Generalized Energy-Based Models

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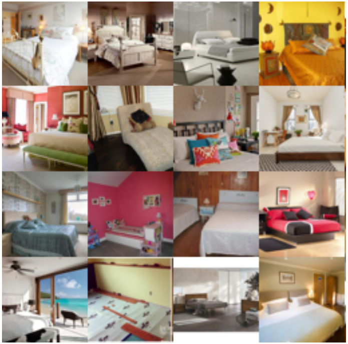
Simons Institute, Berkeley, 2022

Training generative models

- Have: One collection of samples X from unknown distribution P.
- Goal: generate samples Q that look like P

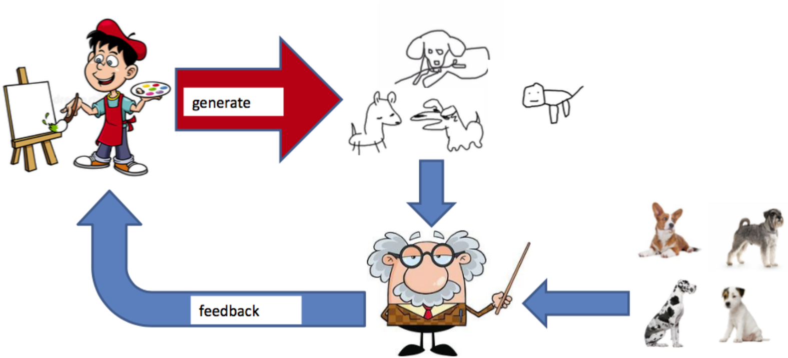


LSUN bedroom samples P

Generated Q, MMD GAN

Role of divergence $D(P, Q)$?

Visual notation: GAN setting



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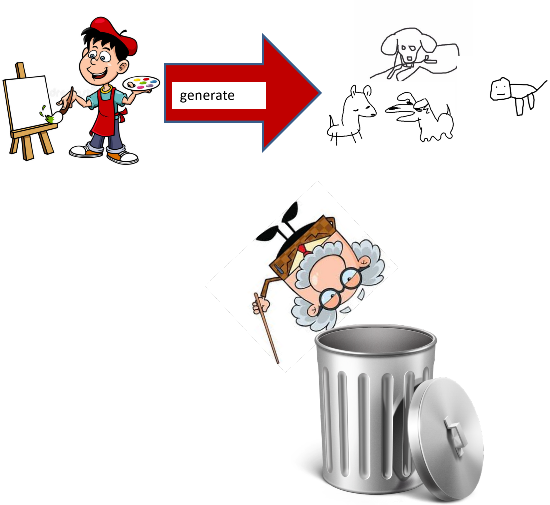

Outline

Divergences $D(P, Q)$

- Integral probability metrics (MMD, Wasserstein)
- ϕ-divergences (f-divergences) and a variational lower bound (KL)

Generalized energy-based models:

- Energy-based model supported on a low-dimenional subspace
- Robust to mismatch in model/data support

Arbel, Zhou, G., Generalized Energy Based Models (ICLR 2021)

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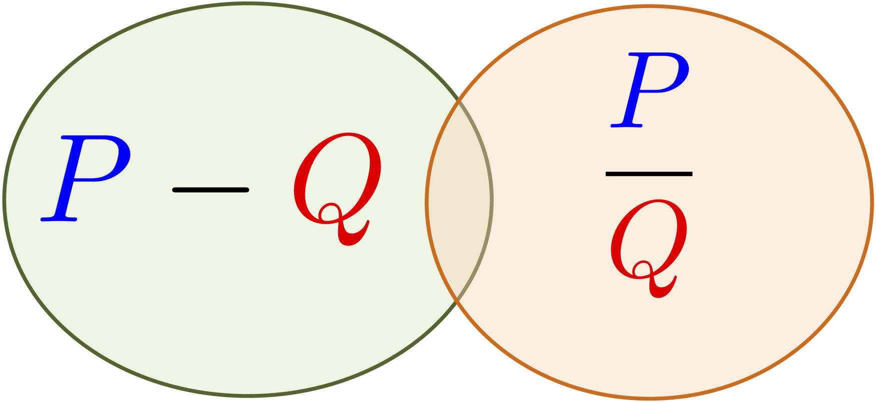
GEBM vs GAN:

- "Like a GAN" but incorporate critic into sample generation
- Perform better than using generator alone

Arbel, Zhou, G., Generalized Energy Based Models (ICLR 2021)

Divergence measures (critics)

