Sparse Random Graphs-I

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Graph Limits and Processes on Networks:
From Epidemics to Misinformation Boot Camp
Let's start with a few questions...

How Disease becomes Epidemic?
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How **Disease** becomes **Epidemic**?

What causes **Internet** to **breakdown**?
Let's start with a few questions...

How **Disease** becomes **Epidemic**?

What causes **Internet** to **breakdown**?

When does **Misinformation** reach a **large population**?
What are Random Graphs for?

These seemingly unrelated questions have a few commonalities:

1. There is an underlying large network with a complex structure
2. There is emergence of behavior having drastic impact, a.k.a. phase transition

- Random Graphs provide a simplified probabilistic representation to model these complex systems.

  ➡ Capture structural properties (degree distribution, communities)
  ➡ Provide insight into emergence of different types of behavior such as phase transition
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➢ Random Graphs provide a *simplified probabilistic representation* to model these complex system.
   ➢ Capture structural properties (degree distribution, communities)
   ➢ Provide insight into emergence of different types of behavior such as phase transition
Why are Random Graphs useful?

Random Graphs are good graphs: General graphs are too messy and Random Graph is a way to pose regularity properties. Example: Expansion, Convergence (Graphon, Local-weak) Often reveal core properties responsible for phenomena of interest. Example: Will see how local neighborhood structure impact global properties like phase transition, typical distances. Random Graphs serve to get provable guarantees for graph algorithms. Example: Heuristic algorithms for NP-hard problems such as graph partitioning, coloring.
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Plan

Today:

➢ Local Branching Process approximation technique on random graphs
➢ Explore its relation to *Giant Component Problem* on different models

Tomorrow:

➢ Applications to *Percolation, Epidemics*
➢ Using *Stochastic Process convergence* in Random Graphs
Let’s start with the most elementary yet fundamental model...

Erdős-Rényi Random Graph
Erdős-Rényi Random Graph

Definition
➢ Given \( n \) nodes \( \{1, 2, \ldots, n\} \)

Historical note:
➢ This model was actually studied by Gilbert (1959) and heuristically by Solomonoff & Rapoport (1951)
➢ Erdős & Rényi (1959) initially worked with a slightly different model where fixed number of edges sampled uniformly. In a sequence of eight papers between 1959-1968 they laid the foundation of Random Graph theory
Erdős-Rényi Random Graph

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Erdős-Rényi Random Graph

Definition

➢ Given $n$ nodes $\{1, 2, \ldots, n\}$
➢ Edge $\{i, j\}$ present w.p. $p$ independently
➢ Denote this graph by $ER_n(p)$

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What are we after?

$\text{ER}_n(p)$ with $n = 1000$

- If $p = \lambda n$, there is a phase transition around $\lambda = 1$
  - $\lambda < 1$: All components are small
  - $\lambda > 1$: There is a unique giant component

Pic source: van der Hofstad (2017)
What are we after?

$\text{ER}_n(p)$ with $n = 1000$

$\begin{align*}
p &= \frac{0.5}{n} \\
p &= \frac{2}{n}
\end{align*}$

➢ If $p = \frac{\lambda}{n}$, then there is \textit{phase transition around} $\lambda = 1$

$\begin{align*}
\Rightarrow \lambda < 1 & : \text{All components are small} \\
\Rightarrow \lambda > 1 & : \text{There is a unique giant component}
\end{align*}$

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Pic source: van der Hofstad (2017)
Local neighborhood structure of $\text{ER}_n(\frac{\lambda}{n})$

➢ To analyze component sizes: Gradually explore graph in BFS starting from any node, e.g., node 1
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![Diagram showing the exploration process](image)

- $\text{Bin}(n - 1, \frac{\lambda}{n})$
- $\text{Bin}(n - 4, \frac{\lambda}{n})$

Two obstacles come up to analyze this process:
- Depletion of vertices
- Conflicts among new vertices
Local neighborhood structure of $\text{ER}_n\left(\frac{\lambda}{n}\right)$

➢ To analyze component sizes: Gradually explore graph in BFS starting from any node, e.g., node 1

![Diagram of graph exploration]

➢ Generally, each vertex at depth $i$ explores $\text{Bin}(n - s_i, \frac{\lambda}{n})$ new vertices at depth $i + 1$, where $s_i$ is the number of vertices explored up to depth $i$
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Local neighborhood structure of \( \text{ER}_n \left( \frac{\lambda}{n} \right) \)

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![Graph visualization]

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Local neighborhood structure of ER\(_n\left(\frac{\lambda}{n}\right)\)

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Two obstacles come up to analyze this process

➡ Depletion of vertices

➡ Conflicts among new vertices
Domination by Branching Process

Let’s consider a random process without Depletion and Conflicts

The object on right is a Branching Process
Domination by Branching Process

Let’s consider a random process without Depletion and Conflicts

Let $N_k = \# \text{ vertices at depth } k$ for $\text{ER}_n(\frac{\lambda}{n})$ exploration, and $\bar{N}_k$ denotes same for Branching process. There is a coupling such that w.p. 1

$$N_k \leq \bar{N}_k \quad \forall k \geq 0$$
Small component sizes for $\lambda < 1$

