Global community detection using individual-centered partial information networks

Rachel Wang

Joint work with Xiao Han and Xin Tong

School of Mathematics and Statistics, University of Sydney

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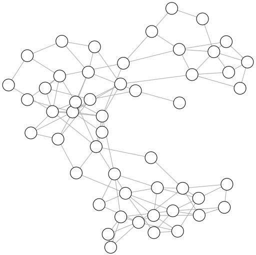
Social network modeling

- $G = (\mathcal{V}, \mathcal{E})$
  - $\mathcal{V}$: set of individuals $[n] := \{1, \ldots, n\}$
  - $\mathcal{E}$: set of edges, assumed to be binary and undirected for simplicity
- Statistical problems: estimating community memberships, subgraph counts, node covariates, ...
- Most of the current literature assumes either a global view of the network or multiple subgraphs (Mukherjee et al. 2021) can be sampled, from which information can be combined.



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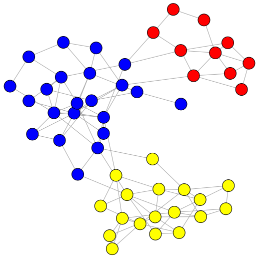

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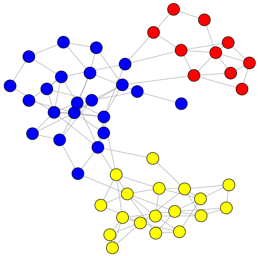
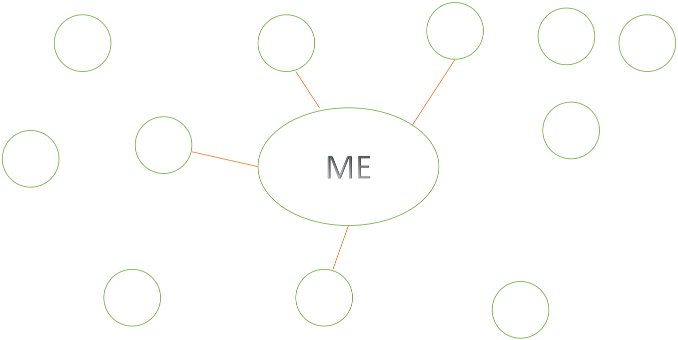
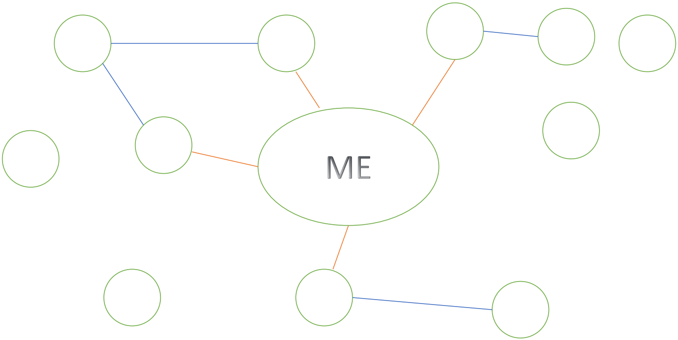


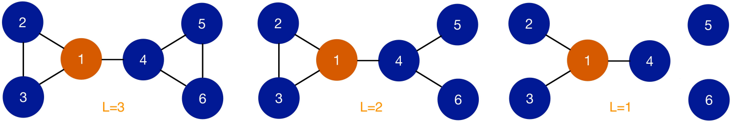
How much does a person understand about the connections in the full network?



Beyond friends?



A toy example of individual-centered partial information networks



- An illustration of a network consisting of 6 individuals.
- The left panel is the full network.
- Suppose individual 1 is the person of interest.
- The left, middle and right panel show individual 1's view of the network when their knowledge depth is $L = 3, 2, 1$, respectively.

Key: Characterizing the amount of partial (local) information by path length. We focus on $L = 2$.