Divergences



Divergences

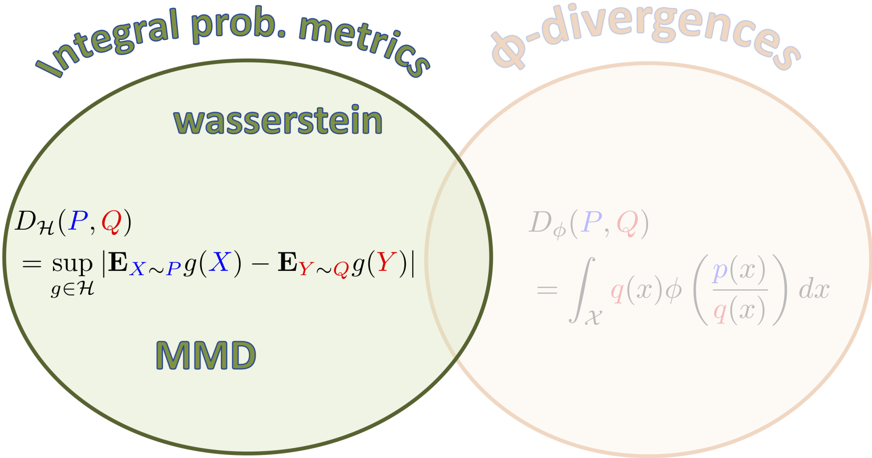
Integral prob. metrics

$$D_{\mathcal{H}}(P, Q) = \sup_{g \in \mathcal{H}} \left| \mathbf{E}_{X \sim P} g(X) - \mathbf{E}_{Y \sim Q} g(Y) \right|$$

Φ-divergences

$$D_{\phi}(P, Q) = \int_{\mathcal{X}} q(x) \phi\left(\frac{p(x)}{q(x)}\right) dx$$

The Integral Probability Metrics



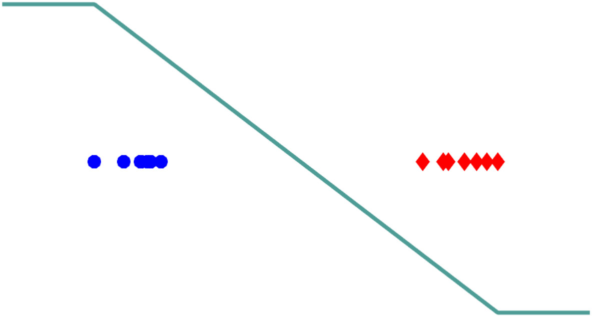
Wasserstein distance

A helpful critic witness:

$$W_1(P, Q) = \sup_{\|f\|_L \leq 1} E_P f(X) - E_Q f(Y).$$

$$\|f\|_L := \sup_{x \neq y} |f(x) - f(y)| \,/ \|x - y\|$$

W_1=0.88



Santambrogio, Optimal Transport for Applied Mathematicians (2015, Section 5.4)

G Peyré, M Cuturi, Computational Optimal Transport (2019)

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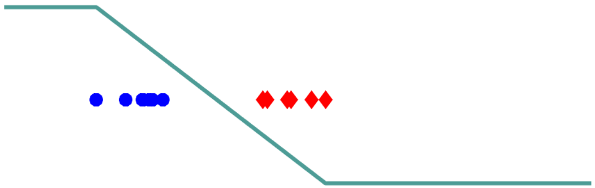

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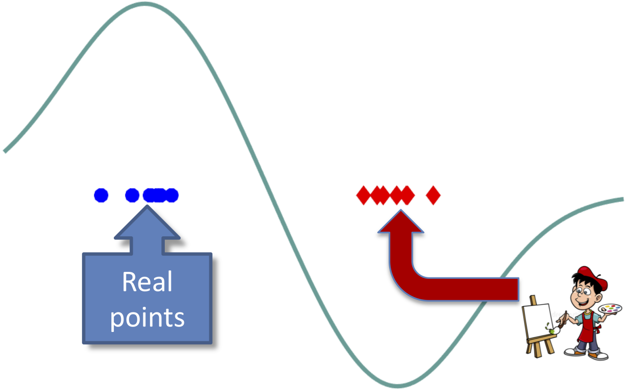
Maximum mean discrepancy

A helpful critic witness:

$$MMD(P, Q) = \sup_{\|f\|_{\mathcal{F}} \leq 1} E_P f(X) - E_Q f(Y).$$

MMD=1.8



Maximum mean discrepancy

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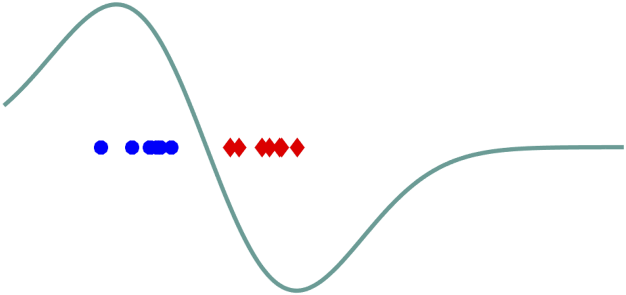
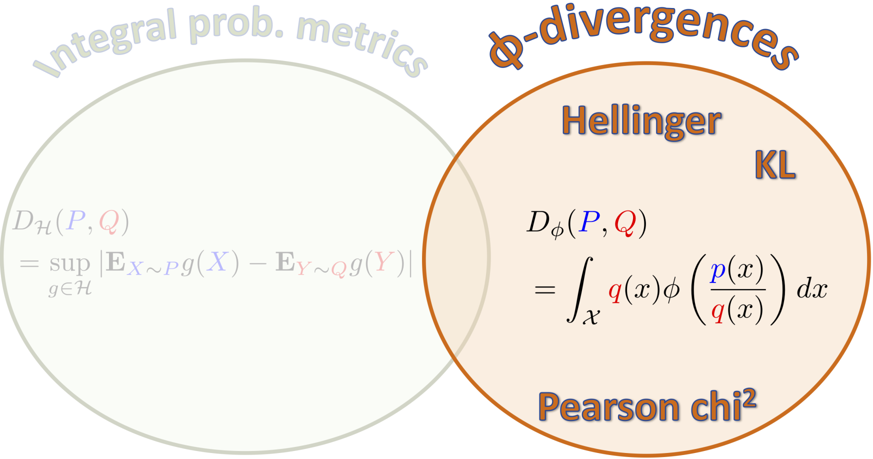
MMD=1.1



The ϕ-divergences



The ϕ-divergences

Define the ϕ-divergence(f-divergence):

$$D_\phi(P, Q) = \int \phi\left(\frac{p(z)}{q(z)}\right) q(z)dz$$

where ϕ is convex, lower-semicontinuous, $\phi(1) = 0$.

- Example: $\phi(u) = u\log(u)$ gives KL divergence.

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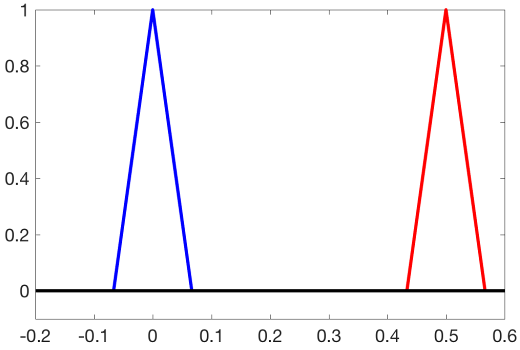
Are ϕ-divergences good critics?

Simple example: disjoint support.

Goodfellow et al. (NeurIPS 2014), Arjovsky and Bottou [ICLR 2017]

$$D_{KL}(P, Q) = \infty \qquad D_{JS}(P, Q) = \log 2$$



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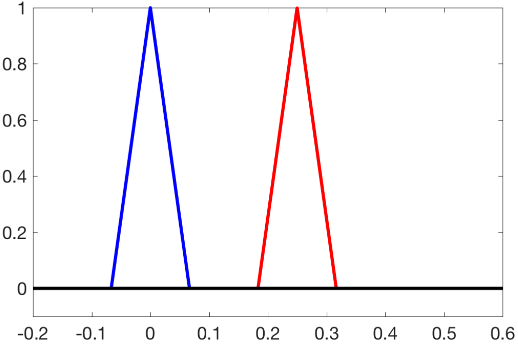

A variational lower bound

A lower-bound ϕ-divergence approximation:

$$D_\phi(P, Q) = \int q(z)\phi\left(\frac{p(z)}{q(z)}\right) dz$$

Nguyen, Wainwright, Jordan, IEEE Transactions on Information Theory (2010);
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$$D_\phi(P, Q) = \int q(z)\phi\left(\frac{p(z)}{q(z)}\right) dz$$
$$= \int q(z)\underbrace{\sup_{f_z}\left(\frac{p(z)}{q(z)} f_z - \phi^*(f_z)\right)}_{\phi\left(\frac{p(z)}{q(z)}\right)}$$

$\phi^*(v)$ is dual of $\phi(x)$.

A variational lower bound

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D_\phi(P, Q) &= \int q(z)\phi\left(\frac{p(z)}{q(z)}\right) dz \\
&= \int q(z) \sup_{f_z} \left(\frac{p(z)}{q(z)} f_z - \phi^*(f_z)\right) \\
&\geq \sup_{f \in \mathcal{H}} \mathrm{E}_P f(X) - \mathrm{E}_Q \phi^* \left(f(Y)\right)
\end{aligned}
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(restrict the function class)

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(restrict the function class)

Bound tight when:

$$
f^\diamond(z) = \partial\phi\left(\frac{p(z)}{q(z)}\right)
$$

if ratio defined.

