Geometric Complexity Theory and Matrix Multiplication (Tutorial)

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Workshop on Geometric Complexity Theory
Simons Institute for the Theory of Computing
Berkeley, September 15, 2014
Background and motivation
Goals

- Tensor rank is a natural math. concept arising in various places.
- It is intimately related to the computational complexity of evaluating bilinear maps, in particular to the multiplication of matrices.
- To determine the (asymptotic) complexity for multiplying matrices is a major open question in algebraic complexity theory.
- GCT was proposed for the permanent vs determinant problem by Mulmuley and Sohoni in 2001.
- In joint work with Christian Ikenmeyer, we further developed the ideas of GCT in the setting of tensors (STOC 11, STOC 13).
- We managed to prove lower bounds on the border rank of matrix multiplication by exhibiting representation theoretic “occurrence obstructions”.
- Our bounds are not as good as Landsberg and Ottaviani’s recent bounds (’11), but they have the same order of magnitude as Strassen and Lickteig’s bounds (’83).
- This talk: set the ground. More details on Wednesday (Christian).
Tensor rank

- Consider finite dimensional complex vector spaces $W_i$ for $i = 1, 2, 3$ and put $W := W_1 \otimes W_2 \otimes W_3$. Elements $w \in W$ are called tensors.

- The rank $R(w)$ of $w \in W$ is defined as the minimum $r \in \mathbb{N}$ s.t. there are $w_{1i}, \ldots, w_{ri} \in W_i$, $i = 1, 2, 3$, with

  $$w = \sum_{\rho=1}^r w_{\rho 1} \otimes w_{\rho 2} \otimes w_{\rho 3}.$$ 

- Special case $W_3 = \mathbb{C}$: $R(w)$ equals the rank of the corr. linear map $W_1^* \to W_2$. In this case we know everything about $R(w)$.

- General case much harder: comp. of $R(w)$ is NP-hard (Hastad).

- To $w \in W$ there corresponds a bilinear map $\varphi : W_1^* \times W_2^* \to W_3$. The nonscalar complexity $L(\varphi)$ is defined as the minimum number of nonscalar multiplications sufficient to evaluate the map $\varphi$ by an arithmetic circuit.

- Strassen: $L(\varphi) \leq R(w) \leq 2L(\varphi)$. 
Complexity of matrix multiplication: the records

- Consider the tensor $M(n) \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{n \times n} \otimes \mathbb{C}^{n \times n}$ of the matrix multiplication map

$$\mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}, (A, B) \mapsto AB.$$ 

- Best known lower bound (Landsberg '12):

$$R(M(n)) \geq 3 n^2 + o(n^2).$$

(Before, $R(M(n)) \geq 2.5 n^2 + o(n^2)$ due to Bläser’s ('99).)

- Asymptotic upper bounds: the exponent $\omega$ of matrix multiplication is defined as

$$\omega := \lim_{n \rightarrow \infty} \log_n R(M(n)),$$

- Coppersmith & Winograd 1990: $\omega \leq 2.376$. Recent improvements by Davie & Stothers, Williams, Le Gall ('14):

$$\omega \leq 2.3728639.$$
Border rank ...

- The border rank $R(w)$ of a tensor $w \in W$ is defined as the minimum $r \in \mathbb{N}$ such that there exists a sequence $w_k \in W$ with $\lim_{k \to \infty} w_k = w$ and $R(w_k) \leq r$ for all $k$.
- $R(w) \leq R(w')$
- Fact: $\omega = \lim_{n \to \infty} \log_n R(M(n))$.
- Best known lower bound (Landsberg and Ottaviani '11)
  \[ R(M(n)) \geq 2n^2 - n. \]
  (Before, Lickteig '84: $R(M(n)) \geq 1.5n^2 + 0.5n - 1.$)
as orbit closure problem

- The group

\[ G := \text{GL}(W_1) \times \text{GL}(W_2) \times \text{GL}(W_3) \]  

acts on \( W = W_1 \otimes W_2 \otimes W_3 \) via

\[ (g_1, g_2, g_3)(w_1 \otimes w_2 \otimes w_3) := g_1(w_1) \otimes g_2(w_2) \otimes g_3(w_3). \]

- Tensor \( w \in W \) defines orbit \( Gw \) and orbit closure \( \overline{Gw} \). The same for euclidean topology and Zariski topology!