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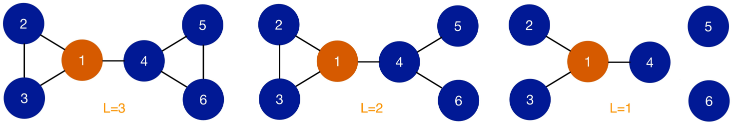

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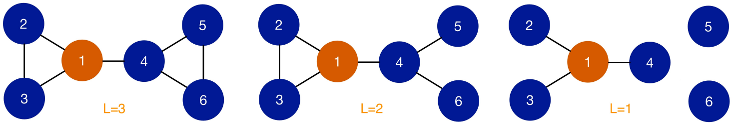

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Q: Can one learn the global community memberships based on their partial knowledge of the full network?

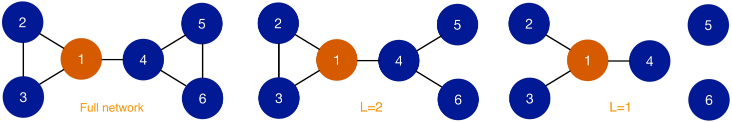
Existing literature

- The structure of our partial network is related to existing network sampling schemes, e.g., egocentric sampling, snowball sampling and respondent driven sampling (RDS).
- Main differences:
  - We are interested in what one instance of partial network can offer, which allows us to compare what network structure is visible to each individual.
  - Most RDS based methods are focused on estimating node covariates, while we are interested in latent community structure.
  - Multiple sampling may not be feasible in networks with restricted access (e.g., a terrorist network)

Preliminaries

- Recall $G = (\mathcal{V}, \mathcal{E})$ is the full network of n individuals.
- G can be represented by a $n \times n$ binary, symmetric adjacency matrix $\mathbf{A} = (a_{ij})$, where $a_{ij} = \begin{cases} 1, & (i,j) \in \mathcal{E}\,, \\ 0, & (i,j) \notin \mathcal{E}\,. \end{cases}$
- Let $\mathbf{B} = (b_{ij})$ be individual 1's perceived adjacency matrix based on knowledge depth $L = 2$.

How is **B** related to **A**?



In this toy example, for individual 1 with knowledge depth $L = 2$:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

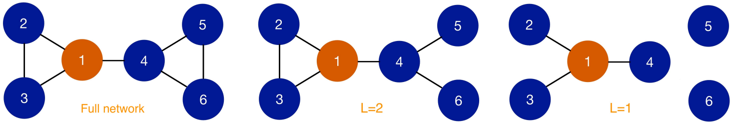
In general,

$$b_{ij} = a_{ij}(1 - \mathbb{I}(a_{1i} = 0)\mathbb{I}(a_{1j} = 0)).$$

It follows that

$$\mathbf{B} = -\mathbf{SAS} + \mathbf{AS} + \mathbf{SA}, \quad \text{where} \quad \mathbf{S} = \operatorname{diag}(a_{11}, \ldots, a_{1n}).$$

How is $\mathbf{B}$ related to $\mathbf{A}$?



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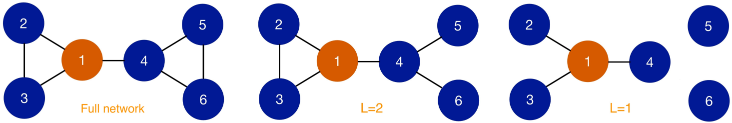

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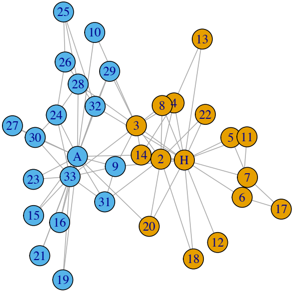
A peak at the results

individual of interest	H	2	3	A	20	32
degrees	16	9	10	17	3	6
fraction of edges	.654	.513	.705	.641	.526	.654
detection accuracy	.559	.706	.941	.706	.941	.794

Zachary's karate club



Preliminaries - low rank assumption

- We assume $\text{rank}(\mathbb{E}\mathbf{A}) = K$ and
- the eigen decomposition (reduced form) $\mathbb{E}\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^\top$.
- $\mathbf{D} = \text{diag}(d_1, \ldots, d_K)$, in which d_i is the i-th largest eigenvalue (by magnitude) of $\mathbb{E}\mathbf{A}$,
- $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_K)$ is the corresponding eigenvector matrix.
- Generate $\mathbf{A}$ as independent Bernoulli from $\mathbb{E}\mathbf{A}$.
- Usually theoretical analysis usually proceeds by noting

$$\mathbf{A} = \underbrace{\mathbb{E}\mathbf{A}}_{\text{signal}} + \underbrace{(\mathbf{A} - \mathbb{E}\mathbf{A})}_{\text{noise}}.$$

