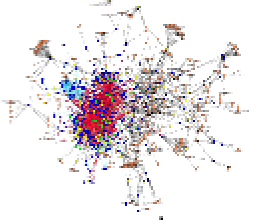

Local Convergence for Sparse Graphs

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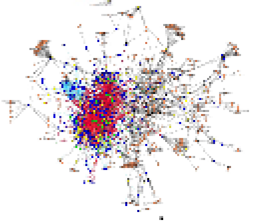


Introduction

Questions: Given a sequence G_n of graphs with $|V(G_n)| = n \to \infty$, what is the "right" notion of convergence? What is the limit?

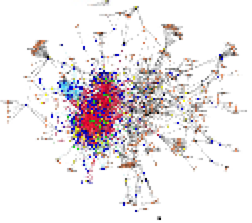
Equivalent notions for dense graphs:

- Convergence of sampled subgraphs
- Convergence of subgraph counts
- Convergence of multi-way cuts
- Convergence of micro-canonical free energies

What about Sparse graphs, with

$$\bar{d}_n = \frac{2}{n} |\mathrm{E}(\mathrm{G_n})|,$$

uniformly bounded for all n



Introduction

Problem: The dense notions are not suitable for bounded degree graphs.

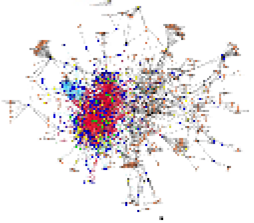
Example: If we sample x_1, x_2 uniformly at random

$$\Pr(x_1 x_2 \in E) = \frac{1}{n^2} 2|E| = \frac{1}{n} \bar{d}_n \to 0$$

Thus sampling gives empty graphs if we sample k vertices at random (with or without replacement)

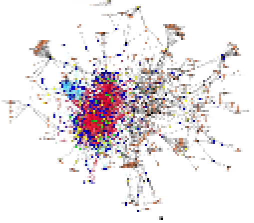
Solution: Sample only one vertex, and look at local neighborhoods

Def [BS'01]: A sequence G_n is called locally convergent, or Benjamini-Schramm convergent if for all $R < \infty$, the distribution of the R-neighborhood around a uniformly randomly vertex $x \in V(G_n)$ is convergent



Continue on White Board

- Examples
- random vs. deterministic graphs
- unimodularity,
- Existence questions



Examples

Ex1: The sequences $\{1,2,\ldots,n\}^d$ and $(\mathbb{Z}/n\mathbb{Z})^d$ converge to the rooted graph $(0,\mathbb{Z}^d)$

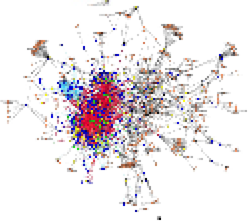
Ex2: Let $G_{n,d}$ be the d-regular random graph and $B_{n,d}$ be the d-regular bipartite random graph. Both sequences are BS-convergence, and converge to the infinite d-regular tree, but have very different global properties:

- $\mathrm{MaxCut}(B_{n,d}) = \frac{dn}{2}$
- $\mathrm{MaxCut}(G_{n,d}) \approx \frac{dn}{4}$

Relation to Notions from Dense Convergence

- Subgraph Convergence
- Convergence of weighted multi-way cuts
- Convergence of free energies

For simplicity, I will assume that the maximal degree of G_n is uniformly bounded by some $D < \infty$



2a) Subgraph counts

Def: Subgraph frequencies: Given a graph $G = (V, E)$ with adjacency matrix A and a connected graph H on k nodes, define

$$\nu(H,G) = \frac{1}{|V|} \sum_{v_1,\dots,v_k \in V} \prod_{ij \in E(H)} A_{v_i v_j} \prod_{ij \notin E(H)} (1 - A_{v_i v_j})$$

where the sum goes over distinct sets of vertices

Def: Subgraph Count Convergence:

- For all finite graphs H, $\nu(H, G_n)$ converges to some $\nu(H) \in \mathbb{R}$

Rem: It is easy to see that for bounded degree graphs, this is equivalent to BS-convergence. We call these two notions left-convergence

2c) Multiway Cuts

Multiway-cuts: Given $J \in \mathbb{R}^{k\times k}$ and $\sigma: V \to [k]$ define*

$$E_{G,J}(\sigma) = \frac{1}{n} \sum_{x,y:\{x,y\}\in E} J_{\sigma(x)\sigma(y)}$$

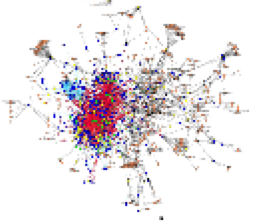
and for $\alpha \in \Delta_k$, set

$$MinCut_{J,\alpha}(G) = \min_{\sigma} E_{G,J}(\sigma)$$

where the minimum goes over all maps $\sigma: V \to [k]$ such that

$$\big|\, |\sigma^{-1}(\{i\})| - n\alpha_i \big| \leq 1 \text{ for all } i \in [k]$$

*) Note the different normalization by $\frac{1}{n}$



2c) Multiway Cuts

Q: Does left-convergence imply convergence of multi-way cuts?