- Could interpret \( Gw, \overline{Gw} \) as subsets of \( \mathbb{P}(W) \) as both are cones.

- Let \( r \in \mathbb{N}, \ r \leq \min_i \dim W_i \). Define \( r \)-th unit tensor in \( W \):

\[ \langle r \rangle := \sum_{\rho=1}^{r} e_{\rho 1} \otimes e_{\rho 2} \otimes e_{\rho 3}, \]

where \( e_{1i}, \ldots, e_{ri} \) are part of a basis of \( W_i \).

- The \( G \)-orbit of \( \langle r \rangle \) is a basis independent notion.

- Strassen (1987):

\[ R(w) \leq r \iff w \in \overline{G\langle r \rangle} \iff \overline{Gw} \subseteq \overline{G\langle r \rangle}. \]
Basic ideas for lower bounds
Orbit closure problem

- Reductive algebraic group $G$ acts linearly on vector space $W$ (eg. $G = \text{GL}_m(\mathbb{C})$ or products thereof).
- $\mathcal{O}(W)$ ring of polynomial functions $W \to \mathbb{C}$.
- degree grading: $\mathcal{O}(W) = \bigoplus_{d \in \mathbb{N}} \mathcal{O}(W)_d$
- Vanishing ideal of $Gw$ for $w \in W$

$$ I(Gw) := \{ f \in \mathcal{O}(W) \mid \forall v \in Gw \quad f(v) = 0 \}. $$

- Elementary fact:

$$ v \notin Gw \iff \exists f \in I(Gw) \quad f(v) \neq 0. $$

- Such $f$ may serve as a witness for $v \notin Gw$.
- In which degree $d$ to search for such $f$? $\mathcal{O}(W)_d$ has huge dimension even for small $d$!
- Representation theory allows for guided search for $f$. 
Representations in rings of regular functions

- The group $G$ acts on the ring $\mathcal{O}(W)$ of polynomial functions on $W$:
  \[(gf)(w) := f(g^{-1}w), \quad f \in \mathcal{O}(W), \ w \in W.\]

- The vanishing ideal $I(\text{Gw})$ is $G$-invariant.
- Representation theory: $I(\text{Gw})$ splits into a direct sum of irreducible modules (as $G$ is reductive).
- The isomorphy types of irreducible $G$-modules in $\mathcal{O}(W)_d$ are determined by discrete data called highest weights $\lambda$. Those are triples $\lambda$ of partitions of $d$ (Schur, Young, Weyl).
- Irreducible $G$-modules in $\mathcal{O}(W)_d$ are generated their highest weight functions (unique up to scaling). They have a “weight” $\lambda$.
- Recall:
  \[v \not\in \text{Gw} \iff \exists f \in I(\text{Gw}) \ f(Gv) \neq 0.\]
- One may take for $f$ a highest weight function! See Christian’s talk.
Strassen’s resultant for 3-slice tensors

- \( W = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^3 \simeq \bigoplus_3 \mathbb{C}^{m \times m}, \ m \geq 3. \)
- Interpret \( w \in W \) as a triple \((A, B, C)\) of \( m \times m \) matrices.
- Strassen (1983):

\[
f_m(A, B, C) := (\det A)^2 \det(BA^{-1}C - CA^{-1}B)
\]

is a semi-invariant: for \((g_1, g_2, g_3) \in \text{GL}_m \times \text{GL}_m \times \text{GL}_3, \ w \in W, \)

\[
f_m((g_1 \otimes g_2 \otimes g_3)w) = (\det g_1 \det g_2)^3 (\det g_3)^m f(w).
\]