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Case of the KL

$$D_{KL}(P, Q) = \int \log \left(\frac{p(z)}{q(z)} \right) p(z) dz$$

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Case of the KL

$$D_{KL}(P, Q) = \int \log\left(\frac{p(z)}{q(z)}\right) p(z)\, dz$$

$$\geq \sup_{f \in \mathcal{H}} -\mathrm{E}_P f(X) + 1 - \mathrm{E}_Q \underbrace{\exp\left(-f(Y)\right)}_{\phi^*(-f(Y)+1)}$$

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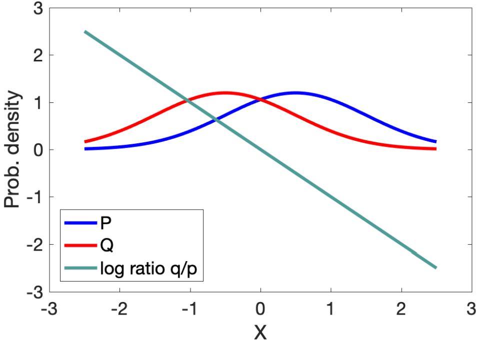

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&\approx \sup_{f \in \mathcal{H}} \left[-\frac{1}{n} \sum_{j=1}^{n} f(x_i) - \frac{1}{n} \sum_{i=1}^{n} \exp(-f(y_i)) \right] + 1
\end{aligned}
$$

$x_i \overset{\text{i.i.d.}}{\sim} P$

$y_i \overset{\text{i.i.d.}}{\sim} Q$

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This is a

KL

Approximate

Lower-bound

Estimator.

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The KALE divergence

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Empirical properties of KALE

$$KALE(P, Q; \mathcal{H}) = \sup_{f \in \mathcal{H}} -E_P f(X) - E_Q \exp(-f(Y)) + 1$$

$$f = \langle w, \phi(x) \rangle_{\mathcal{H}} \qquad \mathcal{H} \text{ an RKHS}$$

$$\|w\|^2_{\mathcal{H}} \quad \text{penalized :}$$

Glaser, Arbel, G. "KALE Flow: A Relaxed KL Gradient Flow for Probabilities with Disjoint Support," (NeurIPS 2021, Section 2)

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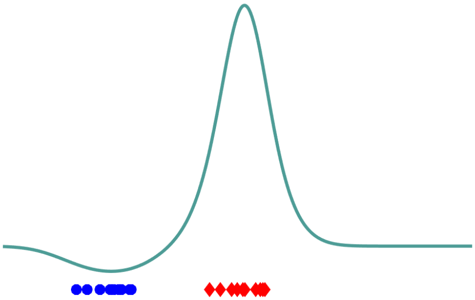
$$KALE(Q, P; \mathcal{H}) = 0.18$$



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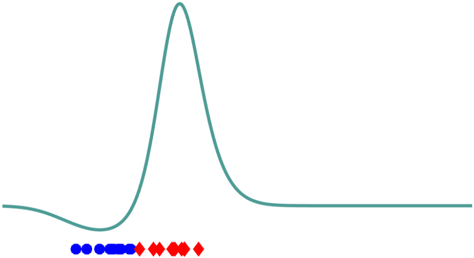
$\|w\|^2_{\mathcal{H}}$ penalized : KALE smoothie

$$KALE(Q, P; \mathcal{H}) = 0.12$$



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Topological properties of KALE (1)

Key requirements on $\mathcal{H}$ and $\mathcal{X}$:

- Compact domain $\mathcal{X}$,
- $\mathcal{H}$ dense in the space $C(\mathcal{X})$ of continuous functions on $\mathcal{X}$ wrt $\|\cdot\|_\infty$.
- If $f \in \mathcal{H}$ then $-f \in \mathcal{H}$ and $cf \in \mathcal{H}$ for $0 \leq c \leq C_{\max}$.

Theorem: $KALE(P, Q; \mathcal{H}) \geq 0$ and $KALE(P, Q; \mathcal{H}) = 0$ iff $P = Q$.

Zhang, Liu, Zhou, Xu, and He. "On the Discrimination-Generalization Tradeoff in GANs"

(ICLR 2018, Corollary 2.4; Theorem B.1)

Arbel, Liang, G. (ICLR 2021, Proposition 1)

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$\mathcal{H}$ dense in $C(\mathcal{X})$ for $\mathcal{X} \subset \mathbb{R}^d$ when:

$$\mathcal{H} = \operatorname{span}\{\sigma(w \top x + b) : [w, b] \in \Theta\}$$

$\sigma(u) = \max\{u, 0\}^\alpha$, $\alpha \in \mathbb{N}$, and $\{\lambda\theta : \lambda \geq 0, \theta \in \Theta\} = \mathbb{R}^{d+1}$.