Ex: Take G_n to be $G_{n,d}$ for odd n and $B_{n,d}$ for even n. Both sequences are BS-convergent to the infinite d-regular tree.

But for d large, we have that

$$\mathrm{MaxCut}(B_{n,d}) = \frac{dn}{2}$$

$$\mathrm{MaxCut}(G_{n,d}) \approx \frac{dn}{4}$$

As a consequence, the multi-way cuts of G_n are not convergent. Thus left convergence does NOT imply convergence of multi-way cuts

2c) Multiway Cuts

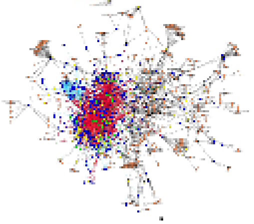
Q: Does convergence of multi-way cuts imply left convergence?

Ex: Take G_n to be

- a union of $\lceil \frac{n}{4} \rceil$ 4-cycles for odd n and
- a union of $\lceil \frac{n}{6} \rceil$ 6-cycles for even n.

Then $\mathrm{MaxCut}(G_n) = \frac{1}{2}\,|V(G_n)|$

More general, it is not hard to show that all multi-way cuts of G_n are convergent. But G_n is clearly not left convergent, so convergence of multi-way cuts does NOT imply left convergence either.



2d) Statistical Physics

Def: Free energy

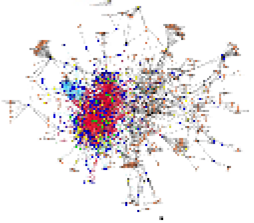
$$F_J(G) = -\frac{1}{n}\log Z_J(G)$$

where $Z_J(G)$ is the partition function

$$Z_J(G) = \sum_{\sigma:V\to[k]} e^{-nE_{G,J}(\sigma)} = \sum_{\sigma:V\to[k]} e^{-\sum_{x,y:\{x,y\}\in E} J\sigma(x)\sigma(y)}$$

and the sum is over all $\sigma: V \to [k]$

Rem: We call this right convergence



2c) Statistical Physics

Q: Does left convergence imply right convergence?

A: NO. Take $k = 2$; $J_{11} = J_{22} = 0$ and $J_{12} = -1$. Then $-nE_{G,J}(\sigma)$ is just the cut between sets of vertices of different color, and

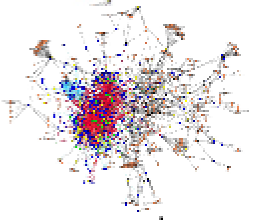
$$e^{MaxCut(G)} \leq Z_J(G) = \sum_{\sigma:V\to[k]} e^{-nE_{G,J}(\sigma)} \leq 2^n e^{MaxCut(G)}$$

Take the left convergent sequence G_n that is $G_{n,d}$ for odd n and $B_{n,d}$ for even n. For large d,

$$\mathrm{MaxCut}(B_{n,d}) = \frac{dn}{2} \text{ and } \mathrm{MaxCut}(G_{n,d}) \approx \frac{dn}{4}.$$

Thus the free energy, $F_J(G) = -\frac{1}{n}\log Z_J(G)$ will not converge if d is large

Thm [BCKL'13]: Right Convergence implies left convergence



2d) Statistical Physics

Thm [BCKL'13]: Right Convergence implies left convergence

Proof Idea: Recall that $F_J(G) = -\frac{1}{n}\log Z_J(G)$ where we can write $Z_J(G)$ as

$$Z_J(G) = \sum_{\sigma: V \to [k]} e^{-\sum_{x,y:\{x,y\}\in E} J_{\sigma(x)\sigma(y)}}$$

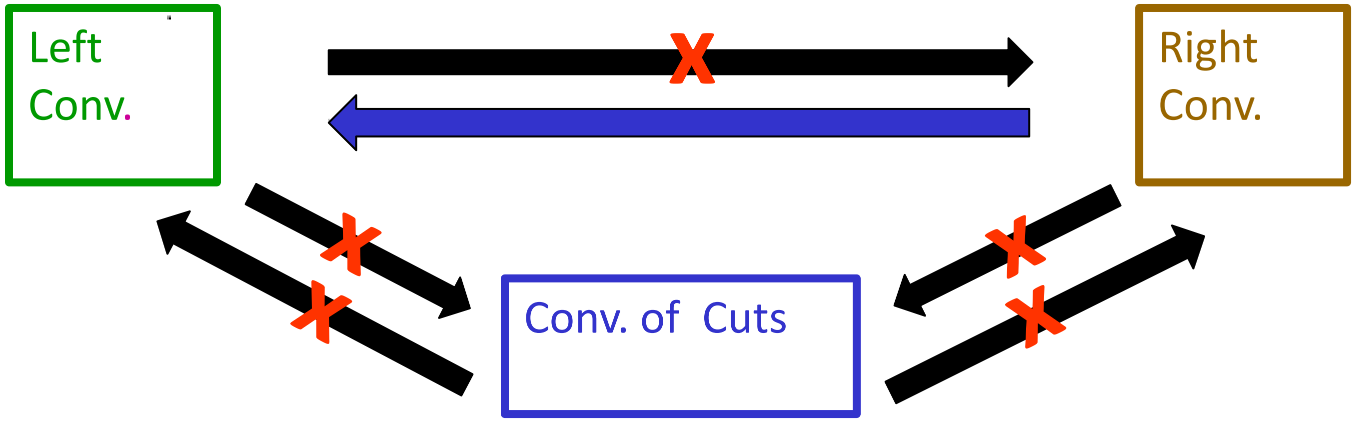
For small J, we can expand $F_J(G)$ into a convergent power series in the J_{ij}'s which converges uniformly in G as long as G has a uniform bound on the maximal degree.