- $C(1) := $ component size of vertex 1
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$\mathbb{E}[C(1)]$
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$$\mathbb{E}[C(1)] = \mathbb{E} \left[ \sum_{k \geq 1} N_k \right] \leq \mathbb{E} \left[ \sum_{k \geq 1} \tilde{N}_k \right] = \sum_{k \geq 1} \mathbb{E}[\tilde{N}_k]$$
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For a Branching Process: $\mathbb{E}[\tilde{N}_k] = \lambda^k$
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*If $\lambda < 1$, then BP has size $O(1)$ so expected component size is $O(1)$*
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**Theorem: Subcritical $ER_n(\frac{\lambda}{n})$**

If $\lambda < 1$, then

$$
\max_u \frac{C(u)}{\log n} \xrightarrow{p} \frac{1}{I_\lambda}, \quad \text{where } I_\lambda = \lambda - 1 - \log \lambda
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Small component sizes for $\lambda < 1$

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**Theorem: Subcritical** $\text{ER}_n(\frac{\lambda}{n})$

If $\lambda < 1$, then

$$\max_u \frac{C(u)}{\log n} \overset{p}{\longrightarrow} \frac{1}{I_\lambda}, \quad \text{where } I_\lambda = \lambda - 1 - \log \lambda$$

➢ Proof uses Large Deviation estimates for branching process survival prob
What happens to the BP for $\lambda > 1$

Let's again look at upper bounding Branching Process (BP)

\[
\text{Bin}(n, \frac{\lambda}{n}) \approx \text{Poisson}(\lambda)
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$\lambda > 1 \Rightarrow P(\text{BP survives up to infinite generations}) = \zeta_{\lambda > 0}$

$\zeta_{\lambda}$ satisfies is a positive solution of

\[
1 - \zeta = e^{-\lambda \zeta}
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What happens to the BP for $\lambda > 1$

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Existence of a Giant for $\lambda > 1$

➢ For $\lambda > 1$, $P(BP \text{ survives up to infinite generations}) = \zeta_\lambda > 0$

➢ As we will see, exploration and BP remain close together for a long time

⇒ When BP survives, exploration continues for a long time giving rise to a large component
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- $C(v)$ is large w.p. $\zeta_\lambda \implies E[#\{v : C(v) \text{ is large}\}] \approx n\zeta_\lambda$
Existence of a Giant for $\lambda > 1$

➢ For $\lambda > 1$, $\mathbb{P}(\text{BP survives up to infinite generations}) = \zeta_{\lambda} > 0$

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⇒ When BP survives, exploration continues for a long time giving rise to a large component

➢ $C(v)$ is large w.p. $\zeta_{\lambda} \implies \mathbb{E}[\#\{v : C(v) \text{ is large}\}] \approx n\zeta_{\lambda}$

Theorem: Supercritical $\text{ER}_n\left(\frac{\lambda}{n}\right)$

Let $C_{(i)} := i$-th largest component of $\text{ER}_n\left(\frac{\lambda}{n}\right)$. If $\lambda > 1$, then as $n \to \infty$

$$\frac{C_{(1)}}{n} \xrightarrow{\mathbb{P}} \zeta_{\lambda} \quad \text{and} \quad \frac{C_{(2)}}{n} \xrightarrow{\mathbb{P}} 0$$

A unique giant component emerges...
Moments of large component sizes

Lemma 1: First moment

\[ X_i \geq 1 \implies C(i) \geq L = \zeta \lambda + o(L), \]

Lemma 2: Second moment

\[ X_i \geq 1 \implies C^2(i) \geq L = \zeta^2 \lambda + o(L), \]

Two lemmas directly imply

\[ C(1) \approx \zeta \lambda \text{ and } C(2) \approx 0, \]

which shows existence and uniqueness of giant component.
Moments of large component sizes

Lemma 1: First moment

\[
\sum_{i \geq 1} \frac{C^{(i)}}{n} \mathbb{1}\{C^{(i)} \geq L\} = \zeta \lambda + o_{L,n}(1)
\]
Moments of large component sizes

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**Lemma 2: Second moment**

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\sum_{i \geq 1} \frac{C^{2(i)}}{n^2} \mathbb{1}\{C^{(i)} \geq L\} = \zeta^2_\lambda + o_{L,n}(1)
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which shows existence and uniqueness of giant
Moments of large component sizes

**Lemma 1: First moment**

\[
\sum_{i \geq 1} \frac{C^{(i)}_n n}{n} \mathbbm{1}\{C^{(i)}_n \geq L\} = \zeta_\lambda + o_{L,n}(1)
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**Lemma 2: Second moment**

\[
\sum_{i \geq 1} \frac{C^{(i)}^2_n n^2}{n^2} \mathbbm{1}\{C^{(i)}_n \geq L\} = \zeta^2_\lambda + o_{L,n}(1)
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Two lemmas directly imply

\[\frac{C^{(1)}_n}{n} \approx \zeta_\lambda \quad \text{and} \quad \frac{C^{(2)}_n}{n} \approx 0\]

which shows existence and uniqueness of giant

*Next, prove two lemmas but before that...*
Proximity to upper bounding branching process

**Will show:** $N_k \approx \bar{N}_k$ until $s$ vertices explored for $s = n^a$, $a < 1$

*Exploration and BP remain close together for a long time*
Proximity to upper bounding branching process

**Will show:** $N_k \approx \tilde{N}_k$ until $s$ vertices explored for $s = n^a$, $a < 1$

*Exploration and BP remain close together for a long time*

Two sources of discrepancy

1. **Depletion:**
Proximity to upper bounding branching process

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Two sources of discrepancy

1. **Depletion:** $\text{Bin}(n - s, \frac{\lambda}{n}) \approx \text{Poisson}(\lambda)$ for $s = o(n)$
Proximity to upper bounding branching process

**Will show:** \( N_k \approx \bar{N}_k \) until \( s \) vertices explored for \( s = n^a, \ a < 1 \)

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Two sources of discrepancy

1. **Depletion:** Bin\((n - s, \frac{\lambda}{n})\) \( \approx \) Poisson\((\lambda)\) for \( s = o(n) \)
2. What about **conflicts**?