- \( f_m \) vanishes on the tensors of border rank < \( 3m/2 \).
- Semi-invariants are highest weight functions of rectangular weights \( \lambda \).
- Bläser’s bound relied on Strassen’s resultant.
Splitting into irreducible representations

- The ring $\mathcal{O}(\overline{Gw})$ of regular functions on $\overline{Gw}$ consists of the restrictions of polynomial functions to $\overline{Gw}$.
- Have induced $G$-action and surjective $G$-equivariant restriction $\mathcal{O}(W) \rightarrow \mathcal{O}(\overline{Gw})$.
- $\mathcal{O}(\overline{Gw}) = \bigoplus_{d \in \mathbb{N}} \mathcal{O}(\overline{Gw})_d$ is graded, $\mathcal{O}(\overline{Gw})_d$ is a (f.d.) $G$-module.
- $G$ is reductive, so any (rational) $G$-module splits into irreducible $G$-modules.
- Let $V_{\lambda}(G)$ denote the irreducible $G$-modules of highest weight $\lambda$.
- The splitting into irreducibles can be written as
  \[ \mathcal{O}(\overline{Gw})_d = \bigoplus_{\lambda} \text{mult}_{\lambda}(w) V_{\lambda}(G)^*. \]
- We are interested in the multiplicities $\text{mult}_{\lambda}(w)$. 
The idea of comparing multiplicities

- Observation:

\[ \overline{Gv} \subseteq \overline{Gw} \implies \forall \lambda \quad \text{mult}_\lambda(v) \leq \text{mult}_\lambda(w). \]

- Proof: Restriction of regular functions yields, for all degrees \( d \), a surjective \( G \)-module morphism \( \mathcal{O}(\overline{Gw})_d \rightarrow \mathcal{O}(\overline{Gv})_d \). Use Schur’s lemma. \( \square \)

- A representation theoretic obstruction consists of \( \lambda \) violating the above inequality of multiplicities.

- Christandl et al. ’12: If \( \dim \overline{Gv} < \dim \overline{Gw} \) and \( \overline{Gv} \subseteq \overline{Gw} \), then \( k \mapsto \text{mult}_{k\lambda}(w) \) grows at a faster rate than \( k \mapsto \text{mult}_{\lambda}(kv) \).

- Therefore, asymptotic considerations of cannot help. This supports the following concept:
Occurrence obstructions

- An occurrence obstruction consists of $\lambda$ such that
  \[ \text{mult}_\lambda(w) = 0 \text{ and } \text{mult}_\lambda(v) > 0. \]

- Reformulation: $\text{mult}_\lambda(w) = 0$ means that all highest weight functions of weight $\lambda$ vanish on $G_w$. This is a very strong condition!

- Strassen’s example is not an occurrence obstruction: for $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$ there is another semi-invariant of the same weight, but which doesn’t vanish on tensors of rank $\leq 5$.

- Warning: while it is true that, in principle, orbit closure problems $G_v \not\subseteq G_w$ can always be disproved using highest weight functions, it is not clear that one can always do so with occurrence obstructions!

- But we will see at least one family of occurrence obstructions.
Towards determining $\text{mult}_\lambda(w)$
Decomposition of $\mathcal{O}(W)$ and Kronecker coefficients

- The space $W = W_1 \otimes W_1 \otimes W_3$ decomposes as

$$\mathcal{O}(W_1 \otimes W_2 \otimes W_3)_d = \bigoplus_{\lambda} k(\lambda) \ V_{\lambda}(G)^*,$$

the sum being over the triples $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ of partitions of the same size $d$.

- **Schur-Weyl duality**: the multiplicities $k(\lambda)$ are the Kronecker coefficients.

- Characterization in terms of representations of the symmetric group $S_d$:

$$k(\lambda) := \dim \left( [\lambda_1] \otimes [\lambda_2] \otimes [\lambda_3] \right)_{S_d}$$

(2)

Here $[\lambda_i]$ denotes the irreducible $S_d$-module labeled by $\lambda_i$.

