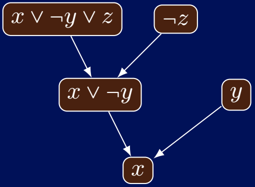
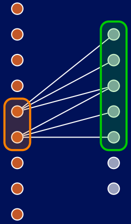
A Lower Bound for k-DNF Resolution on Random CNF Formulas via Expansion





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joint work with Anastasia Sofronova

Simons Institute
June 17, 2022

St Petersburg University

PDMI
RAS

Proof Systems

Definition[Cook, Reckhow 79]

Proof system for $L \Leftrightarrow$ poly-time algorithm Π: $\{0,1\}^* \times \{0,1\}^* \to \{0,1\}$:

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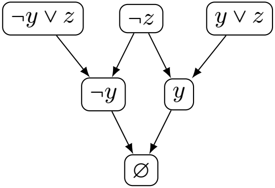

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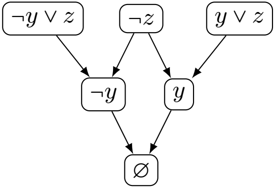
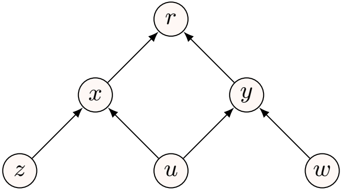


Cutting Planes: proof is a sequence of inequalities over $\mathbb{Z}$ $(p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, \ldots, p_\ell \geq 0)$:

- p_i is an encoding of $C \in \varphi$, $x_k \geq 0$ or $-x_k + 1 \geq 0$;
- $\frac{p_i \quad p_j}{p_k}$, $(p_i \geq 0) \wedge (p_j \geq 0)$ imply $(p_k \geq 0)$ over $\mathbb{Z}^n$;
- $p_\ell = 1$.

Pebbling



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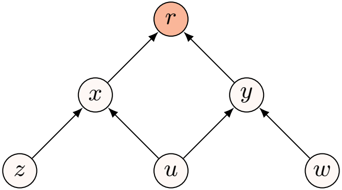

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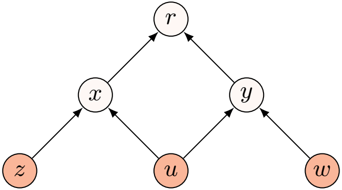

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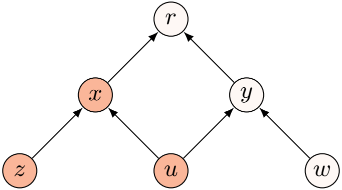

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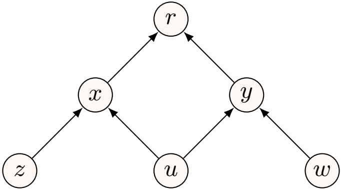

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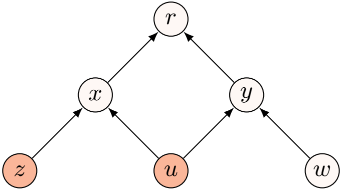

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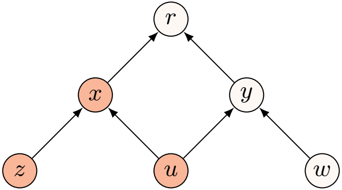

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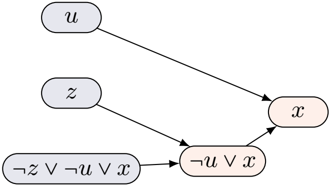

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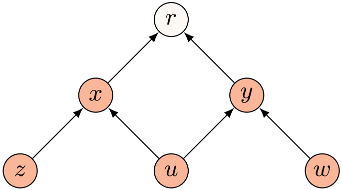

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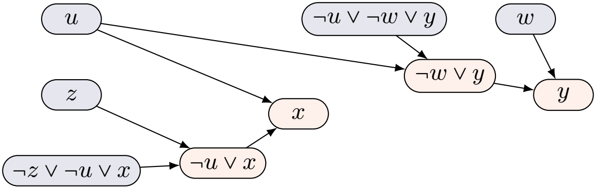

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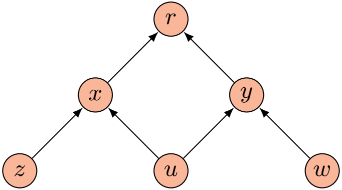

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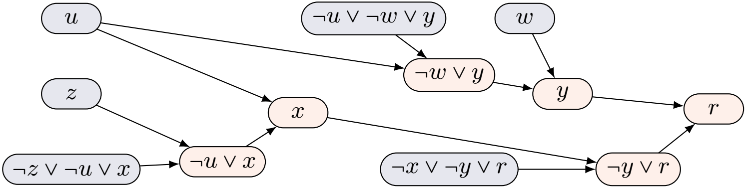

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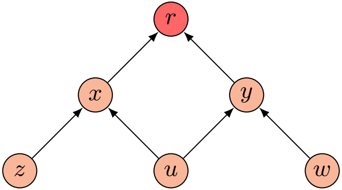