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Preliminaries - low rank assumption

What about $\mathbf{B} = -\mathbf{SAS} + \mathbf{AS} + \mathbf{SA}$? We can show that

$$\mathbf{B}_E = -\mathbf{S}(\mathbb{E}\mathbf{A})\mathbf{S} + (\mathbb{E}\mathbf{A})\mathbf{S} + \mathbf{S}(\mathbb{E}\mathbf{A})$$

is the "signal" term in the sense that

$$\|\mathbf{B} - \mathbf{B}_E\| \leq \text{smallest singular value of } \mathbf{B}_E$$

.

Theoretical properties of $\mathbf{B}_E$

Theorem 1 (Informal, eigenvalues and eigenvectors)

Suppose that $\mathbf{V}^\top\mathbf{S}\mathbf{V}$ and $\mathbf{I}-\mathbf{V}^\top\mathbf{S}\mathbf{V}$ are invertible. We have $\mathrm{rank}(\mathbf{B}_E)=2K$. Then for $i=-K,\ldots,-1,1,\ldots,K$, $(x_i^{-1},\mathbf{q}_i)$ is an eigenvalue / eigenvector pair of $\mathbf{B}_E$, iff

- x_i is a solution of $\det(\mathbf{H}(x))=0$, where $\mathbf{H}(x)=\mathbf{I}-x\mathbf{D}\mathbf{V}^\top\mathbf{S}\mathbf{V}-x^2\mathbf{D}(\mathbf{I}-\mathbf{V}^\top\mathbf{S}\mathbf{V})\mathbf{D}\mathbf{V}^\top\mathbf{S}\mathbf{V}$.
- $\mathbf{q}_i=\mathbf{S}\mathbf{V}\mathbf{q}_{1i}+(\mathbf{I}-\mathbf{S})\mathbf{V}\mathbf{q}_{2i}$, where $\mathbf{q}_{1i}$ is an eigenvector of $\mathbf{H}(x_i)$ corresponding to the zero eigenvalue, and

$$\mathbf{q}_{2i}=x_i\mathbf{D}\mathbf{V}^\top\mathbf{S}\mathbf{V}\mathbf{q}_{1i}.$$

Let $\mathbf{Q}=(\mathbf{q}_K,\ldots,\mathbf{q}_1,\mathbf{q}_{-1},\ldots,\mathbf{q}_{-K})=\mathbf{S}\mathbf{V}\mathcal{Q}_1+(\mathbf{I}-\mathbf{S})\mathbf{V}\mathcal{Q}_2$.
We can choose $\mathbf{q}_i$'s such that

$$\mathbf{Q}^\top\mathbf{Q}=\mathbf{I}_{2K\times 2K}.$$

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Theoretical properties of $\mathbf{B}_E$

Let $p_n = \max_{i,j} \mathbb{P}(a_{ij} = 1)$, assume $\min_{j \geq 2} \mathbb{P}(a_{1j} = 1) \sim p_n$, $1 - c > p_n \gg \log n/n$ for some constant $c > 0$.

Theorem 2 (Informal, form of eigenvalues)

With mild conditions on $\mathbf{D}$, $\mathbf{V}$, w.h.p., we have

- $|x_i|^{-1} \sim np_n^{3/2}$ for $i \in [\pm K]$;
- if $p_n \to 0$, with additional conditions, we obtain the high probability expressions of x_i^{-1};
- the expressions of x_i^{-1} suggest $\lambda_{\min} = \lambda_K(\mathbf{V}^\top \mathbf{S} \mathbf{V})$ determines the gap between the smallest eigenvalue (in magnitude) and 0.