- **Ex.** $W = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, $\lambda = ((2,2), (2,2), (2,2))$. Then $k(\lambda) = 1$. Hence there is semi-invariant $f$ (Cayley’s hyperdeterminant) s.t.

$$f((g_1 \otimes g_2 \otimes g_3)w) = (\det g_1 \det g_2 \det g_3)^2 \ f(w).$$
Monoids of representations

- The Kronecker monoids are defined as

\[ K(m_1, m_2, m_3) := \{ \lambda \mid \mathbb{N}, d \in \lambda, d \vdash m, k(\lambda) > 0 \}. \]

- For \( w \in \mathcal{W} \) we consider the monoid of representations of \( w \)

\[ S(w) := \{ \lambda \mid \text{mult}_\lambda(w) > 0 \}. \]

- General principles (finiteness of ring of \( U \)-invariants) imply that these monoids are finitely generated.

- The surjective morphism \( \mathcal{O}(\mathcal{W}) \to \mathcal{O}(\overline{Gw}) \) implies \( S(w) \subseteq K(m_1, m_2, m_3) \).

- \( S(w) = K(m_1, m_2, m_3) \) for almost all \( w \in \mathcal{W} \).

- The real cone generated by \( K(m_1, m_2, m_3) \) is polyhedral. It is complicated, but understood to a certain extent, see Ressayre’s talk.

- Occurrence obstructions \( \lambda \) are contained in \( K(m_1, m_2, m_3) \setminus S(w) \).
Inheritance

- What happens to $\text{mult}_\lambda(w)$ when we enlarge the ambient space $W$ and the group $G$ of symmetries?

$$W_i \subseteq W'_i, \ W' := W'_1 \otimes W'_2 \otimes W'_3, \ G' := \text{GL}(W'_1) \times \text{GL}(W'_2) \times \text{GL}(W'_3).$$

- NOTHING!

- Can interpret a highest $G$-weight $\lambda$ with nonnegative entries as a highest $G'$-weight $\lambda$ (appending zeros to partitions $\lambda_i$).

- **Inheritance Theorem (GCT2, Weyman)**
  Let $w \in W$ and $\lambda$ be a highest $G'$-weight.
  
  1. If $V_\lambda(G')^*$ occurs in $O(G'w)$, then $\lambda$ is a highest $G$-weight, i.e., $\lambda_i$ has at most $\dim W_i$ parts.
  2. If $\lambda$ is a highest $G$-weight, then

$$\text{mult}(V_\lambda(G)^*, O(Gw)) = \text{mult}(V_\lambda(G')^*, O(G'w)).$$

- Proof based on method of $U$-invariants.
Coordinate rings of orbits

- Orbits are considerably easier to understand than orbit closures.
- $\mathcal{O}(Gw)$ denotes the ring of functions that can be locally written as the quotient of two polynomial functions. (This way, $Gw$ becomes an algebraic variety.)
- $\mathcal{O}(\overline{Gw})$ is a subring of $\mathcal{O}(Gw)$.
- Since the inclusion $\mathcal{O}(\overline{Gw}) \hookrightarrow \mathcal{O}(Gw)$ is $G$-equivariant, we get
  \[
  \text{mult}_\lambda(w) := \text{mult}_\lambda(\mathcal{O}(\overline{Gw})) \leq \text{mult}_\lambda(\mathcal{O}(Gw)).
  \]
- We have (quite complicated) formulas for the multiplicities on the right hand side.
- **But currently, we have no systematic way to compute $\text{mult}_\lambda(w)$**.
- In our example of an occurrence obstruction we even have $\text{mult}_\lambda(\mathcal{O}(Gw)) = 0$, hence $\text{mult}_\lambda(w) = 0$. 
Peter-Weyl Theorem

