# On the Intersection of a Sparse Curve and a Low-degree Curve: A Polynomial Version of the Lost Theorem

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Joint work with Pascal Koiran and Natacha Portier

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# Descartes' rule

## Fundamental theorem of algebra

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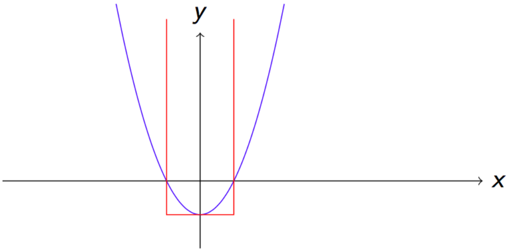

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## Descartes' estimate

If $P$ is a real $t$-sparse polynomial, then $P$ has at most $t - 1$ positive roots (counted with multiplicity).
So, at most $2t - 1$ distinct roots on $\mathbb{R}$.

In fact Descartes' rule is more precise.

# Bézout's Theorem

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The following $n \times n$ system

$$\begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ \ldots \\ f_n(x_1, \ldots, x_n) = 0 \end{cases}$$

has at most $d_1 d_2 \ldots d_n$ nondegenerate complex solutions.

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- Similar result for sparse polynomials and real solutions?
- Kushnirenko's question (1977): bounded by a function $N(t_1, \ldots, t_n)$? What is the optimal bound?

# An initial case

**Sevostyanov's problem (1977)**

Let $f$ and $g$ be two real bivariate polynomials.
$f$ is of degree $d$ and $g$ is $t$-sparse.
Is the number of distinct isolated real solutions of the system

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$$

bounded by a function $N(d,t)$? If so, what is this function?

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According to Kushnirenko, Sevostyanov proved the existence of $N(d,t)$ in 1978.

# Fewnomial bounds

**Theorem (Khovanskiĭ (1983))**

*System of $n$ equations and $n$ variables*
*with ony $n + l + 1$ distinct monomials.*
*Then, number of positive real solutions bounded by*

$$2^{\binom{l+n}{2}}(n+1)^{l+n}$$

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In particular,

- Kushnirenko's question: $N(t_1,\ldots,t_n) \leq 2^{\binom{t_1+\ldots+t_n}{2}}(n+1)^{t_1+\ldots+t_n}$

# Improvement

Khovanskiĭ's Theorem was improved by Bihan and Sottile.

**Theorem (Bihan, Sottile (2007))**

*System of $n$ equations and $n$ variables*
*with only $n + l + 1$ distinct monomials.*
*Then, number of positive real solutions bounded by*

$$\frac{e^2 + 3}{4} 2^{\binom{l}{2}} n^l.$$

# Intersection of a trinomial curve with a sparse curve

**Theorem (Li, Rojas, Wang (2003))**

*$f(x, y)$ is a trinomial and $g(x, y)$ is $t$-sparse.*
*Then the system $f(x, y) = g(x, y) = 0$ has at most $2^t - 2$ real solutions.*

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**Theorem (Koiran, Portier, T. )**

*$f(x,y)$ is a trinomial and $g(x,y)$ is t-sparse.*
*Then the system $f(x,y) = g(x,y) = 0$ has at most $\frac{2}{3}t^3 + 5t$ positive real solutions.*

# Sevostyanov's problem

**Theorem (Koiran, Portier, T.)**

*$f(x, y)$ non-zero polynomial of degree $d$ and $g(x, y)$ $t$-sparse.*
*Then the system $f(x, y) = g(x, y) = 0$ has at most $O(d^3 t + d^2 t^3)$ real solutions.*

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The best previous bound was $d(2+d)^{t+1} 2^{\binom{t+1}{2}}$.
The constraint $f \neq 0$ is important.

# Stronger hypothesis

## Theorem

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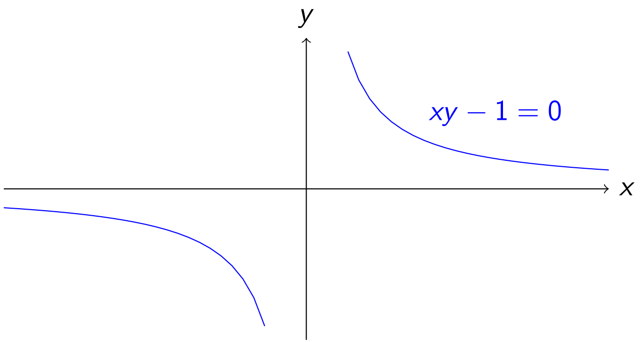

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## Lemma

*$f(x,y)$ non-zero polynomial of degree $d$ and $g(x,y)$ $t$-sparse.*
*Assume that:*