Theoretical properties of $\mathbf{B}_E$

Summary so far

- The eigenvectors take the form
  $\mathbf{Q} = (\mathbf{q}_K, \ldots, \mathbf{q}_1, \mathbf{q}_{-1}, \ldots, \mathbf{q}_{-K}) = \underbrace{\mathbf{S}\mathbf{V}\mathcal{Q}_1}_{\text{neighbors of node 1}} + \underbrace{(\mathbf{I} - \mathbf{S})\mathbf{V}\mathcal{Q}_2}_{\text{non-neighbors}}.$
- Order of the eigenvalues, $|x_i|^{-1} \sim np_n^{3/2}$ w.h.p.
- $\lambda_{\min} = \lambda_K(\mathbf{V}^\top \mathbf{S}\mathbf{V})$ detemines the signal strength.
  - $\lambda_{\min}$ influences the performance of spectral clustering $\Rightarrow$ measure of how important individual 1 is $\Rightarrow$ centrality measure
  - $\lambda_{\min}$ lies between 0 and 1.
  - Can be estimated using empirical version of $\mathbf{V}$ from $\mathbf{A}$.
  - When $K = 1$, $\lambda_{\min}$ bears connections to both degree centrality and eigenvector centrality.

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Introducing a concrete model

The stochastic block model (SBM, Holland et al. 1983)

- $\mathbb{E}\mathbf{A} = \mathbf{\Pi}\mathbf{P}\mathbf{\Pi}^\top$, where $\mathbf{P} = (p_{kl})$ is a symmetric $K \times K$ matrix, $\mathbf{\Pi} = (\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_n)^\top \in \mathbb{R}^{n\times K}$ and individual i's membership vector $\boldsymbol{\pi}_i \in \{\mathbf{e}_1, \ldots, \mathbf{e}_K\}$.
- In this model, when individual i belongs to community k and individual j belongs to community l, we have

$$\mathbb{P}(a_{ij} = 1) = \mathbb{E}a_{ij} = \boldsymbol{\pi}_i^\top \mathbf{P}\boldsymbol{\pi}_j = p_{kl}\,.$$

- Idea: there are $2K$ types of rows in $\mathbf{Q}$ (eigenvectors of $\mathbf{B}_E$); $\mathbf{Q}$ is close to the empirical version $\mathbf{W}$, whose columns are eigenvectors of observed $\mathbf{B}$.

Rationale behind our algorithm

Recall $p_n = \max_{i,j} \mathbb{P}(a_{ij} = 1)$.

Condition

$\min_{k\in[K]} p_{1k} \sim p_n$. $\min_{k\in[K]} \sum_{j\in[n]} \mathbb{I}(\boldsymbol{\pi}_j = \mathbf{e}_k) \geq cn$ and $\sigma_K(\mathbf{P}) \geq cp_n$. Moreover, $1 - c \geq p_n \gg (1/n)^{1/2}$.

Lemma 1 ($2K$ different rows in $\mathbf{Q}$)

For any $2K \times 2K$ orthogonal matrix $\mathbf{O}$, it holds w.h.p. that for $i, j \in [n]$,

$$\boldsymbol{\pi}_i \neq \boldsymbol{\pi}_j \Longrightarrow \left\| \underbrace{\mathbf{Q}(i)}_{1\times 2K} \mathbf{O} - \underbrace{\mathbf{Q}(j)}_{1\times 2K} \mathbf{O} \right\|_2 \geq \sqrt{\frac{2}{cn}},$$

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_j, a_{1i} \neq a_{1j} \Longrightarrow \|\mathbf{Q}(i)\mathbf{O} - \mathbf{Q}(j)\mathbf{O}\|_2 \geq \sqrt{\frac{2}{cn}},$$

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_j, a_{1i} = a_{1j} \Longrightarrow \|\mathbf{Q}(i)\mathbf{O} - \mathbf{Q}(j)\mathbf{O}\|_2 = 0.$$

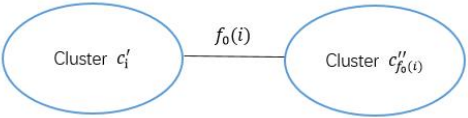
Main algorithm for SBM

1. $\underbrace{\{\mathbf{W}(i) : a_{1i} = 1\}}_{\text{apply } k-\text{means}}$ and $\underbrace{\{\mathbf{W}(i) : a_{1i} = 0\}}_{\text{apply } k-\text{means}}$ $\rightarrow$ return $2K = K + K$ clusters.
2. Merge the $2K$ clusters $\{c'_1, \ldots, c'_K\}$ and $\{c''_1, \ldots, c''_K\}$ into K communities.