- The stabilizer group $\text{stab}(w) := \{ g \in G \mid gw = w \}$ describes the symmetries of $w \in W$.
- Space of $\text{stab}(w)$-invariants in $V_\Lambda(G)$:

  $$ V_\Lambda(G)^{\text{stab}(w)} := \{ \nu \in V_\Lambda(G) \mid \forall g \in \text{stab}(w) \, gv = \nu \} $$

- Theorem. For any $\Lambda$

  $$ \text{mult}_\Lambda(\mathcal{O}(Gw)) = \dim V_\Lambda(G)^{\text{stab}(w)}. $$

  (Consequence of alg. Peter-Weyl Thm. on decomposition of $\mathcal{O}(G)$.)
Example: generic tensor

- **Thm. (?)** Let $m \geq 3$. The stabilizer of almost all $w \in (\mathbb{C}^m)^\otimes 3$ is trivial: it equals $\{(a \text{id}, b \text{id}, c \text{id}) \mid a, b, c \in \mathbb{C}^\times, abc = 1\}$.

- This implies via Peter-Weyl that
  \[ \{\Lambda \mid \text{mult}_\Lambda(\mathcal{O}(Gw)) > 0\} \]
  is very large: it consists of all triples of partitions of the same size.

- By contrast, for generic $w$,
  \[ S(w) = K(m_1, m_2, m_3) \]
  is much smaller.

- The example of generic tensors impressively shows that monoids of representations for orbit and orbit closure can differ considerably!
Example: stabilizer of unit tensor

- Group $G := \text{GL}_m \times \text{GL}_m \times \text{GL}_m$, unit tensor

$$\langle m \rangle := \sum_{\rho=1}^{m} e_\rho \otimes e_\rho \otimes e_\rho \in (\mathbb{C}^m)^{\otimes 3}$$

- Recall $G\langle m \rangle = \{ w \in (\mathbb{C}^m)^{\otimes 3} | R(w) \leq m \}$. 
- What is $H := \text{stab}(\langle m \rangle)$? 
- The torus

$$T := \{ (\text{diag}(a), \text{diag}(b), \text{diag}(c)) \in G_m \mid \forall \rho \ a_\rho b_\rho c_\rho = 1 \}$$

is contained in $H$. 
- Symmetric group $S_m$ is embedded in $G$ via $\pi \mapsto (P_\pi, P_\pi, P_\pi)$ (simultaneous permutation of standard bases). Clearly, $S_m \leq H$. 
- Proposition. $\text{stab}(\langle m \rangle)$ is the semidirect product of $T$ and $S_m$. 
- $\langle m \rangle$ is uniquely determined by its stabilizer $H$ (up to a scalar).
Orbit versus orbit closure
Stability

- Consider the subgroup \( G_s := \text{SL}(W_1) \times \text{SL}(W_2) \times \text{SL}(W_3) \).
- We call \( w \in W \) **polystable** if \( G_s w \) is closed (and \( w \neq 0 \)).
- Polystability can be shown with the Hilbert-Mumford criterion. The unit tensors \( \langle m \rangle \) are polystable.

- Essential: It turns out that if \( w \) is polystable, then there is a close connection between \( O(Gw) \) and \( O(\overline{Gw}) \).
The period of tensors

- We obtain a group homomorphism \( \det: G \to \mathbb{C}^\times \) by composing the representation \( D: G \to \text{GL}(W) \) with the determinant:
  \[
  \det(g) := \det(D(g)).
  \]

- Specifically,
  \[
  \det(g_1, g_2, g_3) = (\det g_1)^{m_2m_3} \cdot (\det g_2)^{m_1m_3} \cdot (\det g_3)^{m_1m_2}.
  \]

- Let \( w \in W \) be polystable and assume that \( \det(\text{stab}(w)) = \mu_a \) is the group of \( a \)-th roots of unity. We call \( a \) the period of \( w \).

- \( \langle m \rangle \) has period 1 if \( m \) is even and period 2 otherwise.