![Diagram showing the relationship between \( N_k \) and \( \bar{N}_k \)]
Proximity to upper bounding branching process

**Will show:** $N_k \approx \bar{N}_k$ until $s$ vertices explored for $s = n^a, a < 1$

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Two sources of discrepancy

1. **Depletion:** $\text{Bin}(n - s, \frac{\lambda}{n}) \approx \text{Poisson}(\lambda)$ for $s = o(n)$

2. What about **conflicts**?

**Fact:** $\mathbb{E}[\# \text{conflicts}] \leq C \frac{\lambda^{2k}}{n}$
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Proximity to upper bounding branching process contd.

**Fact:** \( \mathbb{E}[\# \text{ conflicts}] \leq C \frac{\lambda^{2k}}{n} \)

➢ If \( k \) is a *large* constant, then \( \mathbb{E}[\# \text{ conflicts}] \approx 0 \quad \text{Exploration} = \text{BP w.p.} \approx 1 \)
Proximity to upper bounding branching process contd.

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➢ If $k$ is a *large constant*, then $\mathbb{E}[\# \text{ conflicts}] \approx 0$ \hspace{1cm} *Exploration = BP w.p. $\approx 1$*

➢ Let $k = a \log_\lambda n$ and $0 \leq a < 1$. 
Proximity to upper bounding branching process contd.

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➢ Let \( k = a \log_\lambda n \) and \( 0 \leq a < 1 \). Then \( \lambda^k = n^a \) and
Proximity to upper bounding branching process contd.

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➢ Let \( k = a \log_\lambda n \) and \( 0 \leq a < 1 \). Then \( \lambda^k = n^a \) and

\[ \Rightarrow \mathbb{E}[\# \text{ conflicts}] \leq Cn^{2a-1} \]
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  - $\mathbb{E}[\# \text{ conflicts}] \leq C n^{2a-1}$
  - Conditioned on BP survives for $r$ depth ($r$ large), $\bar{N}_k \approx \lambda^k = n^a$ for $k \geq r$
Proximity to upper bounding branching process contd.

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\[
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\[
\# \text{ conflicts} = o(\bar{N}_k) \text{ whp}
\]
Proximity to upper bounding branching process contd.

**Fact:** $\mathbb{E}[\# \text{ conflicts}] \leq C \frac{\lambda^{2k}}{n}$

➢ If $k$ is a large constant, then $\mathbb{E}[\# \text{conflicts}] \approx 0 \quad \text{Exploration } = \text{BP w.p. } \approx 1$

➢ Let $k = a \log \lambda n$ and $0 \leq a < 1$. Then $\lambda^k = n^a$ and

$\Rightarrow \mathbb{E}[\# \text{conflicts}] \leq Cn^{2a-1}$

$\Rightarrow$ Conditioned on BP survives for $r$ depth ($r$ large), $\tilde{N}_k \approx \lambda^k = n^a$ for $k \geq r$

$\# \text{conflicts} = o(\tilde{N}_k) \text{ whp } \implies \begin{array}{c} N_k \approx \tilde{N}_k \approx \lambda^k \end{array}$
Proximity to upper bounding branching process contd.

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\[ \mathbb{E}[\# \text{ conflicts}] \leq C n^{2a-1} \]

➢ Conditioned on BP survives for \( r \) depth (\( r \) large), \( \bar{N}_k \approx \lambda^k = n^a \) for \( k \geq r \)

\[ \# \text{ conflicts} = o(\bar{N}_k) \text{ whp} \implies \bar{N}_k \approx N_k \approx \lambda^k \]

When exploration survives for long time, growth rate of \( N_k \) becomes exponential in \( \lambda \)
Proximity to upper bounding branching process contd.

**Fact:** \( \mathbb{E}[\# \text{ conflicts}] \leq C \frac{\lambda^{2k}}{n} \)

- If \( k \) is a *large constant*, then \( \mathbb{E}[\# \text{ conflicts}] \approx 0 \quad \text{Exploration} = \text{BP w.p.} \approx 1 

- Let \( k = a \log_\lambda n \) and \( 0 \leq a < 1 \). Then \( \lambda^k = n^a \) and
  - \( \mathbb{E}[\# \text{ conflicts}] \leq C n^{2a-1} \)
  - Conditioned on BP survives for \( r \) depth (\( r \) large), \( \bar{N}_k \approx \lambda^k = n^a \) for \( k \geq r \)

\[ \# \text{ conflicts} = o(\bar{N}_k) \quad \text{whp} \implies \bar{N}_k \approx \bar{N}_k \approx \lambda^k \]

