Sign Conditions for Injectivity of Generalized Polynomial Maps with Applications to Chemical Reaction Networks and Real Algebraic Geometry

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Solving Polynomial Equations workshop
Simons Institute
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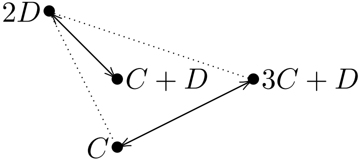
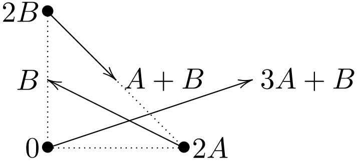
Outline of talk

- Steady states
- **Theorem**: sign conditions for ≤ 1 positive real solution
- Connection to Descartes' Rule of Signs
- *Four questions about steady states*

Question 1: inward-pointing networks

- **Theorem** (Gopalkrishnan, Miller, AS 2014):
  "Inward-pointing" networks like the one on the right (below) *always have* ≥ 1 *positive steady state.*





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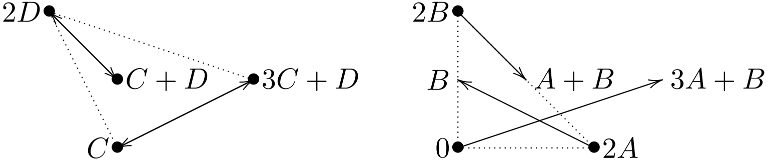

- **Question:** What about networks like the one on the left?
- *Wild speculation*: The above pictures are Newton polytopes. Is there any connection to the Bernstein bound on the number of solutions in $(\mathbb{C}^*)^n$ arising from a mixed-volume computation?

QUESTION 2: MULTISITE PHOSPHORYLATION

The n-**site (sequential and distributive) phosphorylation network** is:

$$S_0 + E \underset{k_{\text{off}_0}}{\overset{k_{\text{on}_0}}{\rightleftarrows}} ES_0 \overset{k_{\text{cat}_0}}{\rightarrow} S_1 + E \underset{k_{\text{off}_1}}{\overset{k_{\text{on}_1}}{\rightleftarrows}} \rightarrow \cdots \rightarrow S_{n-1} + E \underset{k_{\text{off}_{n-1}}}{\overset{k_{\text{on}_{n-1}}}{\rightleftarrows}} ES_{n-1} \rightarrow S_n + E$$

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- The rates $k_{\mathrm{cat}_0}, \ldots$ which yield multiple steady states are characterized by sign conditions, but what is the max #?
- **Theorem** (Wang and Sontag 2008): The n-site phosphorylation system has $\leq 2n - 1$ steady states.

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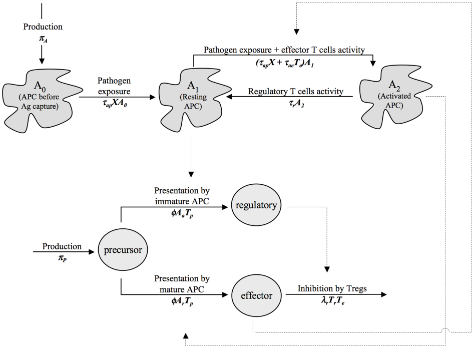
- **Theorem** (Wang and Sontag 2008): The n-site phosphorylation system has $\leq 2n-1$ steady states.
- **Conjecture**: The n-site phosphorylation system has $\leq n+1$ steady states for even n and $\leq n$ for odd n.
- Conjecture is true for $n = 1, 2$, and disproven by Flockerzi, Holstein, and Conradi for odd $n \geq 3$ and $n = 4$.
  What about even $n \geq 6$?

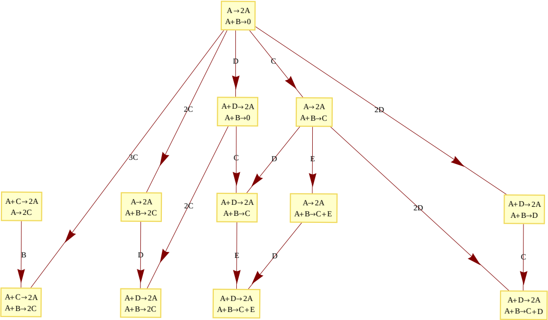
Question 3: immune system network



- Can we do a *direct analysis* of the (multiple) steady states of this network (Fouchet and Regoes, Plos One 2008)?
- Alternate approach: lifting steady states from small to large networks (next slide)

Question 4: lifting steady states

- **Theorem** (Joshi and AS 2013): Assume that G and N are networks with inflows/outflows, such that N *is obtained from* G *by removing reactions and species.* Then if N admits multiple steady states, then G does too.



Question: For which other pairs (G, N) can we lift (multiple) steady states?

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- *I am recruiting a postdoc.*

Thank you.