Zhang, Liu, Zhou, Xu, and He. "On the Discrimination-Generalization Tradeoff in GANs" (ICLR 2018, Corollary 2.4; Theorem B.1)
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Topological properties of KALE (2)

Additional requirement: all functions in $\mathcal{H}$ Lipschitz in their inputs with constant L

Theorem: $KALE(P, Q^n; \mathcal{H}) \to 0$ iff $Q^n \to P$ under the weak topology.

Liu, Bousquet, Chaudhuri. "Approximation and Convergence Properties of Generative Adversarial Learning" (NeurIPS 2017); Arbel, Liang, G. (ICLR 2021, Proposition 1)

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Additional requirement: all functions in $\mathcal{H}$ Lipschitz in their inputs with constant L

Theorem: $KALE(P, Q^n; \mathcal{H}) \to 0$ iff $Q^n \to P$ under the weak topology.

Partial proof idea:

$$\begin{aligned} KALE(P, Q; \mathcal{H}) &= -\int f \, dP - \int \exp(-f) \, dQ + 1 \\ &= \int f(x) \, dQ(x) - f(x') \, dP(x') \\ &\quad - \int \underbrace{(\exp(-f) + f - 1)}_{\geq 0} \, dQ \\ &\leq \int f(x) \, dQ(x) - f(x') \, dP(x') \leq L W_1(P, Q) \end{aligned}$$

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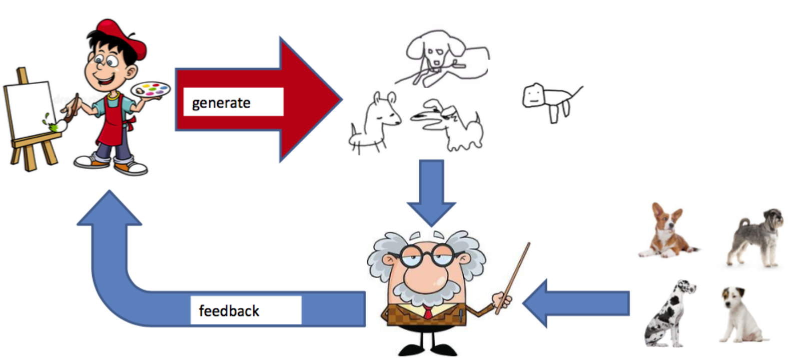
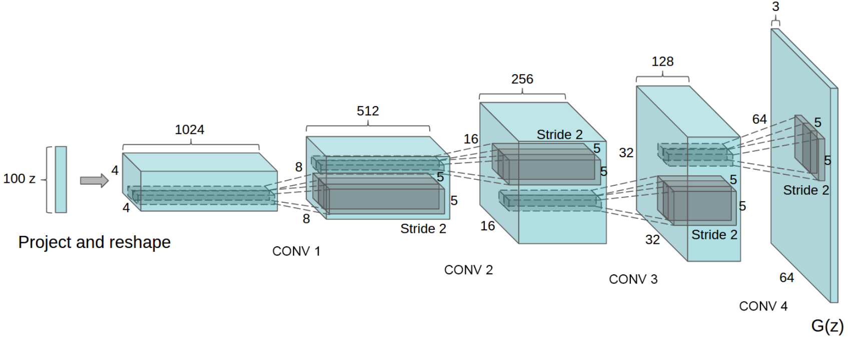


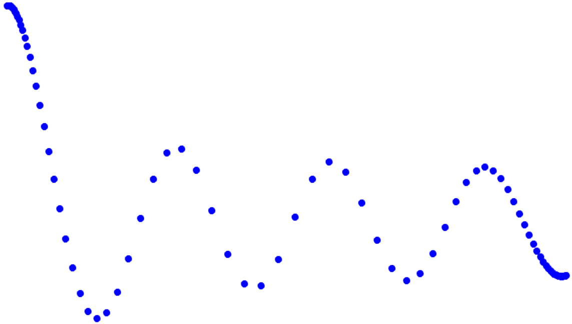
Reminder: the generator



Radford, Metz, Chintala, ICLR 2016

Generalized Energy-Based Models - the idea

Target distribution P



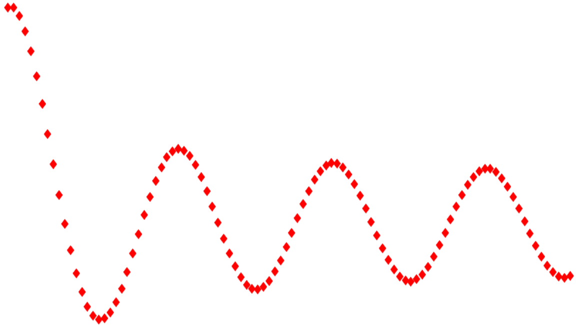
Arbel, Zhou, G. (ICLR 2021)

Generalized Energy-Based Models - the idea

GAN (generator)

$$X \sim Q_\theta \quad \Longleftrightarrow \quad X = B_\theta(Z), \quad Z \sim \eta,$$