*When exploration survives for long time, growth rate of \( N_k \) becomes exponential in \( \lambda \)*

Next, lets prove two lemmas
Proving first moment lemma

To prove: \( \sum_{i \geq 1} \frac{C^{(i)}}{n} \mathbb{1}\{C^{(i)} \geq L\} \approx \zeta \lambda \)
To prove: \( \sum_{i \geq 1} \frac{C(i)}{n} \mathbb{1}\{C(i) \geq L\} \approx \zeta_{\lambda} \)

\[
\sum_{i \geq 1} \frac{C(i)}{n} \mathbb{1}\{C(i) \geq L\} = \mathbb{P}(u \text{ falls in a component of size } \geq L \mid G) \quad (u \text{ is a uniform vertex})
\]
To prove: \( \sum_{i \geq 1} \frac{C(i)}{n} \mathbb{I}\{C(i) \geq L\} \approx \zeta_\lambda \)

\[
\sum_{i \geq 1} \frac{C(i)}{n} \mathbb{I}\{C(i) \geq L\}
\]

\[
\approx \mathbb{P}(u \text{ falls in a component of size } \geq L \mid G) \quad (u \text{ is a uniform vertex})
\]

\[
\approx \mathbb{P}(C(u) \geq L \mid G)
\]
Proving first moment lemma

To prove: \( \sum_{i \geq 1} \frac{C(i)}{n} \mathbb{1}\{C(i) \geq L\} \approx \zeta \lambda \)

\[ \sum_{i \geq 1} \frac{C(i)}{n} \mathbb{1}\{C(i) \geq L\} \]

\[ = \mathbb{P}(u \text{ falls in a component of size } \geq L \mid G) \quad (u \text{ is a uniform vertex}) \]
\[ = \mathbb{P}(C(u) \geq L \mid G) \]

Now,

\[ \mathbb{P}(C(u) \geq L) \]
Proving first moment lemma

To prove: \( \sum_{i \geq 1} \frac{C(i)}{n} \mathbb{1}\{C(i) \geq L\} \approx \zeta_{\lambda} \)

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= \mathbb{P}(C(u) \geq L \mid G)
\]

Now,

\( \mathbb{P}(C(u) \geq L) \approx \mathbb{P}(BP \geq L) \quad (\text{Exploration of } u = \text{BP w.p. } \approx 1) \)
Proving first moment lemma

To prove: \( \sum_{i \geq 1} \frac{C(i)}{n} \mathbb{1}\{C(i) \geq L\} \approx \zeta\lambda \)

\[ \sum_{i \geq 1} \frac{C(i)}{n} \mathbb{1}\{C(i) \geq L\} = \mathbb{P}(u \text{ falls in a component of size } \geq L \mid G) \quad (u \text{ is a uniform vertex}) \]
\[ \mathbb{P}(C(u) \geq L \mid G) \]

Now,
\[ \mathbb{P}(C(u) \geq L) \approx \mathbb{P}(BP \geq L) \quad (\text{Exploration of } u = BP \text{ w.p. } \approx 1) \]
\[ \approx \mathbb{P}(BP \text{ survives}) \quad (\text{holds for large enough } L) \]
\[ = \zeta\lambda \]
Proving first moment lemma

To prove: \( \sum_{i \geq 1} \frac{C(i)}{n} \mathbb{1}\{C(i) \geq L\} \approx \zeta_\lambda \)

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\sum_{i \geq 1} \frac{C(i)}{n} \mathbb{1}\{C(i) \geq L\} \\
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= \mathbb{P}(C(u) \geq L \mid G)
\]

Now,

\[
\mathbb{P}(C(u) \geq L) \approx \mathbb{P}(\text{BP } \geq L) \quad \text{(Exploration of } u = \text{BP w.p. } \approx 1) \\
\approx \mathbb{P}(\text{BP survives}) \quad \text{(holds for large enough } L) \\
= \zeta_\lambda
\]

**Key fact 1:** Local neighborhood of \( u \) is approximately BP whp
Proving first moment lemma

To prove: \( \sum_{i \geq 1} \frac{C^{(i)}}{n} \mathbb{I}\{C^{(i)} \geq L\} \approx \zeta_{\lambda} \)

\[
\begin{align*}
\sum_{i \geq 1} \frac{C^{(i)}}{n} \mathbb{I}\{C^{(i)} \geq L\} \\
= \mathbb{P}(u \text{ falls in a component of size } \geq L \mid G) \quad (u \text{ is a uniform vertex}) \\
= \mathbb{P}(C(u) \geq L \mid G)
\end{align*}
\]

Now,
\[
\begin{align*}
\mathbb{P}(C(u) \geq L) &\approx \mathbb{P}(BP \geq L) \quad \text{(Exploration of } u = BP \text{ w.p. } \approx 1) \\
&\approx \mathbb{P}(BP \text{ survives}) \quad \text{(holds for large enough } L) \\
&= \zeta_{\lambda}
\end{align*}
\]

**Key fact 1:** Local neighborhood of \( u \) is approximately BP whp

➢ Theory of approximating local neighborhood of graphs is called **Local-weak convergence** (Christian’s talk)
Proving second moment lemma