- *$f$ irreducible in $\mathbb{C}[X,Y]$*
- *and finite number of solutions.*

*Then the system $f(x,y) = g(x,y) = 0$ has at most $O(d^3t + d^2t^3)$ distinct real solutions.*

# From the lemma to the theorem

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2. Irreducibility of $f$.
   - $f = f_1 f_2 \ldots f_k$

$$\bigcup_i \text{solutions of } \begin{cases} f_i = 0 \\ g = 0 \end{cases}$$

# Outline of the proof

## Lemma

*$f(x, y)$ irreducible polynomial of degree $d$ and $g(x, y)$ $t$-sparse.*
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1. Rewrite $f(x, y) = 0$ as $y = h(x)$ ($h$ nice).
   To bound the number of roots of $g(x, (h(x))) = \sum^k a_i x^{\alpha_i} h(x)^{\beta_i}$.

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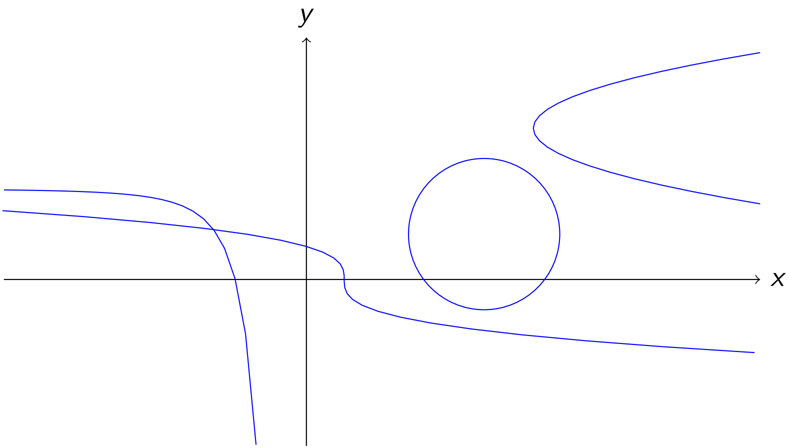

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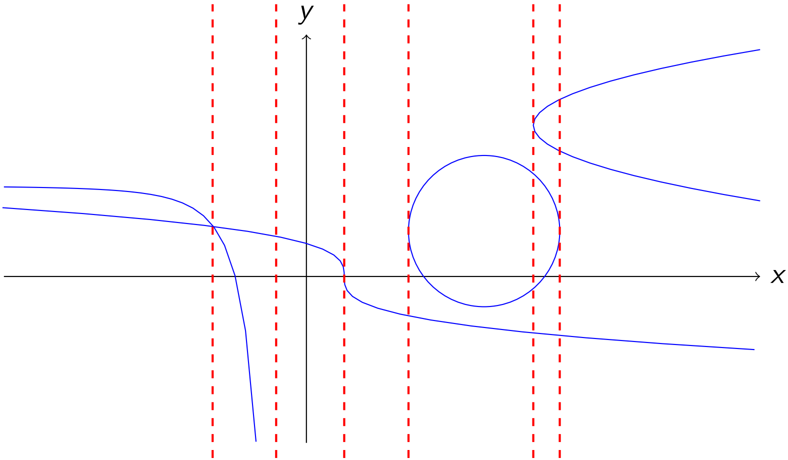

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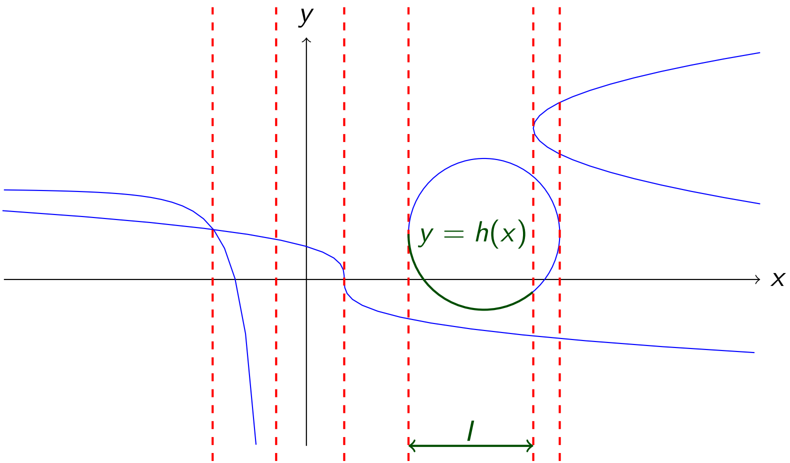