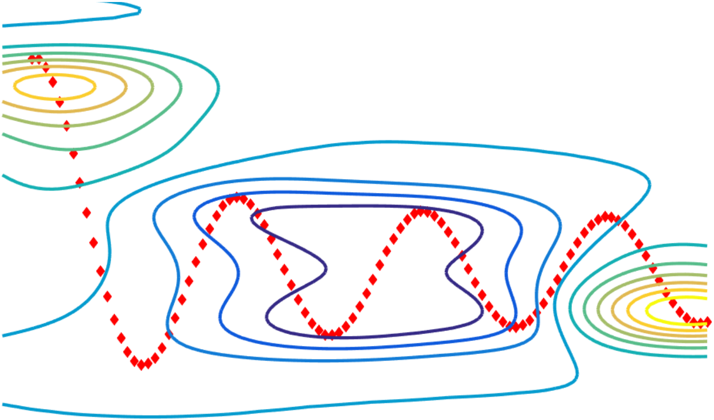
correct support but wrong mass



Arbel, Zhou, G. (ICLR 2021)

Generalized Energy-Based Models - the idea

Log energy function and Q_θ



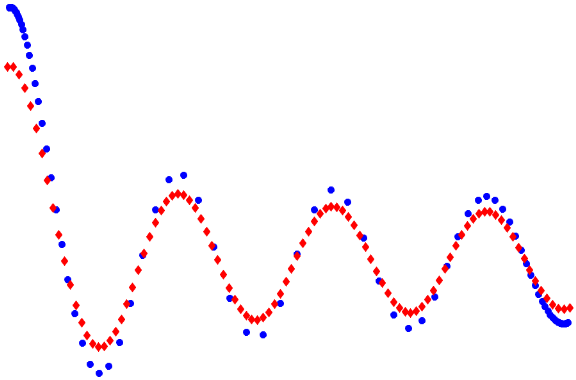
Key:

- Orange: increase mass
- Blue: reduce mass

Arbel, Zhou, G. (ICLR 2021)

Generalized Energy-Based Models - the idea

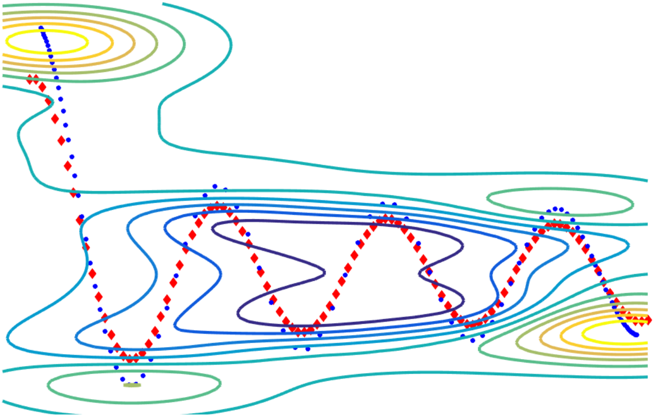
Target distribution P and GAN (generator) Q_θ, wrong support and wrong mass



Arbel, Zhou, G. (ICLR 2021)

Generalized Energy-Based Models - the idea

Log energy function, P, and Q_θ



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Generalized energy-based models

Define a model $Q_{B_\theta,E}$ as follows:

- Sample from generator with parameters θ

$$X \sim Q_\theta \quad \Longleftrightarrow \quad X = B_\theta(Z), \quad Z \sim \eta$$

- Reweight the samples according to importance weights:

$$f_{Q,E}(x) = \frac{\exp(-E(x))}{Z_{Q_\theta,E}}, \qquad Z_{Q,E} = \int \exp(-E(x))\, dQ_\theta(x),$$

where $E \in \mathcal{E}$, the energy function class.

$f_{Q,E}(x)$ is Radon-Nikodym derivative of $Q_{B_\theta,E}$ wrt Q_θ.

- When Q_θ has density wrt Lebesgue on $\mathcal{X}$, standard energy-based model (special case)
- Sample from model via HMC on posterior of Z.

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- When Q_θ has density wrt Lebesgue on $\mathcal{X}$, standard energy-based model (special case)
- Sample from model via HMC on posterior of Z.

Arbel, Zhou, G. (ICLR 2021)

Generalized energy-based models

Define a model $Q_{B_\theta,E}$ as follows:

- Sample from generator with parameters θ

$$X \sim Q_\theta \quad \Longleftrightarrow \quad X = B_\theta(Z), \quad Z \sim \eta$$

- Reweight the samples according to importance weights:

$$f_{Q,E}(x) = \frac{\exp(-E(x))}{Z_{Q_\theta,E}}, \qquad Z_{Q,E} = \int \exp(-E(x))\, dQ_\theta(x),$$

where $E \in \mathcal{E}$, the energy function class.