To prove: \( \sum_{i \geq 1} \frac{C^2(i)}{n^2} \mathbb{1}\{C(i) \geq L\} \approx \zeta^2 \lambda \)
Proving second moment lemma

To prove: \[ \sum_{i \geq 1} \frac{C^2(i)}{n^2} \mathbb{1}\{C(i) \geq L\} \approx \zeta^2 \]

\[ \zeta^2 \lambda \approx \left( \sum_{i \geq 1} \frac{C(i)}{n} \mathbb{1}\{C(i) \geq L\} \right)^2 \text{ by previous lemma} \]
Proving second moment lemma

To prove: \( \sum_{i \geq 1} \frac{C_i^2}{n^2} \mathbb{1}\{C_i \geq L\} \approx \zeta^2 \)

\[
\zeta^2 \approx \left( \sum_{i \geq 1} \frac{C_i}{n} \mathbb{1}\{C_i \geq L\} \right)^2 \text{ by previous lemma}
\]

\[
= \sum_{i \geq 1} \frac{C_i^2}{n^2} \mathbb{1}\{C_i \geq L\} + \sum_{i \neq j} \frac{C_i C_j}{n^2} \mathbb{1}\{C_i \geq L, C_j \geq L\}
\]
Proving second moment lemma

To prove: \[ \sum_{i \geq 1} \frac{C_{(i)}^2}{n^2} \mathbb{1}\{C_{(i)} \geq L\} \approx \zeta^2 \]

\[ \zeta^2 \approx \left( \sum_{i \geq 1} \frac{C_{(i)}}{n} \mathbb{1}\{C_{(i)} \geq L\} \right)^2 \text{ by previous lemma} \]

\[ = \sum_{i \geq 1} \frac{C_{(i)}^2}{n^2} \mathbb{1}\{C_{(i)} \geq L\} + \sum_{i \neq j} \frac{C_{(i)} C_{(j)}}{n^2} \mathbb{1}\{C_{(i)} \geq L, C_{(j)} \geq L\} \]

Take two uniform vertices \( u_1, u_2 \). The second term equals

\[ \mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \not\sim u_2 \mid G) \]
Proving second moment lemma

To prove: \( \sum_{i \geq 1} \frac{c^2_{i}}{n^2} \mathbb{1}\{C_{i} \geq L\} \approx \zeta^{2}_{\lambda} \)

\[ \zeta^{2}_{\lambda} \approx \left( \sum_{i \geq 1} \frac{c_{i}}{n} \mathbb{1}\{C_{i} \geq L\} \right)^{2} \text{ by previous lemma} \]

\[ = \sum_{i \geq 1} \frac{c^2_{i}}{n^2} \mathbb{1}\{C_{i} \geq L\} + \sum_{i \neq j} \frac{c_{i} c_{j}}{n^2} \mathbb{1}\{C_{i} \geq L, C_{j} \geq L\} \]

Take two uniform vertices \( u_1, u_2 \). The second term equals

\[ \mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \not\sim u_2 \mid G) \]

Enough to show:

\[ \lim_{L \to \infty} \lim_{n \to \infty} \sup \mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \not\sim u_2) = 0 \]

Two large components cannot be disjoint...
Two large components cannot be disjoint, why?

To show $\mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \not\leftrightarrow u_2) \approx 0$, suffices to prove

$$\mathbb{P}(u_1 \not\leftrightarrow u_2 \mid C(u_1) \geq L, C(u_2) \geq L) \approx 0$$
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Unfortunately, this is quite hard to show,
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**Idea:** Replace the conditioning event by $\{\partial_r(u_1), \partial_r(u_2) \neq \emptyset\}$
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$\Rightarrow \{\partial_r(u_1), \partial_r(u_2) \neq \emptyset\} \approx \{C(u_1), C(u_2) \geq L\}$ in prob. for some large $r$
Two large components cannot be disjoint, why?

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**Idea:** Replace the conditioning event by $\{\partial_r(u_1), \partial_r(u_2) \neq \emptyset\}$

$\Rightarrow \{\partial_r(u_1), \partial_r(u_2) \neq \emptyset\} \approx \{C(u_1), C(u_2) \geq L\}$ in prob. for some large $r$

$\Rightarrow$ Advantage with conditioning on $\{\partial_r(u_1), \partial_r(u_2) \neq \emptyset\}$ is we can now explore rest of the graph
Two large components cannot be disjoint, why?

➢ Condition on \( \{ \partial_r(u_1), \partial_r(u_2) \neq \emptyset \} \) for large \( r \)

Takeaway: If there are two components with large boundary, we can grow them until boundary has size \( \sqrt{n} \) and then they intersect.
Two large components cannot be disjoint, why?

Condition on \( \{ \partial_r(u_1), \partial_r(u_2) \neq \emptyset \} \) for large \( r \)

Recall: After exploration survives up to \( r \), further growth is exponential.
Two large components cannot be disjoint, why?

➢ Condition on \( \{\partial_r(u_1), \partial_r(u_2) \neq \emptyset\} \) for large \( r \)

➢ **Recall:** After exploration survives up to \( r \), further growth is exponential

➢ **Grow** \( u_1 \) **neighborhood** up to \( k_1 \) s.t. boundary size \( N_{k_1} \approx \lambda^{k_1} = \omega_n \sqrt{n} \)
Two large components cannot be disjoint, why?