Some intuition about the merging step: for $i, j > 1$ and $i \neq j$,

$$\underbrace{\mathbb{P}(b_{ij} = 1 | a_{1i} = a_{1j} = 1)}_{\rightarrow \mathbf{P}^{\mathbf{S},\mathbf{S}}} = \underbrace{\mathbb{P}(b_{ij} = 1 | a_{1i} = 1, a_{1j} = 0)}_{\rightarrow \mathbf{P}^{\mathbf{S},\mathbf{I}-\mathbf{S}}} = \underbrace{\mathbb{P}(a_{ij} = 1)}_{\rightarrow \mathbf{P}} .$$

- Clusters are identifiable only up to label permutation.
- Find the "best" permutation $f_0 : [K] \rightarrow [K]$ to *match* $\widehat{\mathbf{P}}^{\mathbf{S},\mathbf{S}}$ and $\widehat{\mathbf{P}}^{\mathbf{S},\mathbf{I}-\mathbf{S}}$.
- Merge the $2K$ clusters according to f_0.



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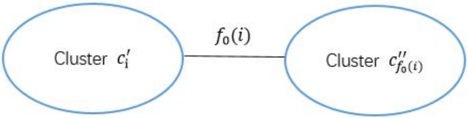

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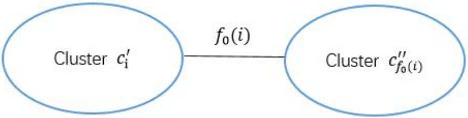

Consistency of algorithm

Theorem 3 (consistency under SBM)

Under some additional separation condition on $\mathbf{P}$ and $p_n \gg (\log n/n)^{1/4}$, w.h.p. ,

$$\text{Proportion of misclustered nodes} = O(\frac{1}{np_n^2}).$$

That is, the algorithm has the almost exact recovery property.

Extension to the degree-corrected SBM

Adding degree heterogeneity, the degree-corrected stochastic block model (DCSBM, Karrer and Newman 2011)

- $\mathbb{E}(\mathbf{A}|\Theta) = \Theta\mathbf{\Pi}\mathbf{P}\mathbf{\Pi}^\top\Theta$, where $\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_n)$ is the set of degree parameters associated with the nodes.
- When individual i belongs to community k and individual j belongs to community l, we have

$$\mathbb{P}(a_{ij} = 1|\Theta) = \theta_i\theta_j p_{kl}\ .$$

- Assume $\theta_i \in (0, 1]$, $i \in [n]$ are i.i.d. random variables with $\mathbb{E}(\theta_i) = \theta \sim 1$.

Main algorithm for DCSBM

1. $\underbrace{\{\mathbf{W}(i) : a_{1i} = 1\}}_{\text{apply spherical } k-\text{median}}$ and $\underbrace{\{\mathbf{W}(i) : a_{1i} = 0\}}_{\text{apply spherical } k-\text{median}}$ $\rightarrow$ return $2K = K + K$ clusters.
   Spherical k-median (Lei and Rinaldo 2015): e.g., for $\{i \in [n] : a_{1i} = 1\}$,

$$\operatorname{argmin}_{\left\{\substack{\mathbf{x}_i : \mathbf{x}_i \text{ is } 2K\text{-dimensional row vector} \\ \text{with } a_{1i} = r \text{ and } |\{\mathbf{x}_i\}_{a_{1i}=1}| \leq K}\right\}} \sum_{i : a_{1i} = 1, \mathbf{W}(i) \neq 0} \left\| \frac{\mathbf{W}(i)}{\|\mathbf{W}(i)\|_2} - \mathbf{x}_i \right\|_2$$

2. Merge the $2K$ clusters $\{c'_1, \ldots, c'_K\}$ and $\{c''_1, \ldots, c''_K\}$ into K communities, accounting for degree heterogeneity.
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A conditional setting emphasizing individual differences

- Can the clustering error bound reflect neighborhood features of individual 1? Consider conditioning on the neighborhood **S**.
- θ_i generated from a two-component mixture with CDF $yF_1(x) + (1-y)F_2(x)$, $y \in (0,1)$. $\mu_2 \sim 1$ and $\mu_1 \leq \mu_2$. (non-hub vs. hub nodes)
- Let

$$n_{jk} = |\{i : a_{1i} = 1, \theta_i \text{ generated from } F_j, \text{ and } i \in \text{Community } k\}|$$

for $k = 1, \ldots, K$ and $j = 1, 2$.

