Generalized Knuth equivalence for Schur positivity

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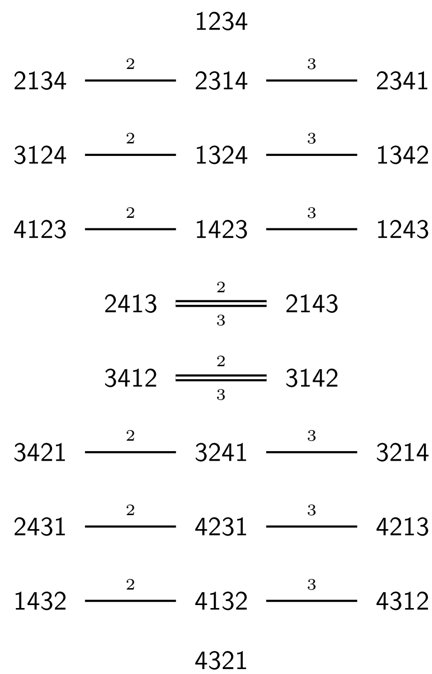

Figure 1: The Knuth equivalence graph on $\mathcal{S}_4$.

plactic	nilCoxeter
$\mathsf{acb} \equiv \mathsf{cab}$	$\mathsf{aa} \equiv 0$
$\mathsf{bac} \equiv \mathsf{bca}$	$\mathsf{ac} \equiv \mathsf{ca}$ for $c - a > 1$
$\mathsf{aba} \equiv \mathsf{baa}$	$\mathsf{a\acute{a}a} \equiv \mathsf{\acute{a}a\acute{a}}$
$\mathsf{bab} \equiv \mathsf{bba}$	

Fomin-Greene	I_k^{Lam}
$\mathsf{acb} \equiv \mathsf{cab}$	$\mathsf{ac} \equiv \mathsf{ca}$ for $c - a > k$
$\mathsf{bac} \equiv \mathsf{bca}$	$\mathsf{acb} \equiv \mathsf{bac}$ for $c - a \leq k$
$\mathsf{aca} \equiv \mathsf{caa}$ for $c - a > 1$	$\mathsf{cab} \equiv \mathsf{bca}$ for $c - a \leq k$
$\mathsf{cac} \equiv \mathsf{cca}$ for $c - a > 1$	$\mathsf{v} \equiv 0$ if v has a repeated letter
$\mathsf{a\acute{a}a} + \mathsf{\acute{a}\acute{a}a} \equiv \mathsf{\acute{a}aa} + \mathsf{\acute{a}a\acute{a}}$	

Figure 2: Relations for some quotient algebras of the free associative algebra $\mathcal{U}$, where $a < b < c$, $\acute{a} = a + 1$, and v is a word. The elementary symmetric functions commute in all these algebras.

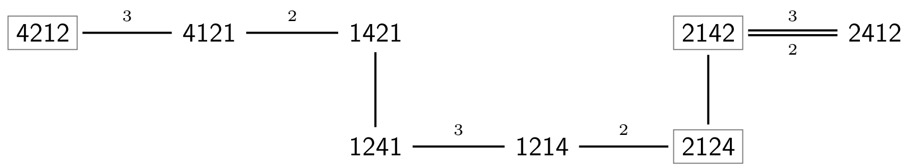

Figure 3: A component of the equivalence graph of the nilCoxeter algebra. Its generating function is the Stanley symmetric function $F_{32154} = s_{(3,1)}(\mathbf{x}) + s_{(2,2)}(\mathbf{x}) + s_{(2,1,1)}(\mathbf{x})$, which can be read off from the outlined words.

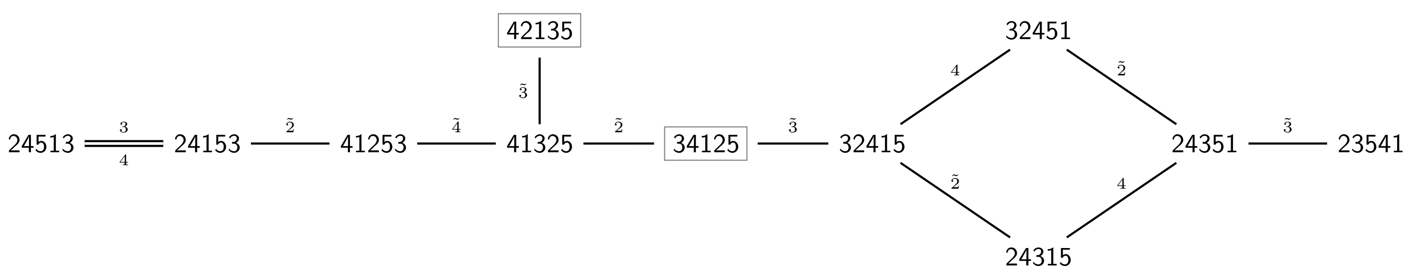

Figure 4: A component of Assaf's D graph for 3-column Macdonald polynomials. It is a component of the equivalence graph of $\mathcal{U}/I_3^{\mathrm{Lam}}$. Its generating function is $s_{(3,2)}(\mathbf{x})+s_{(3,1,1)}(\mathbf{x})$.

Example 1. The noncommutative Schur function $\mathfrak{J}_{(3,2)}$ in $\mathcal{U}/I_3^{\mathrm{Lam}}$ is given by

$$\mathfrak{J}_{(3,2)}(u_1, \ldots, u_5) \;= 21435 + 34125 + 21534 + 31524 + 41523.$$

T	$\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}$	$\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}$	$\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}$	$\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}$	$\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}$
sqread(T)	21435	34125	21534	31524	41523

Letting f be the sum of the words in Figure 4, the generating function of f is given by

$$D(f) = \sum_{\lambda} s_\lambda(\mathbf{x})\langle \mathfrak{J}_\lambda(\mathbf{u}), f\rangle = s_{(3,2)}(\mathbf{x}) + s_{(3,1,1)}(\mathbf{x}).$$