- Condition on \( \{ \partial_r(u_1), \partial_r(u_2) \neq \emptyset \} \) for large \( r \)
- **Recall:** After exploration survives up to \( r \), further growth is exponential
- **Grow** \( u_1 \) **neighborhood** up to \( k_1 \) s.t. boundary size \( N_{k_1} \approx \lambda^{k_1} = \omega_n \sqrt{n} \)
- **Grow** \( u_2 \) **neighborhood** s.t. boundary size is \( \sqrt{n} \)
Two large components cannot be disjoint, why?

➢ Condition on \( \{\partial_r(u_1), \partial_r(u_2) \neq \emptyset\} \) for large \( r \)

➢ Recall: After exploration survives up to \( r \), further growth is exponential

➢ Grow \( u_1 \) neighborhood up to \( k_1 \) s.t. boundary size \( N_{k_1} \approx \lambda^{k_1} = \omega_n \sqrt{n} \)

➢ Grow \( u_2 \) neighborhood s.t. boundary size is \( \sqrt{n} \)

\[
\mathbb{P}(\text{no edge between boundaries}) = \left(1 - \frac{\lambda}{n}\right)^{\omega_n \sqrt{n} \times \sqrt{n}} \approx e^{-\lambda \omega_n} \to 0
\]
Two large components cannot be disjoint, why?

- Condition on \( \{ \partial_r(u_1), \partial_r(u_2) \neq \emptyset \} \) for large \( r \)
- Recall: After exploration survives up to \( r \), further growth is exponential
- Grow \( u_1 \) neighborhood up to \( k_1 \) s.t. boundary size \( N_{k_1} \approx \lambda^{k_1} = \omega_n \sqrt{n} \)
- Grow \( u_2 \) neighborhood s.t. boundary size is \( \sqrt{n} \)

\[
P(\text{no edge between boundaries}) = \left(1 - \frac{\lambda}{n}\right)^{\omega_n \sqrt{n} \times \sqrt{n}} \approx e^{-\lambda \omega_n} \to 0
\]

\[
\implies P(u_1 \not\leftrightarrow u_2 \mid \partial_r(u_1), \partial_r(u_1) \neq \emptyset) \approx 0
\]
Two large components cannot be disjoint, why?

➢ Condition on \( \{ \partial_r(u_1), \partial_r(u_2) \neq \emptyset \} \) for large \( r \)

➢ **Recall:** After exploration survives up to \( r \), further growth is exponential

➢ Grow \( u_1 \) neighborhood up to \( k_1 \) s.t. boundary size \( N_{k_1} \approx \lambda^{k_1} = \omega_n \sqrt{n} \)

➢ Grow \( u_2 \) neighborhood s.t. boundary size is \( \sqrt{n} \)

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\mathbb{P}(\text{no edge between boundaries}) = \left(1 - \frac{\lambda}{n}\right)^{\omega_n \sqrt{n} \times \sqrt{n}} \approx e^{-\lambda \omega_n} \rightarrow 0
\]

\[
\implies \mathbb{P}(u_1 \not\leftrightarrow u_2 \mid \partial_r(u_1), \partial_r(u_1) \neq \emptyset) \approx 0
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**Takeaway:** If there are two components with large boundary, we can grow them until boundary has size \( \sqrt{n} \) and then they intersect
To prove: \[ \sum_{i \geq 1} \frac{C_i^2}{n^2} \mathbb{1}\{C_i \geq L\} \approx \zeta\lambda^2 \]
To prove: \( \sum_{i \geq 1} \frac{c_{(i)}^2}{n^2} \mathbb{1}\{C_{(i)} \geq L\} \approx \zeta_\lambda^2 \)

➢ Reduced to prove

\[
\lim_{L \to \infty} \lim_{n \to \infty} \mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \not\leftrightarrow u_2) = 0
\]

Two large components cannot be disjoint...
To prove: $\sum_{i \geq 1} \frac{C^2_{(i)}}{n^2} \mathbb{1}\{C_{(i)} \geq L\} \approx \zeta^2_\lambda$

➢ Reduced to prove

$$\lim_{L \to \infty} \limsup_{n \to \infty} \mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \not\leftrightarrow u_2) = 0$$

Two large components cannot be disjoint...

➢ Reduced to prove

$$\mathbb{P}(u_1 \not\leftrightarrow u_2 | \partial_r(u_1), \partial_r(u_2) \neq \emptyset) \approx 0$$
Proving second moment lemma contd

To prove: \[ \sum_{i \geq 1} \frac{C^2_{(i)}}{n^2} \mathbb{1}\{C_{(i)} \geq L\} \approx \zeta^2 \]

➢ Reduced to prove

\[ \lim_{L \to \infty} \limsup_{n \to \infty} \mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \not\sim u_2) = 0 \]

Two large components cannot be disjoint...

