Characterization of the affine solutions of sparse polynomial systems

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Sparse polynomial systems

Given a polynomial system $f_1, \ldots, f_m \in \mathbb{Q}[X_1, \ldots, X_n]$, we describe algorithmically the algebraic variety $V(f_1, \ldots, f_m)$ of all common zeros in $\mathbb{C}^n$ of the system

$$f_1(X_1, \ldots, X_n) = 0, \ldots, f_m(X_1, \ldots, X_n) = 0.$$

A sparse polynomial system in the variables $X = (X_1, \ldots, X_n)$ over $\mathbb{Q}$ with support the finite sets $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m)$ in $(\mathbb{Z}_{\geq 0})^n$ is a collection of polynomials

$$f_j(X) = \sum_{\alpha \in \mathcal{A}_j} a_{j,\alpha} X^\alpha \qquad j = 1, \ldots, m$$

such that for all $a_{j,\alpha} \in \mathbb{Q} \setminus \{0\}$, $\alpha \in \mathcal{A}_j$.

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Geometric resolution

Let $V = \{\xi_1, \ldots, \xi_D\} \subset \mathbb{C}^n$ be an algebraic variety definable over $\mathbb{Q}$. Let $\mu \in \mathbb{Q}[X_1, \ldots, X_n]$ be a linear form such that $\mu(\xi_i) \neq \mu(\xi_j)$ if $i \neq j$. Then a geometric resolution of V with respect to μ is a family of polynomials $(q, v_1, \ldots, v_n) \in (\mathbb{Q}[U])^{n+1}$ such that

- $q = \prod_{i=1}^{D} (U - \mu(\xi_i)) \in \mathbb{Q}[U]$, and
- the polynomials $v_1, \ldots, v_n \in \mathbb{Q}[U]$ fulfill $\deg(v_j) < D$ for all $1 \leq j \leq n$ and $V = \{(v_1(u), \ldots, v_n(u)) \in \mathbb{C}^n \mid u \in \mathbb{C}, q(u) = 0\}$.

Let $V \subset \mathbb{C}^n$ be an equidimensional variety of dimension r defined by polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$ such that, for each irreducible component W of V, the identity $I(W) \cap \mathbb{Q}[X_1, \ldots, X_r] = \{0\}$ holds. By considering $\mathbb{Q}(X_1, \ldots, X_r) \otimes \mathbb{Q}[V]$, we are in a zero-dimensional situation.

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Sets of zeros of sparse polynomial systems

Even a generic square sparse system can have positive dimensional sets of zeros:

$$\text{Let } F = \begin{cases} f_1 &= aX_1X_2X_3^2 + bX_1X_2X_3 \\ f_2 &= cX_1^2X_3 + dX_1X_3 \\ f_3 &= eX_2^2X_3 + fX_2X_3 \end{cases}$$

The zero set $V(F) \subseteq \mathbb{C}^3$ has 5 components:

- 1 point: $(-\frac{d}{c}, -\frac{f}{e}, -\frac{b}{a})$
- 3 lines: $\{X_1 = 0,\, X_2 = -\frac{f}{e}\}$, $\{X_1 = -\frac{d}{c},\, X_2 = 0\}$ and $\{X_1 = 0,\, X_2 = 0\}$
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$$\text{Let } F = \begin{cases} f_1 = aX_1X_2X_3^2 + bX_1X_2X_3 = X_1X_2X_3(aX_3+b) \\ f_2 = cX_1^2X_3 + dX_1X_3 = X_1X_3(cX_1+d) \\ f_3 = eX_2^2X_3 + fX_2X_3 = X_2X_3(eX_2+f) \end{cases}$$

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Generic sparse systems

Components in the torus

If $V \subset \mathbb{C}^n$ is an irreducible variety of dimension r, the degree of V is $\deg(V) = \max\{\#(H_1 \cap \cdots \cap H_r \cap V) \mid H_1, \ldots, H_r$ are affine hyperplanes in $\mathbb{C}^n$ such that $H_1 \cap \cdots \cap H_r \cap V$ is a finite set$\}$.

If $V \subset \mathbb{C}^n$ is an arbitrary variety, the degree of V is the sum of the degrees of every irreducible component of V.

