsilon$

[Barmpalias, Elwes, Lewis-Pye, 2014]