The "standard" stochastic SIR epidemic

(Tom Britton, Aug 30, 2022)

Susceptible → Infectious → Recovered (+ immune)

(SIS, SIRS, SEIR, ...)
$\quad$ E: Latent

Standard:
- Fixed closed community of size n (large)
- No changing behavior or preventions
- all individuals are "identical"
- all individuals mix uniformly

Model (continuous time model)

- An individual who gets infected becomes infectious for a duration $I \sim F$ (i.i.d)
  - a) Markovian $I \sim Exp(\gamma)$
  - b) Reed-Frost $I \equiv 1$
- While infectious an individual has "inf contacts" at rate β (with a uniformly selected individual). If contacted is susceptible → infectious, otherwise nothing happens.

Initiation: $S(0) = n-1$, $I(0) = 1$, $R(0) = 0$

$(S(t), I(t), R(t)$ = # susc, inf rec at t
$S+I+R=n)$

Epidemic goes until $T=\inf\{t; I(t)=0\}$
$P(T<\infty)=1$. $R(T)=n-S(T)$ = final size

R_0 = basic repr. number $\beta \cdot E(I)=$
$= E$(inf contacts during inf-period)
$\approx E$(infections in beginning of epidemic)

Questions of interest | Answers

a) How does epidemic behave in beginning? | $\approx$ Br pr.

b) — " — in main phase | (Markovian model) $\approx$ deterministic process with Gaussian deviations

only super-critical situation

c) — " — in the end | $\approx$ Br pr starting with many individuals having drift

d) Final size $Z = R(T)$? $\frac{Z}{n} \to \begin{cases} 0 \\ \mathcal{I} \end{cases}$

$\mathcal{I}$ solves $1-x = e^{-R_0 x}$

~~Rob~~ $P(\frac{Z}{n} \to \mathcal{I})$ depends on $I \sim F$

R-F: $P(\frac{Z}{n} \to \mathcal{I}) = \mathcal{I}$
Exp $= 1 - \frac{1}{R_0}$

e) How long will epidemic last?

$\underbrace{C_1 \cdot \log n}_{\text{beginning}} + \underset{\uparrow}{C_2}_{\text{main phase}} + \underset{\uparrow}{C_3 \cdot \log n}_{\text{end}}$

My goal is to sketch proofs of a, d, b

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a) Initial phase Epid $\approx$ Br pr

Why? "all" infections results infection in beginning

$\Rightarrow$ Br pr regime

Coupling of Br pr & epidemic (for all n)

Random quantities

iid Po pr with rate β: $X_1(\cdot), X_2(\cdot), \ldots$

iid inf-periods $\sim F$ $\quad I_1, I_2, \ldots$

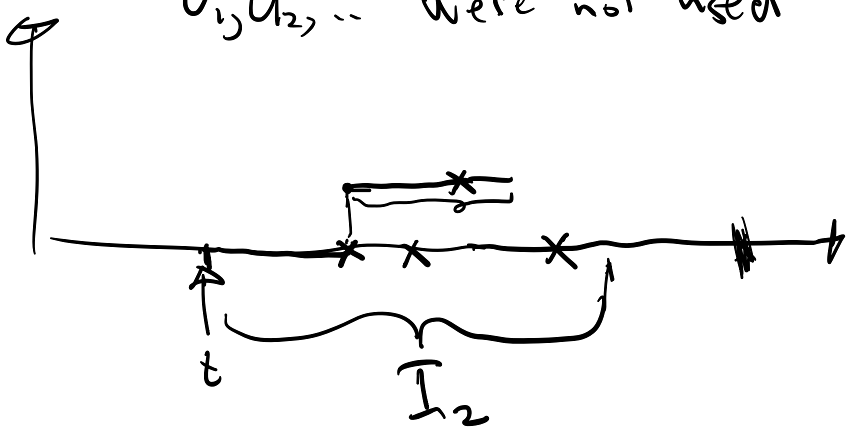
iid $U[0,1]$ $\quad U_1, U_2, \ldots$

Constr of Br pro

1. First individual infects (= give birth) according to $X_1(\cdot)$ up until $t = I_1$ recovers (dies)
2. As new people get infected (= born) they start infecting according to $X_2(\cdot)$ until $s = I_2$ and so on

$\Rightarrow$ Br pr with constant birth rate β and life length I

$U_1, U_2, \ldots$ were not used



Construct epidemic for size n

Label individuals $1, \dots, n$

Index case has label 1

Index case has inf. contacts according $\Xi_1(\cdot)$ up until $t = I_1$

Each new infection is with individual

$\beta' = \frac{n-1}{n}\beta$

$\lceil U_i n \rceil \in \{1, 2, \dots, n\}$. If this individual has not been infected before → infection other do nothing "ghost event"