➢ Reduced to prove

\[ \mathbb{P}(u_1 \not\sim u_2 \mid \partial_r(u_1), \partial_r(u_2) \neq \emptyset) \approx 0 \]

➢ Proved this by growing both the neighborhoods of \( u_1, u_2 \) and they intersect when the boundary size grows to size \( \sqrt{n} \)
To prove: $\sum_{i \geq 1} \frac{C_{(i)}^2}{n^2} 1\{C_{(i)} \geq L\} \approx \zeta^2$ 

➢ Reduced to prove

$$\lim_{L \to \infty} \lim_{n \to \infty} \sup \mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \not\leftrightarrow u_2) = 0$$

*Two large components cannot be disjoint...*

➢ Reduced to prove

$$\mathbb{P}(u_1 \not\leftrightarrow u_2 \mid \partial_r(u_1), \partial_r(u_2) \neq \emptyset) \approx 0$$

➢ Proved this by growing both the neighborhoods of $u_1, u_2$ and they intersect when the boundary size grows to size $\sqrt{n}$

*Completes the proof of Lemma 2*
Proving second moment lemma contd

To prove: \( \sum_{i \geq 1} \frac{C_i^2}{n^2} \mathbb{1}_{\{C_i \geq L\}} \approx \zeta^2 \lambda \)

➢ Reduced to prove

**Key fact 2:** \( \lim_{L \to \infty} \limsup_{n \to \infty} \mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \nleftrightarrow u_2) = 0 \)

*Two large components cannot be disjoint...*

➢ Reduced to prove

\[ \mathbb{P}(u_1 \nleftrightarrow u_2 \mid \partial_r(u_1), \partial_r(u_2) \neq \emptyset) \approx 0 \]

➢ Proved this by growing both the neighborhoods of \( u_1, u_2 \) and they intersect when the boundary size grows to size \( \sqrt{n} \)

*Complements the proof of Lemma 2*
We proved

**Theorem: Supercritical** $\text{ER}_n\left(\frac{\lambda}{n}\right)$

Let $C_{(i)} :=$ $i$-th largest component of $\text{ER}_n\left(\frac{\lambda}{n}\right)$. If $\lambda > 1$, then

$$
\frac{C_{(1)}}{n} \xrightarrow{P} \zeta_\lambda \quad \text{and} \quad \frac{C_{(2)}}{n} \xrightarrow{P} 0
$$
Summary: Giant for Erdős-Rényi

We proved

**Theorem: Supercritical** $\text{ER}_n\left(\frac{\lambda}{n}\right)$

Let $C_{(i)} := i$-th largest component of $\text{ER}_n\left(\frac{\lambda}{n}\right)$. If $\lambda > 1$, then

$$\frac{C_{(1)}}{n} \xrightarrow{\text{IP}} \zeta_{\lambda} \quad \text{and} \quad \frac{C_{(2)}}{n} \xrightarrow{\text{IP}} 0$$

The two main ingredients to prove this were...

1. Local neighborhood approximation: Local neighborhood of $u$ is approximately BP whp and when BP survives, $C_u$ is large
2. Two large components intersect: $\lim_{L \to \infty} \lim_{n \to \infty} P(C_u \geq L, C_{u_2} \geq L, u_1 \not\leftrightarrow u_2) = 0$

Was shown by growing two neighborhoods, and they must intersect when neighborhoods become large enough $O(\sqrt{n})$.

➢ van der Hofstad (2021) proved this for general graphs that converge in local-weak convergence sense
Summary: Giant for Erdős-Rényi

We proved

**Theorem: Supercritical** $\text{ER}_n\left(\frac{\lambda}{n}\right)$

Let $C_{(i)} := i$-th largest component of $\text{ER}_n\left(\frac{\lambda}{n}\right)$. If $\lambda > 1$, then

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The two main ingredients to prove this were...

1. **Local neighborhood approximation:** Local neighborhood of $u$ is approximately BP whp and when BP survives, $C(u)$ is large
Summary: Giant for Erdős-Rényi

We proved

**Theorem: Supercritical** $\text{ER}_n(\lambda \frac{n}{\lambda})$

Let $C_{(i)} := i$-th largest component of $\text{ER}_n(\lambda \frac{n}{\lambda})$. If $\lambda > 1$, then

$$\frac{C_{(1)}}{n} \xrightarrow{\text{P}} \zeta_\lambda \quad \text{and} \quad \frac{C_{(2)}}{n} \xrightarrow{\text{P}} 0$$

The two main ingredients to prove this were...

1. **Local neighborhood approximation:** Local neighborhood of $u$ is approximately BP whp and when BP survives, $C(u)$ is large

2. **Two large components intersect:**

$$\lim_{L \to \infty} \lim_{n \to \infty} \mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \not\leftrightarrow u_2) = 0$$
Summary: Giant for Erdős-Rényi

We proved

**Theorem: Supercritical ER**$_n$(\(\frac{\lambda}{n}\))

Let \(C_{(i)} := i\)-th largest component of ER$_n$(\(\frac{\lambda}{n}\)). If \(\lambda > 1\), then

\[
\frac{C_{(1)}}{n} \xrightarrow{\text{IP}} \zeta_\lambda \quad \text{and} \quad \frac{C_{(2)}}{n} \xrightarrow{\text{IP}} 0
\]

The two main ingredients to prove this were...