$f_{Q,E}(x)$ is Radon-Nikodym derivative of $Q_{B_\theta,E}$ wrt Q_θ.

- When Q_θ has density wrt Lebesgue on $\mathcal{X}$, standard energy-based model (special case)
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Arbel, Zhou, G. (ICLR 2021)

How do we learn the energy E?

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Fit the model using Generalized Log-Likelihood:

$$\mathcal{L}_{P,Q}(E) := \int \log(f_{Q,E})\,dP = -\int E\,dP - \log Z_{Q,E}$$

- When $KL(P, Q_\theta)$ well defined, above is Donsker-Varadhan lower bound on KL
  - tight when $E(z) = -\log(p(z)/q(z))$.
- However, Generalized Log-Likelihood still defined when P and Q_θ mutually singular (as long as E smooth)!

KALE and the energy function

Fit the model using Generalized Log-Likelihood:

$$\mathcal{L}_{P,Q}(E) := \int \log(f_{Q,E})\, dP = -\int E\, dP - \log \int \exp(-E)\, dQ_\theta$$

KALE and the energy function

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One last trick...(convexity of exponential)

$$-\log \int \exp(-E)\, dQ_\theta \geq -c - e^{-c} \int \exp(-E)\, dQ_\theta + 1$$

tight whenever $c = \log \int \exp(-E)\, dQ_\theta$.

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Generalized Log-Likelihood has the lower bound:

$$\mathcal{L}_{P,Q}(E) \geq -\int (E + c)\, dP - \int \exp(-E - c)\, dQ_\theta + 1$$

$$:= \mathcal{F}(P, Q_\theta; \mathcal{E} + \mathbb{R})$$

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This is the KALE! with function class $\mathcal{E} + \mathbb{R}$.

KALE and the energy function

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Jointly maximizing yields the maximum likelihood energy E^* and corresponding $c^* = \log \int \exp(-E)\, dQ_\theta$.

Training the base measure (generator)

Recall the generator:

$$X = B_\theta(Z), \quad Z \sim \eta$$

Define: $\mathcal{K}(\theta) := \mathcal{F}(P, Q_\theta; \mathcal{E} + \mathbb{R})$

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$$X = B_\theta(Z), \quad Z \sim \eta$$

Define: $\mathcal{K}(\theta) := \mathcal{F}(P, Q_\theta; \mathcal{E} + \mathbb{R})$

Theorem: $\mathcal{K}$ is lipschitz and differentiable for almost all $\theta \in \Theta$ with:

$$\nabla \mathcal{K}(\theta) = Z_{Q,E^*}^{-1} \int \nabla_x E^*(B_\theta(z)) \nabla_\theta B_\theta(z) \exp(-E^*(B_\theta(z)))\eta(z)dz.$$

where E^* achieves supremum in $\mathcal{F}(P, Q; \mathcal{E} + \mathbb{R})$.

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Assumptions:

- Functions in $\mathcal{E}$ parametrized by $\psi \in \Psi$, where Ψ compact,
  - jointly continous w.r.t. (ψ, x), L-lipschitz and L-smooth w.r.t. x.
- $(\theta, z) \mapsto B_\theta(z)$ jointly continuous wrt (θ, z), $z \mapsto B_\theta(z)$ uniformly Lipschitz w.r.t. z, lipschitz and smooth wrt θ (see paper: constants depend on z)

Sampling from the model

Consider end-to-end model $Q_{B_\theta, E}$, where recall that
$X = B_\theta(Z), \quad Z \sim \eta,$

$$f_{B,E}(x) := \frac{\exp(-E(x))}{Z_{Q,E}}$$

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$X = B_\theta(Z), \quad Z \sim \eta,$

$$f_{B,E}(x) := \frac{\exp(-E(x))}{Z_{Q,E}}$$

For a test function g,

$$\int g(x)dQ_{B,E}(x) = \int g(B(z))f_{B,E}(B(z))\eta(z)dz$$

Posterior latent distribution therefore

$$\nu_{B,E}(z) = \eta(z)f_{B,E}(B(z))$$

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Sample $z \sim \nu_{B,E}$ via Langevin diffusion-derived algorithms (MALA, ULA, HMC,...) to exploit gradient information.