$\Xi_2(\cdot)$, I_2,

and so on

⇒ Produces the standard SIR epidemic

Observation: Br pr & epidemic are identical up until first ghost event

$$P(\text{no ghost event among first } k \text{ infections}) =$$

$$1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right)$$

$$\approx e^0 \cdot e^{-\frac{1}{n}} \cdot e^{-\frac{2}{n}} \cdots e^{-\frac{k-1}{n}}$$

$$= e^{-\frac{1}{n}\frac{k(k-1)}{2}} \longrightarrow 1 \quad \text{if } k = o(\sqrt{n})$$

Epid $\approx$ Br up until $\approx \varepsilon\sqrt{n}$

Epid is subcritical if $R_0 = \beta \cdot E[I] < 1$

cr $= 1$

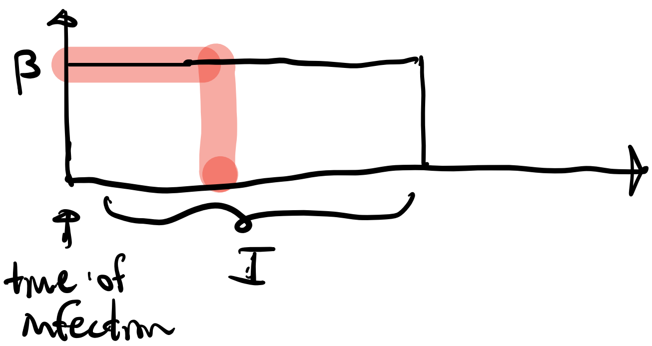
super > 1

From Br pr we can compute

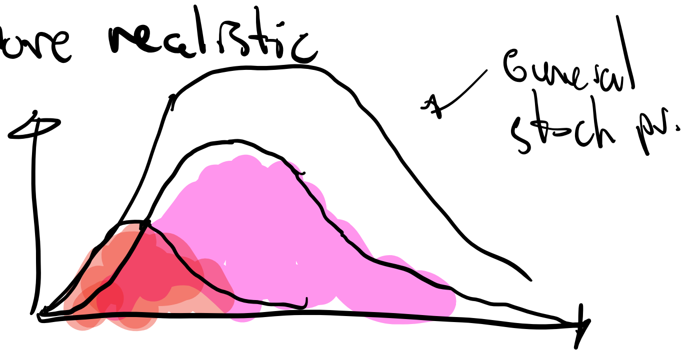
$P(\text{final size} = k)$

Q $\quad P(\text{minor outbreak}) = \sum_i P(\text{final size} = k)$

SIR



More realistic



For the final size Z only the distribution of integral

$\int I(s)\,ds$

Main phase of epidemic (Markovian version)

Density dependent jump Markov processes (Kurtz)

Initiation $S(0) = n - \varepsilon n$, $I(0) = \varepsilon n$ $R(0) = 0$

$$\frac{S(t)}{n}, \frac{I(t)}{n} \approx s(t), i(t)$$

$$Z_n(t) = (S(t), I(t)) \qquad \bar{Z}_n(t) = \frac{1}{n} Z_n(t)$$

Suppose $\bar{Z}_n(t) = (s, i)$

Two possible event

Infection: $(s, i) \to (s, i) + (-1, 1)$ with rate $i \cdot \beta \cdot \frac{s}{n}$

Recovery $(s, i) \to (s, i) + (0, -1)$ $\qquad i \cdot \gamma$

We now construct the epidemic using two indep rate 1 $\mathbb{P}_0$-pr $\underbrace{Y_{(-1,+1)}}_{\text{counts inf.}}(\cdot)$ $Y_{(0,1)}(\cdot)$ counts recoveries

Set $\beta_{(-1,1)}(x, y) = \beta \cdot x \cdot y \qquad \beta_{(0,1)}(x, y) = \gamma \cdot y$

$$\frac{1}{h} P\big(Z_n(t+h) = (S,I)+(-1,1) \mid Z_n(t) = (S,I)\big) = \beta \cdot I \cdot \frac{S}{n} + o(1)$$

$$= n \cdot \beta \cdot \frac{I}{n} \cdot \frac{S}{n} = n \cdot \beta_{(-1,1)}\left(\frac{S}{n}, \frac{I}{n}\right) = n\,\beta_{(-1,1)}(\bar{Z}_n(t))$$

$$\frac{1}{h} P\big(Z_n(t+h) = (S,I)+(0,1) \mid Z_n(t) = (S,I)\big) = \gamma \cdot I = \dots = n \cdot \beta_{(0,1)}(\bar{Z}_n(t))$$

$$Z_n(t) = Z_n(0) + (-1,1)\, Y_{(-1,1)}\left(n \int_0^t \beta_{(-1,1)}(\bar{Z}_n(s))\, ds\right) + (0,-1) \cdot Y_{(0,-1)}\left(n \int_0^t \beta_{(0,-1)}(\bar{Z}_n(s))\, ds\right)$$