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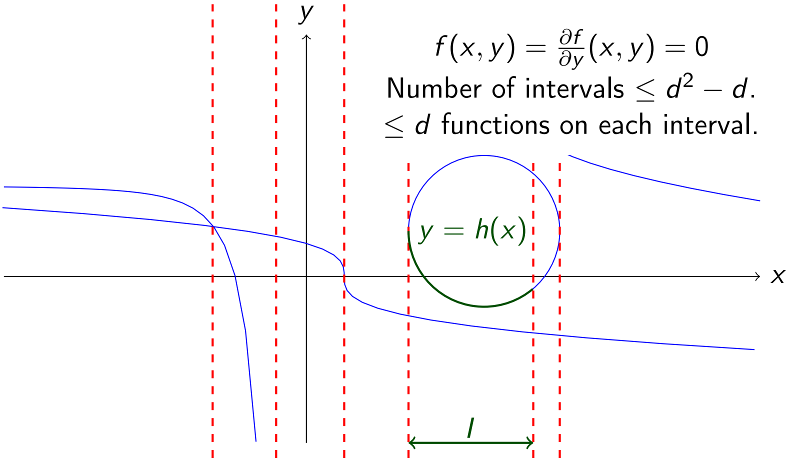

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We will use the Wronskian.

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**Definition:** Let $f_1, \ldots, f_k \in \mathcal{C}^{k-1}(I)$ with $I \subseteq \mathbb{R}$. The *Wronskian* of the family is the determinant of the matrix:

$$W(f_1, \ldots, f_k) = \det \begin{bmatrix} f_1 & f_2 & \ldots & f_k \\ f_1' & f_2' & \ldots & f_k' \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \ldots & f_k^{(k-1)} \end{bmatrix}$$

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- Idea: upper bound the number of roots of a sum by the number of roots of some particular Wronskians.

# From sum to Wronskians

**Lemma**

Let $I$ be a real interval.
If $W(f_1), W(f_1, f_2), \ldots, W(f_1, f_2, \ldots, f_k)$ have no zero on $I$, then $Z_I(f_1 + \ldots + f_k) \leq k - 1$.

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$$W\left(\left(\frac{f_2}{f_1}\right)', \ldots, \left(\frac{f_q}{f_1}\right)'\right) = \left(\frac{1}{f_1}\right)^p W(f_1, \ldots, f_q).$$

# From sum to Wronskians

**Theorem (Koiran, Portier, T.)**

$$Z(f_1 + \ldots + f_k) \leq k - 1 + 2\sum_{j=1}^{k-2} Z(W(f_1, \ldots, f_j))$$

# What remains to be done

**Theorem**

*$f(x,y)$ polynomial of degree $d$ and $g(x,y)$ $t$-sparse.*
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2. To bound the number of roots of
   $W_s = W(a_1 x^{\alpha_1} h^{\beta_1}(x), \ldots, a_s x^{\alpha_s} h^{\beta_s}(x))$.

# Bounds for $W_3$

$$\det \begin{bmatrix} x^{\alpha_1} h^{\beta_1} & x^{\alpha_2} h^{\beta_2} & x^{\alpha_3} h^{\beta_3} \\ \left(x^{\alpha_1} h^{\beta_1}\right)' & \left(x^{\alpha_2} h^{\beta_2}\right)' & \left(x^{\alpha_3} h^{\beta_3}\right)' \\ \left(x^{\alpha_1} h^{\beta_1}\right)'' & \left(x^{\alpha_2} h^{\beta_2}\right)'' & \left(x^{\alpha_3} h^{\beta_3}\right)'' \end{bmatrix}$$

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$$\det \begin{bmatrix} x^{\alpha_1} h^{\beta_1} & x^{\alpha_2} h^{\beta_2} & x^{\alpha_3} h^{\beta_3} \\ \left(x^{\alpha_1 - 1} h^{\beta_1 - 1}\right) (\alpha_1 h + \beta_1 x h') & \dots & \dots \\ \left(x^{\alpha_1 - 2} h^{\beta_1 - 2}\right) (P_2(x, h, h', h'')) & \dots & \dots \end{bmatrix}$$

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$$x^{\alpha_1+\alpha_2+\alpha_3-6}h^{\beta_1+\beta_2+\beta_3-6}\det \begin{bmatrix} x^2h^2 & x^2h^2 & x^2h^2 \\ xh(\alpha_1 h + \beta_1 x h') & \dots & \dots \\ P_2(x,h,h',h'') & \dots & \dots \end{bmatrix}$$

# Consequently

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- Real $\tau$-conjecture (one motivation)

> There exists $c$ such that the univariate polynomial $\sum_{i=1}^{k} \prod_{j=1}^{m} f_{i,j}$ (with $f_{i,j}$ $t$-sparse) has at most $(m+k+t)^c$ real roots.

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This conjecture implies lower bounds on the size of the arithmetic circuits for the permanent.

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  - $f_1 = \ldots = f_n = g = 0$ ($f_i$ dense, $g$ sparse) (almost done)
  - $f = g = 0$ ($f$ and $g$ sparse)
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- Real $\tau$-conjecture (one motivation)

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Thank you!