  Proof. \( \det(P_\pi, P_\pi, P_\pi) = (\text{sgn}\pi)^{3m^2} = \text{sgn}\pi. \) \(\square\)
The determinant of tensors

- If \( w \in W \) is polystable and has period \( a \), then the map

\[
\det^a_w : Gw \rightarrow \mathbb{C}^\times, \ gw \mapsto \det(g)^a
\]

is a well-defined morphism of algebraic varieties. (Recall \( \det(stab(w)) = \mu_a \).) Warning: \( \det^a_w \) is undefined if \( a > 1 \).

- **Lemma.** The extension of \( \det^a_w \) to the boundary of \( Gw \) by zero yields a function \( \overline{Gw} \rightarrow \mathbb{C} \) that is continuous in the \( \mathbb{C} \)-topology.

- However, this extension may not need to be regular. In this case, \( \overline{Gw} \) is not normal.

- **Consider the exponent monoid** \( E_w \)

\[
E(w) := \{ e \in \mathbb{N} \mid (\det^a_w)^e \mid \text{ has a regular extension to } \overline{Gw} \}.
\]

- **Thm.** The group generated by \( E(w) \) equals \( \mathbb{Z} \). Moreover, \( \exists e_0 \in \mathbb{N} \ \forall e \geq e_0 \ e \in E(w) \).
Fundamental invariant of tensors

- We call \( e(w) := \min E(w) \setminus \{0\} \) the regularity of \( w \).
- So the regularity \( e(w) \) is the smallest \( e > 0 \) such that \( (\det_w^a)^e \) is regular.
- We call

\[
\Phi_w := (\det_w^a)^{e(w)} \text{ the fundamental invariant of } w.
\]

- The zero set of \( \Phi_w \) in \( \overline{Gw} \) is the boundary of \( Gw \).
- **Theorem.** Under the above assumptions, \( \mathcal{O}(Gw) \) is the localization of \( \mathcal{O}(\overline{Gw}) \) with respect to \( \Phi_w \):

\[
\mathcal{O}(Gw) = \left\{ \frac{f}{\Phi_w^s} \mid f \in \mathcal{O}(\overline{Gw}), s \in \mathbb{N} \right\}.
\]

- Hence any \( h \in \mathcal{O}(Gw) \), when multiplied with a sufficiently high power of \( \Phi_w \), has a regular extension to \( W \).
Nonnormality of orbit closures

- **Proposition.** (compare Kumar for determinant orbit )
  If \( w \in (\mathbb{C}^m)^{\otimes 3} \) has period \( a < \sqrt{m} \), then \( e(w) > 1 \) and hence \( \overline{Gw} \) is not normal.

- **Proof.** \( \text{det}_w^a \) is a semi-invariant of weight \((m \times a, m \times a, m \times a)\).
  Rules for Kronecker coeff. yield
  \[
  k(m \times a, m \times a, m \times a) = k(m \times a, a \times m, a \times m) = 0 \quad \text{if} \quad m > a^2.
  \]

- Since the unit tensor \( \langle m \rangle \) has period \( a \leq 2 \), we obtain \( e(\langle m \rangle) > 1 \), provided \( m > 2^2 \). So the orbit closure of \( \langle m \rangle \) is not normal in this case.

- **Proposition.** \( e(\langle m \rangle) = 1 \) for \( m \leq 4 \).

- **Problem.** Determine the regularity of unit tensors!

- **Problem.** Write \( \Phi_{\langle m \rangle} \) explicitly as a quotient of two highest weight functions in \( O(W) \). (Such representations must exist.)
Representations for orbit of unit tensors
Representations for orbit of unit tensor

- Recall: \( \text{mult}_\lambda(O(Gw)) = \dim V_\lambda(G)^{\text{stab}(w)} \).
- Recall: stabilizer of unit tensor \( \langle m \rangle \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m \) consists of simultaneous permutations of standard bases and \((\text{diag}(a), \text{diag}(b), \text{diag}(c))\) such that \( a_i b_i c_i = 1 \).