Centered Poisson pr. $\hat{Y}$ $\quad \hat{Y}(t) = Y(t) - t$

$$\Rightarrow \bar{Z}_n(t) = \bar{Z}_n(0) + \underbrace{\frac{1}{n}(-1,1)\, \hat{Y}_{(-1,1)}(\dots)}_{} \xrightarrow{P} 0$$

$$+ \frac{1}{n}(0,-1)\, \hat{Y}_{(0,-1)}(\dots) \xrightarrow{P} 0$$

$$+ \int_0^t F(\bar{Z}_n(s))\, ds$$

$(-1,1)\,\beta_{(-1,1)}(\bar{Z}_n(s)) + (0,-1)\,\beta_{(0,-1)}(\bar{Z}_n(s))$

$\Rightarrow \bar{Z}_n(t) \xRightarrow{P} Z(t)$ & det. defined by + Gronwall inequality assures that deviations don't "explode"

$$Z(t) = Z(0) + \int_0^t F(Z(s))\,ds$$

$\Updownarrow$

— $s(t)$, $i(t)$ fractions $\begin{cases} s'(t) = -\beta\, s(t)\cdot i(t) \\ i'(t) = +\beta\, s(t)\, i(t) - \gamma\, i(t) \end{cases}$

Det SIR epidemic

$$\sqrt{n}\left(\frac{1}{n} Y(n \cdot t) - t\right) \xRightarrow{W} W(t)$$

$$\sqrt{n}\left(\bar{Z}_n(t) - Z(t)\right) \xRightarrow{W} \text{Gaussian process}$$

$\partial F(Z(s))$

None of the results above are "mine"

But all are found in my books (with more details)

Andersson & Britton (2000) (reprinted)

Diekmann, Heesterbeek & Britton (2013)

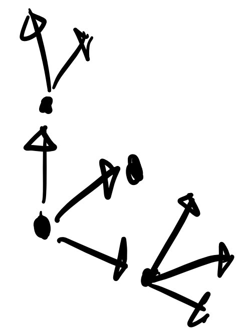
A short follow up:

P(minor outbreak) $= \pi$ and an expression for

$\tau =$ final fraction getting infected in case of major outbreak

Initial phase of epidemic $\approx$ Br. pr.

Offspring distribution?



Given $I=i$

you infect $\mathrm{Bin}(n-1,\ 1-e^{-\frac{\beta}{n}\cdot i})$

$$\approx \mathrm{Po}(\beta\cdot i)$$

Unconditionally you infect $Z \sim \mathrm{Mix}\,\mathrm{Po}(\beta\cdot I)$

$$\left(\begin{array}{l} \text{R-F} : I\equiv 1 \Rightarrow \mathrm{Po}(\beta) = \mathrm{Po}(R_0) \\ \text{Markovian } I\sim \mathrm{Exp}(\gamma) \Rightarrow \mathrm{Geo}\left(\frac{\gamma}{\gamma+\beta}\right) = \mathrm{Geo}\left(\frac{1}{1+R_0}\right) \end{array}\right.$$

$$\pi = P(\text{minor outbreak}) = \sum_k \pi^k P(Z=k)$$

$$= E(\pi^Z) = g(\pi)$$

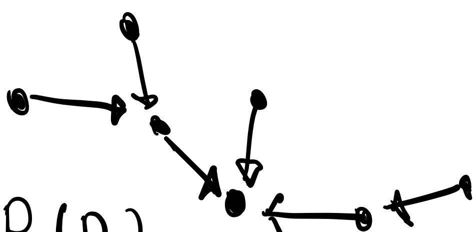
Prob gen fcn

π solves $\pi = g(\pi)$

Final fraction infected (in case of a major outbreak)

Pick a random individual i

Consider her "susceptibility set"



Gen 1: those who would infect me

Gen 2: Those who would inf Gen 1

$\vdots$

~~X~~ $X \sim Po(R_0)$

My susc set $\approx$ Br pr $\Leftrightarrow$ E-R network due to independence in in-degree

Suppose there is a major outbreak

ind i will get infected iff her susceptibility set is infinite

$1 - \bar{z} = \text{Prob}(i \text{ not to get infected})$

$= P$(susc set is finite)

$= P$(Br pr of indegrees goes extinct)

when degree is $Po(R_0)$

$$= \sum_k (1-z)^k \frac{R_0^k}{k!} e^{-R_0}$$

$$= e^{-R_0 z}$$

$$1 - z = e^{-R_0 z}$$

Final size is positive solution