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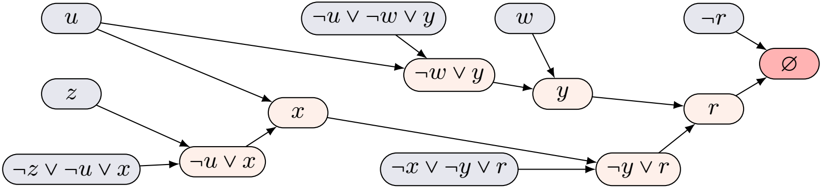
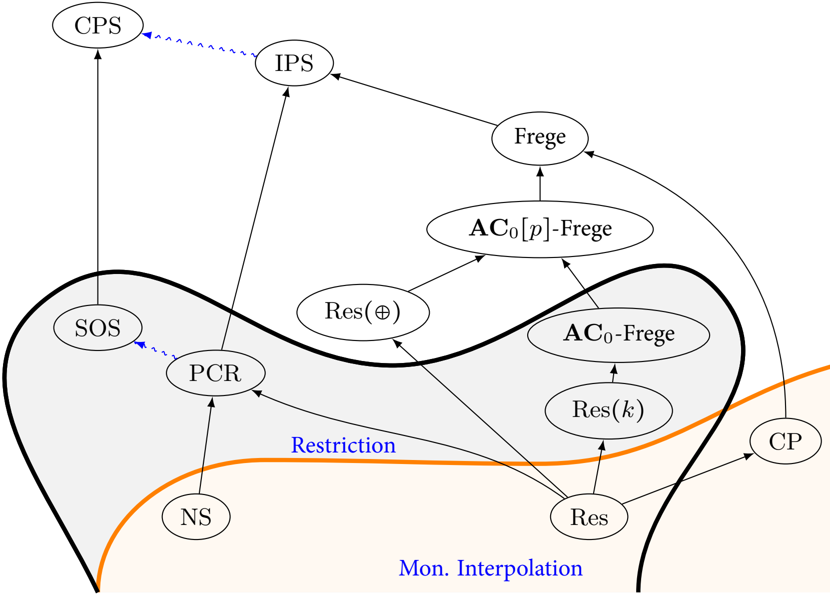


Lower bounds in proof complexity



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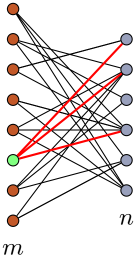
- Random Δ-CNF formulas
- Clique formulas
- Pseudorandom generator formulas

Random Δ-CNF



- m clauses;
- n variables;
- Δ neighbours: $\binom{n}{\Delta}$ possibilities;
- negations (uniformly at random);
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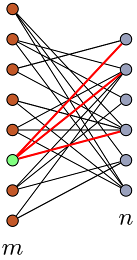

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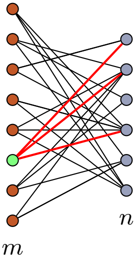

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- Fiege's conjecture: $\mathfrak{D} = \mathcal{O}(1) \Rightarrow$ no poly-time algorithm may "prove" unsatisfiability of random $\mathcal{O}(1)$-CNF.
  - Non-approximability of many problems.

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- Top-down (informal): decision "tree" with conjunctions of k literals.

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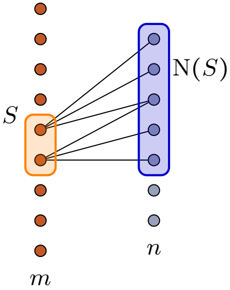

- (r, Δ, c)-expander;
- $\forall S \subseteq L, |S| \leq r \Rightarrow \mathrm{N}(S) \geq c|S|$.

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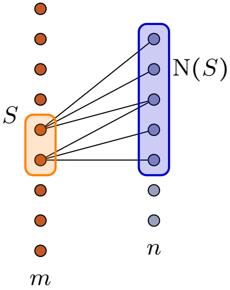

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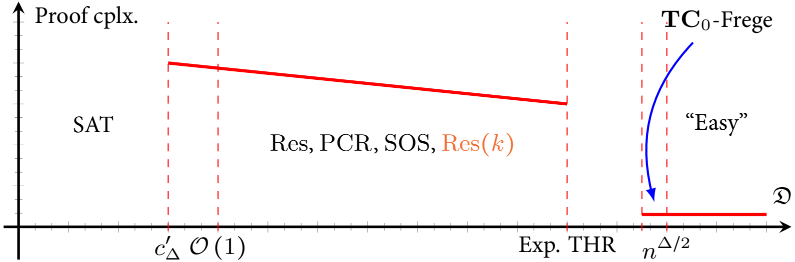

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Theorem

G_φ is an $(r, \Delta, 0.98\Delta)$-expander $\Rightarrow$ $\forall\delta > 0$ if:

$$n^\delta \left(\frac{n}{0.4r}\right)^{20k^2} = o(r/k)$$

then any $\mathrm{Res}(k)$ proof of φ has size at least 2^{n^δ}.

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- Other hard examples.