Using this, one can prove that for a left-convergent sequence

- the limiting free energies F_J exists, and
- can be expressed write it in terms of the limiting subgraph counts $\nu(H)$
- Finally, by comparing terms in the power series in J, one can invert this series, and determine $\nu(H)$ in terms of the limiting free energies F_J

But this implies the statement. Indeed, if it were false, we could find two subsequences converging to different subgraph frequencies $\nu(H)$ with the same limiting free energies F_J, a contradiction.

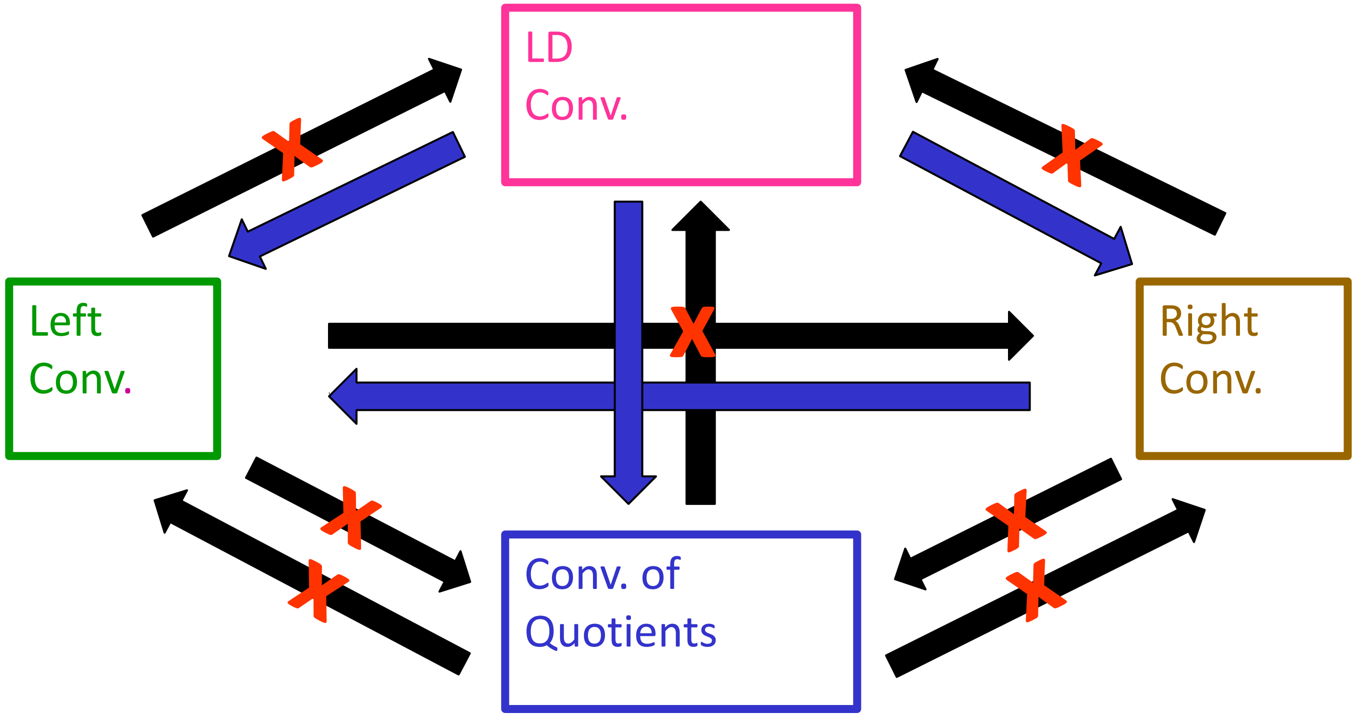
Other implications?

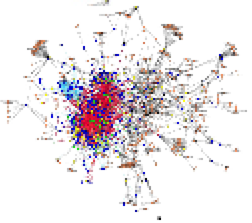


Question: is there a convergence notion which implies all of these?

Answer: Yes. Study random colorings, look at the distribution of all cuts, and study large deviations of these.

Summary
bounded degree convergence





Thank you!