Lemma Let $F = (f_1, \ldots, f_m)$ be a generic system with supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m)$ in $(\mathbb{Z}_{\geq 0})^n$. If $m > n$, F does not have zeros in $(\mathbb{C}^*)^n$. If $m \leq n$ and $\dim\left(\sum_{j \in J} \mathcal{A}_j\right) \geq \#J$ for all $J \subseteq \{1, \ldots, m\}$, the Zariski closure $V^*(F)$ in $\mathbb{C}^n$ of the set of zeros in $(\mathbb{C}^*)^n$ of F is an equidimensional variety of dimension $n - m$ and degree

$$\mathcal{D} = \mathcal{MV}_n(\mathcal{A}_1, \ldots, \mathcal{A}_m, \Delta^{(n-m)}).$$

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Algorithm for the toric case

Proposition Let $F = (f_1, \dots, f_m)$ be a generic system with supports $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_m)$ in $(\mathbb{Z}_{\geq 0})^n$. If $m \leq n$, there exists a probabilistic algorithm that computes a geometric resolution of $V^*(F)$ with complexity

$$O_{\log}(n^3(N + (n-m)n)\mathcal{D}(\mathcal{D}^2 + (\mathcal{D} + \mathcal{E})\Upsilon)),$$

where

- $N = \sum_{j=1}^{n} \#\mathcal{A}_j$,
- $\mathcal{D} = \mathcal{MV}_n(\mathcal{A}, \Delta^{(n-m)})$,
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Components in the affine space

For all $I \subset \{1, \dots, n\}$, let

- F_I be obtained by evaluating F in $X_i = 0$ for all $i \in I$ and discarding the polynomials that vanish,
- $J_I \subset \{1, \dots, m\}$ be the set of indices of F_I,
- $\pi_I : \mathbb{C}^n \to \mathbb{C}^{n-\#I}$, such that $\pi_I(X_1, \dots, X_n) = (X_i)_{i \notin I}$.
- $\mathcal{A}^I$ be the support set of F_I.

Lemma Let W be an irreducible component of $V(F)$. Denote $I_W = \{i \in \{1, \dots, n\} \mid W \subset \{X_i = 0\}\}$. Then,

- $\dim W = n - \#I_W - \#J_{I_W}$,
- $\pi_{I_W}(W)$ is an irreducible component of $V(F_{I_W}) \subset \mathbb{C}^{n-\#I_W}$ that intersects $(\mathbb{C}^*)^{n-\#I_W}$.

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Combinatorial description of $V(F)$

Let $I \subset \{1, \ldots, n\}$. Then $V(F_I) \cap (\mathbb{C}^*)^{n-\#I} \neq \emptyset$ iff for all $J \subset J_I$, $\dim(\sum_{j \in J} \mathcal{A}_j^I) \geq \#J$. In that case, $V^*(F_I)$ has dimension $n - \#I - \#J_I$.

- $\varphi_I : \mathbb{C}^{n-\#I} \to \mathbb{C}^n$, inserts zeros in the coordinates indexed by I.

Proposition If W is an irreducible component of $V^*(F_I)$, then $\varphi_I(W)$ is an irreducible component of $V(F)$ if and only if for all $I' \subset I$, $\#I' + \#J_{I'} \geq \#I + \#J_I$.

Theorem Let $F = (f_1, \ldots, f_m)$ be a generic system with supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m)$ in $(\mathbb{Z}_{\geq 0})^n$. Then,

$$V(F) = \bigcup_I \varphi_I(V^*(F_I)),$$

where the union is over all $I \subset \{1, \ldots, n\}$ that fulfill the previous conditions. Moreover, $\deg(V(F)) = \sum_I \mathcal{MV}_{n-\#I}(\mathcal{A}^I, \Delta^{(n-\#I-\#J_I)})$.

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Algorithm GenericAffineSolve for generic sparse systems

Input: A generic system $F = (f_1, \dots, f_m)$ with supports $\mathcal{A}$ in $(\mathbb{Z}_{\geq 0})^n$.

1. Find all $I \subset \{1, \dots, n\}$ such that $\#I + \#J_I \leq n$ and for all $I' \subset I$, $\#I' + \#J_{I'} \geq \#I + \#J_I$.
2. Find for every I in step 1 a geometric resolution R_I of $V^*(F_I)$.
3. Compute $\varphi_I(R_I)$ and group by dimensions.

Output: A family of geometric resolutions that describe each equidimensional component of $V(F)$.

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The result for generic sparse systems

Theorem Let $F = (f_1, \ldots, f_m)$ be a generic system with supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m)$ in $(\mathbb{Z}_{\geq 0})^n$. The probabilistic algorithm `GenericAffineSolve` computes a list of geometric resolutions that describe each equidimensional component of $V(F)$ within complexity

$$O_{\log}(n2^n N + n^3(N + n^2)\mathcal{D}(\mathcal{D}^2 + (\mathcal{D} + \mathcal{E})\Upsilon)),$$

Example Let F be the generic system of n polynomials in $2n$ variables

$$F = \begin{cases} f_1(X_1, \ldots, X_{2n}) = a_{11}X_1X_2 + a_{12}X_3X_4 + \cdots + a_{1n}X_{2n-1}X_{2n} \\ \vdots \\ f_n(X_1, \ldots, X_{2n}) = a_{n1}X_1X_2 + a_{n2}X_3X_4 + \cdots + a_{nn}X_{2n-1}X_{2n} \end{cases}.$$

$V(F)$ has 2^n irreducible components of dimension n associated to the sets $I_S = \{2k-1 \mid k \in S\} \cup \{2k \mid k \in \{1, \ldots, n\} \setminus S\}$ for all $S \subset \{1, \ldots, n\}$.