1. **Local neighborhood approximation:** Local neighborhood of \(u\) is approximately BP whp and when BP survives, \(C(u)\) is large

2. **Two large components intersect:**

\[
\lim_{L \to \infty} \lim_{n \to \infty} \mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \not\leftrightarrow u_2) = 0
\]

\(\Rightarrow\) Was shown by growing two neighborhoods, and they must intersect when neighborhoods become large enough \(O(\sqrt{n})\)
Summary: Giant for Erdős-Rényi

We proved

**Theorem: Supercritical** $\text{ER}_n\left(\frac{\lambda}{n}\right)$

Let $C_{(i)} := i$-th largest component of $\text{ER}_n\left(\frac{\lambda}{n}\right)$. If $\lambda > 1$, then

$$\frac{C_{(1)}}{n} \xrightarrow{P} \zeta_\lambda \quad \text{and} \quad \frac{C_{(2)}}{n} \xrightarrow{P} 0$$

The two main ingredients to prove this were...

1. **Local neighborhood approximation:** Local neighborhood of $u$ is approximately BP whp and when BP survives, $C(u)$ is large

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$$\lim_{L \to \infty} \lim_{n \to \infty} \mathbb{P}(C(u_1) \geq L, C(u_2) \geq L, u_1 \not\leftrightarrow u_2) = 0$$

⇒ Was shown by growing two neighborhoods, and they must intersect when neighborhoods become large enough $O(\sqrt{n})$

➢ van der Hofstad (2021) proved this for general graphs that converge in local-weak convergence sense
Before moving on to other models, let's see another useful application of the above ideas...

Typical distances in Erdős-Rényi
Typical distances in the giant of Erdős-Rényi

**Typical distance:** Graph distance between two uniform vertices $u_1, u_2$
**Typical distances in the giant of Erdős-Rényi**

**Typical distance:** Graph distance between two uniform vertices $u_1, u_2$

**Theorem:** Typical distances in $\text{ER}_n\left(\frac{\lambda}{n}\right)$

Let $\lambda > 1$. Conditionally on $u_1, u_2$ in same component (i.e., $\text{dist}(u_1, u_2) \neq \infty$)

$$
\frac{\text{dist}(u_1, u_2)}{\log \lambda n} \xrightarrow{p} 1
$$
Typical distances in the giant of Erdős-Rényi

**Typical distance:** Graph distance between two uniform vertices \( u_1, u_2 \)

**Theorem:** Typical distances in \( \text{ER}_n(\frac{\lambda}{n}) \)

Let \( \lambda > 1 \). Conditionally on \( u_1, u_2 \) in same component (i.e., \( \text{dist}(u_1, u_2) \neq \infty \))

\[
\frac{\text{dist}(u_1, u_2)}{\log \lambda n} \xrightarrow{p} 1
\]

**Proof:** Again use neighborhood growth idea...
**Typical distances in the giant of Erdős-Rényi**

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\[
\frac{\text{dist}(u_1, u_2)}{\log \lambda n} \xrightarrow{p} 1
\]

**Proof:** Again use neighborhood growth idea...

1 Keep growing neighborhoods from $u_1, u_2$. 
Typical distances in the giant of Erdős-Rényi

**Typical distance:** Graph distance between two uniform vertices $u_1, u_2$

**Theorem:** Typical distances in $\text{ER}_n(\frac{\lambda}{n})$

Let $\lambda > 1$. Conditionally on $u_1, u_2$ in same component (i.e., $\text{dist}(u_1, u_2) \neq \infty$)

$$\frac{\text{dist}(u_1, u_2)}{\log \lambda n} \overset{p}{\to} 1$$

**Proof:** Again use neighborhood growth idea...

1. Keep growing neighborhoods from $u_1, u_2$. Recall $\mathbb{E}[\#\text{conflicts}] \leq \lambda^{2k - \log \lambda n}$
Typical distances in the giant of Erdős-Rényi

**Typical distance**: Graph distance between two uniform vertices \( u_1, u_2 \)

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Let \( \lambda > 1 \). Conditionally on \( u_1, u_2 \) in same component (i.e., \( \text{dist}(u_1, u_2) \neq \infty \))

\[
\frac{\text{dist}(u_1, u_2)}{\log \lambda n} \xrightarrow{p} 1
\]

**Proof**: Again use neighborhood growth idea...

1. Keep growing neighborhoods from \( u_1, u_2 \). Recall \( \mathbb{E}[\# \text{conflicts}] \leq \lambda^{2k-\log \lambda n} \)

   ➔ They are disjoint until boundary sizes become \( \sqrt{n} \), i.e., \( k \lesssim \frac{1}{2} \log \lambda n \)
**Typical distances in the giant of Erdős-Rényi**

**Typical distance:** Graph distance between two uniform vertices $u_1, u_2$

**Theorem:** Typical distances in $\text{ER}_n(\frac{\lambda}{n})$

Let $\lambda > 1$. Conditionally on $u_1, u_2$ in same component (i.e., $\text{dist}(u_1, u_2) \neq \infty$)

$$\frac{\text{dist}(u_1, u_2)}{\log \lambda n} \xrightarrow{P} 1$$

**Proof:** Again use neighborhood growth idea...

1. Keep growing neighborhoods from $u_1, u_2$. Recall $E[\#\text{conflicts}] \leq \lambda^{2k - \log \lambda n}$

   ➔ They are disjoint until boundary sizes become $\sqrt{n}$, i.e., $k \lesssim \frac{1}{2} \log \lambda n$

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Typical distances in the giant of Erdős-Rényi

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   $$\text{dist}(u_1, u_2) = \frac{1}{2} \log \lambda n + \frac{1}{2} \log \lambda n + o(\log \lambda n) = \log \lambda n + o(\log \lambda n)$$