Consistency of algorithm, conditional setting

Theorem 4 (consistency under DCSBM)

For $\mathbf{S}$ and Θ satisfying some constraints, and

$$\mu_1^{-1} n \sqrt{\frac{1}{p_n \min_{k\in[K]} n_{2k}^2 (n_{2k}\mu_2^2 + n_{1k}\mu_1^2)}} \ll p_n \,,$$

conditioned on $\mathbf{S}$ and Θ w.h.p.,

$$\text{Proportion of misclustered nodes} \\ = O\left(\mu_1^{-1} \sqrt{\frac{1}{p_n \min_{k\in[K]}(n_{2k}\mu_2^2 + n_{1k}\mu_1^2)}}\right).$$

- Knowing more powerful neighbors across all communities helps.
- As a centrality measure for individual, $\lambda_{\min}$ behaves like $\min_{k\in[K]} \frac{n_{1k}\mu_1^2 + n_{2k}\mu_2^2}{n}$.

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Simulation with DCSBM

Setting: $\mathbf{P} = \begin{pmatrix} 3q & q \\ q & 3q \end{pmatrix}$. $K = 2$ groups have equal sizes.
$\theta_i \sim$ i.i.d. Unif(0.5,1.5)

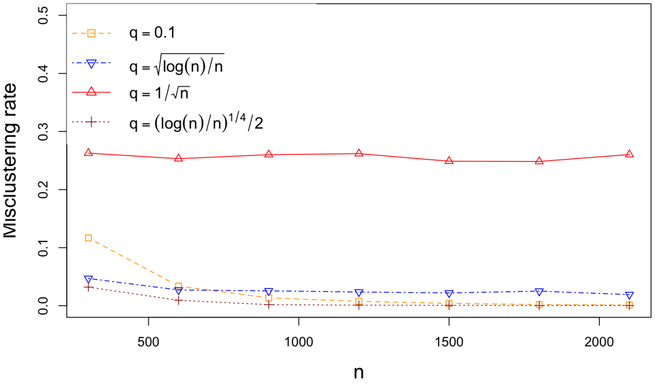

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Table: Correlations between centrality measures and clustering accuracy

	Pearson	Spearman
degree	0.602	0.679
fraction of edges	0.624	0.696
eigen centrality	0.607	0.684
betweenness	0.581	0.700
$\widehat{\lambda}_{\min}$	0.742	0.822

Simulation with DCSBM

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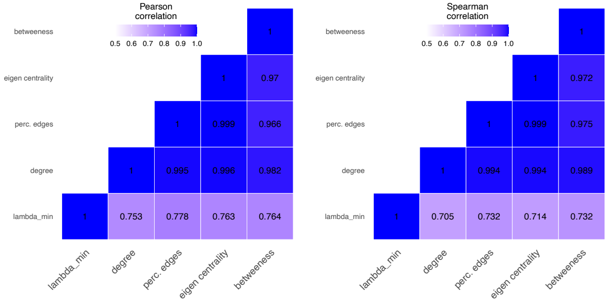
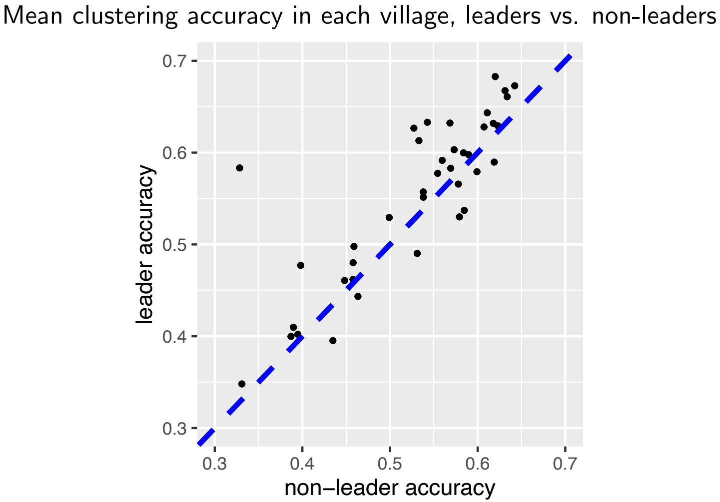