Generate new samples in $\mathcal{X}$ via

$$X \sim Q_{B,E} \quad \iff \quad Z \sim \nu_{B,E}, \quad X = B_\theta(Z).$$

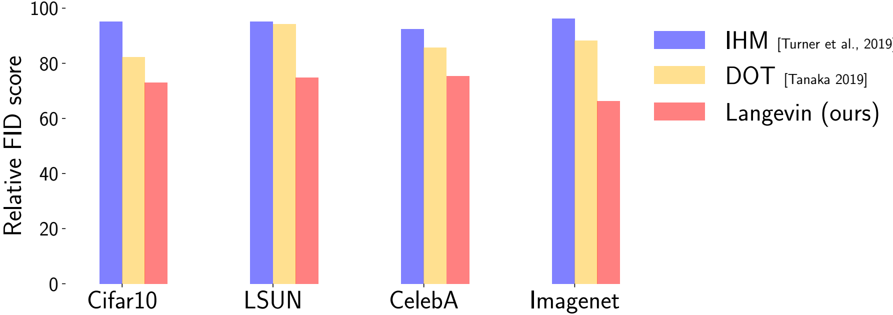
Experiments

Examples: sampling at modes

Tempered GEBM Cifar10 samples at different stages of sampling using a Kinetic Langevin Algorithm (KLA). Early samples $\rightarrow$ late samples. Model run at *low temperature* ($\beta = 100$) for better quality samples.

Sampling at modes: results

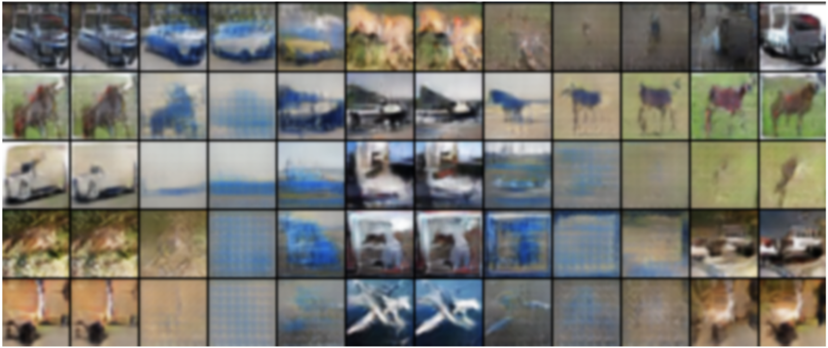
The relative FID score: $\frac{\mathrm{FID}(Q_{B_{\theta},E})}{\mathrm{FID}(B_{\theta})}$



For a given generator B_{θ} and energy E, samples always better (FID score) than generator alone.

Examples: moving between modes

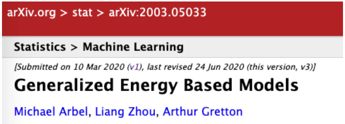
Tempered GEBM Cifar10 samples at different stages of sampling using KLA. Early samples → late samples.
Model run at *lower friction* (but still low temperature, $\beta = 100$) for mode exploration.



Summary

- Generalized energy based model:
  - End-to-end model incorporating generator and critic
  - Always better samples than generator alone.
- ICLR 2021

https://github.com/MichaelArbel/GeneralizedEBM



Summary

- **Generalized energy based model:**
  - End-to-end model incorporating generator and critic
  - Always better samples than generator alone.
- ICLR 2021

https://github.com/MichaelArbel/GeneralizedEBM

arXiv.org > stat > arXiv:2003.05033

Statistics > Machine Learning

[Submitted on 10 Mar 2020 (v1), last revised 24 Jun 2020 (this version, v3)]

Generalized Energy Based Models

Michael Arbel, Liang Zhou, Arthur Gretton

NeurIPS 2020:

arXiv.org > cs > arXiv:2003.06060

Computer Science > Machine Learning

[Submitted on 12 Mar 2020 (v1), last revised 24 Mar 2020 (this version, v2)]

Your GAN is Secretly an Energy-based Model and You Should use Discriminator Driven Latent Sampling

Tong Che, Ruixiang Zhang, Jascha Sohl-Dickstein, Hugo Larochelle, Liam Paull, Yuan Cao, Yoshua Bengio

ICLR 2021:

arXiv.org > cs > arXiv:2012.00780

Computer Science > Machine Learning

[Submitted on 1 Dec 2020 (v1), last revised 5 Jun 2021 (this version, v4)]

Refining Deep Generative Models via Discriminator Gradient Flow

Abdul Fatir Ansari, Ming Liang Ang, Harold Soh

ICLR 2021:

arXiv.org > cs > arXiv:2010.00654

Computer Science > Machine Learning

[Submitted on 1 Oct 2020 (v1), last revised 9 Feb 2021 (this version, v2)]

VAEBM: A Symbiosis between Variational Autoencoders and Energy-based Models

Zhisheng Xiao, Karsten Kreis, Jan Kautz, Arash Vahdat

Questions?


THE KERNEL
BREWERY LONDON
BALTIC PORTER
7.3% ABV
THE KERNEL
EXPORT STOUT

Post-credit scene: MMD flow

From NeurIPS 2019:

Maximum Mean Discrepancy Gradient Flow

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Sanity check: reduction to EBM case

Base measure B_θ is real NVP with closed-form density.