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Unmixed case

When $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m)$ is unmixed (i.e. $\mathcal{A}_1 = \cdots = \mathcal{A}_m$), then we can reformulate our characterization:

Proposition Let $F = (f_1, \ldots, f_m)$ be a generic system with unmixed supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m)$ in $(\mathbb{Z}_{\geq 0})^n$. For $k = 0, \ldots, n-1$, let $V_k(F)$ be the equidimensional component of $V(F)$ of dimension k. Then,

- if $k \neq n - m$, $V_k(F) = \bigcup_I \{x \in \mathbb{C}^n \mid x_i = 0 \text{ for all } i \in I\}$,
- if $m \leq n$, $V_{n-m}(F) = V^*(F) \cup \bigcup_I \{x \in \mathbb{C}^n \mid x_i = 0 \text{ for all } i \in I\}$,

where for each dimension k the union is over all $I \subset \{0, \ldots, n\}$ such that $\#I = n - k$, $\#J_I = 0$ and $\#J_{I'} = m$ for all $I' \subset I$.

Non-generic sparse systems

A bound for the degree of the variety

- A bound for the degree (Krick-Pardo-Sombra'01):

Let $F = (f_1, \dots, f_m)$ be a system of m polynomials in $\mathbb{C}[X_1, \dots, X_n]$ with supports $\mathcal{A}_1, \dots, \mathcal{A}_m$. The degree of the variety $V(F)$ is bounded above by

$$\mathcal{MV}_n((\bigcup_{i=1}^{m} \mathcal{A}_i \cup \Delta)^{(n)}).$$

Proposition Let $F = (f_1, \dots, f_n)$ be a system of n polynomials in $\mathbb{C}[x_1, \dots, x_n]$ with supports $\mathcal{A}_1, \dots, \mathcal{A}_n$. The degree of the variety $V(F) \subset \mathbb{C}^n$ is bounded above by $\mathcal{MV}_n(\mathcal{A}_1 \cup \Delta, \dots, \mathcal{A}_n \cup \Delta)$.

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Solving non-generic square sparse systems

Let $F = (f_1, \ldots, f_n)$ be an arbitrary sparse polynomial system with supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ in $(\mathbb{Z}_{\geq 0})^n$. By intersecting with r generic hyperplanes we obtain $\deg(W)$ points in each irreducible component W of $V(F)$ with dimension r.

The idea is to represent each equidimensional component by this set of points, called witness points.

Let $L_1, \ldots, L_n$ be n generic linear forms and, for each $r = 0, 1, \ldots, n$, take the system $F^{(r)} = (f_1, \ldots, f_n, L_1, \ldots, L_r)$. Taking generic coefficients (b_{ji}), the isolated zeros of $F^{(r)}$ are isolated zeros of the system with n polynomials

$$H_r = (f_1(x) + \sum_{i=1}^{r} b_{1i} L_i, \ldots, f_n(x) + \sum_{i=1}^{r} b_{ni} L_i).$$

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Algorithm PointsInEquidComps for non-generic square sparse systems

Input: A system $F = (f_1, \ldots, f_n)$ with supports $\mathcal{A}$ in $(\mathbb{Z}_{\geq 0})^n$, and the mixed cells in a fine mixed subdivision $S_\omega(\mathcal{A}_1 \cup \Delta, \ldots, \mathcal{A}_n \cup \Delta)$.

1. Choose G with supports $\mathcal{A}_1 \cup \Delta, \ldots, \mathcal{A}_n \cup \Delta$ and random integer coefficients.
2. Find the zeros of G in $\mathbb{C}^n$ using $S_\omega(\mathcal{A}_1 \cup \Delta, \ldots, \mathcal{A}_n \cup \Delta)$.
3. Choose $L_1, \ldots, L_n$ linear forms and random coefficients b_{ji}.
4. For each $0 \leq r \leq n-1$:
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Algorithm PointsInEquidComps for non-generic square sparse systems

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The result for non-generic square sparse systems

Theorem Let $F = (f_1, \ldots, f_n)$ be a system of polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$ with supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ in $(\mathbb{Z}_{\geq 0})^n$. The probabilistic algorithm `PointsInEquidComps` computes a family $(R^{(0)}, \ldots, R^{(n-1)})$ of geometric resolutions of finite sets of points containing a set of witness points of every equidimensional component of $V(F)$. The order of complexity is

$$O_{\log}(n^4 N_\Delta d \mathcal{D}_\Delta^2 \Upsilon_\Delta)$$

where

- $N_\Delta = \sum_{j=1}^{n} \#(\mathcal{A}_j \cup \Delta)$,
- $d = \max_{1 \leq j \leq n}\{\deg(f_j)\}$,
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Thanks!!!