Microfinance in Indian villages

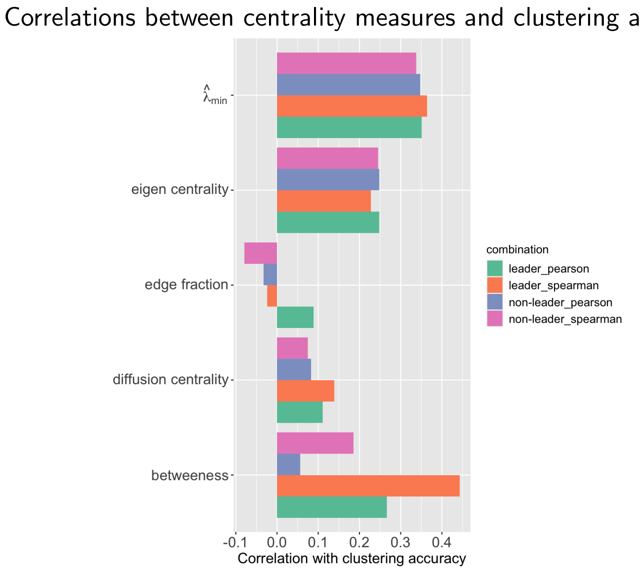
Banerjee et al. *Science* (2013)

- Modeling the spread of information about a microfinance program in Indian villages.
- Social network in each village: households as nodes, each edge is undirected and binary representing any of the 12 relationships collected in the survey (e.g., borrowing / lending money or material goods).
- Caste information as community labels
- Each village has a few predefined leaders serving as "injection" points for information.
- We analyze 39 villages, with the number of households varying between 24-155 and K between 2-4.

Microfinance in Indian villages

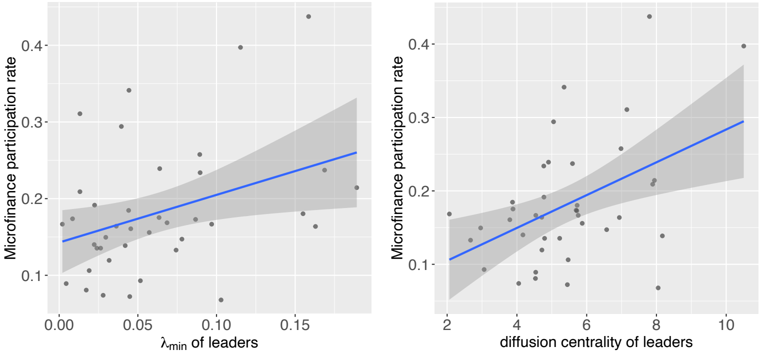


Microfinance in Indian villages



Microfinance in Indian villages

Program participation rate as a function of (left) $\hat{\lambda}_{\min}$ with p-value 0.022; (right) diffusion centrality with p-value 0.003.



Summary and future work

We have introduced an individual-centered partial information framework to study social networks.

- Theoretical properties of the main signal term in the partial adjacency matrix
- Consistent community detection under SBM and DCSBM
- Centrality measure based on eigen gap

Many interesting problems ahead:

- Including only individuals reached by the partial network
- mixed membership block models
- $L = 3$
- Determining K
- Imprecise knowledge about neighbors' neighbors
- Multiple individuals' partial information
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Acknowledgements

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- Dr. Xin Tong, Marshall School of Business, University of Southern California

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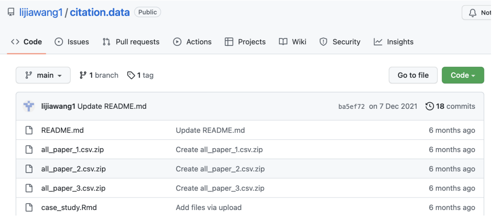
External impact of statistical theory and methodology papers

Patterns

CellPress
OPEN ACCESS

Lijia Wang, Xin Tong, and Y. X. Rachel Wang. "Statistics in everyone's backyard: an impact study via citation network analysis."



Happy Birthday Peter! Thank you for your unwavering support and endless inspiration!