- Let \( V_\lambda = \bigoplus_{\alpha \in \mathbb{Z}^n} V_\lambda^\alpha \) be the decomposition into weight spaces of the irreducible \( \text{GL}_m \)-module \( V_\lambda \) for \( \lambda \vdash_m d \).
- The group \( S_m \) operates on \( \mathbb{Z}^m \) by permutation. Let \( \text{stab}(\alpha) \subseteq S_m \) denote the stabilizer of \( \alpha \in \mathbb{Z}^m \).
- **Theorem (Branching Formula).** If \( \underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \) with partitions \( \lambda_i \) of the same size \( d \),

\[
\text{mult}_{\underline{\lambda}}(O(\text{GL}_m^3 \langle m \rangle)) = \sum_{\alpha} \dim (V_{\lambda_1}^\alpha \otimes V_{\lambda_2}^\alpha \otimes V_{\lambda_3}^\alpha)^{\text{stab}(\alpha)},
\]

where the sum is over all partitions \( \alpha \vdash_m d \) such that \( \alpha \preceq \lambda_i \) for \( i = 1, 2, 3 \) in the dominance order.
An small example

- Branching Formula:

\[
\text{mult}_\lambda(\mathcal{O}(\text{GL}_m^3\langle m \rangle)) = \sum_{\alpha \preceq \lambda} \dim (V_{\lambda_1}^{\alpha} \otimes V_{\lambda_2}^{\alpha} \otimes V_{\lambda_3}^{\alpha})^{\text{stab}(\alpha)}
\]

- We are interested in those \( \lambda \) where this zero: all the summands have to vanish, which is rarely the case.

- Regular partitions \( \alpha \) are those where \( \text{stab}(\alpha) = \{\text{id}\} \), i.e., its components are pairwise distinct. Those \( \alpha \) always contribute.

- The above sum can only vanish if there is no regular \( \alpha \vdash_m d \) such that \( \alpha \preceq \lambda_i \) for \( i = 1, 2, 3 \).

- Example: For \( \lambda = (\begin{array}{c}
\hline
\hline
\end{array}, \begin{array}{c}
\hline
\hline
\end{array}, \begin{array}{c}
\hline
\hline
\end{array}) \), one calculates

\[
\text{mult}_\lambda(\mathcal{O}(\text{GL}_4^3\langle 4 \rangle)) = 0.
\]

Moreover, \( k(\lambda) = 1 \). Consequence: \( \lambda \in K(4, 4, 4) \setminus S(\langle 4 \rangle) \). A generic \( w \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \) satisfies \( R(w) > 4 \) (which is optimal).
A family of occurrence obstructions

- Consider the sequence of triples $\lambda$ consisting of three times the hook partition with a foot of length $\kappa + 1$ and a leg of length $2\kappa + 1$. E.g., for $\kappa = 2$,

$$\lambda = \left(\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}\right).$$

- A nontrivial application of the branching formula implies $\text{mult}_\lambda(\mathcal{O}(\text{GL}_3^{3\kappa}\langle3\kappa\rangle)) = 0$.

- This relies on a criterion due to Rosas, telling us when the Kronecker coefficients of three hooks is positive (in which case it equals 1).

- One can show that $k(\lambda) = 1$. As a consequence, a generic $w \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, $m = 2\kappa + 1$, satisfies

$$R(w) > 3\kappa = \frac{3}{2}(2\kappa + 1) - \frac{3}{2} = \frac{3}{2}m - \frac{3}{2}.$$
Application to matrix multiplication

- We have another, more insightful proof showing why
  \[ \text{mult}_\lambda(\mathcal{O}(\text{GL}_3^3(3\kappa))) = 0. \]

- This statement is weaker, since it refers to orbit closure.

- The other argument relies on the explicit construction of highest weight functions via “obstruction designs”; see Christian’s talk.

- These occurrence obstructions also give lower bounds for matrix multiplication tensors, since we can show that the three hook $\lambda$ from above occurs in $\mathcal{O}(\text{GL}_{n^2}^3 M(n))$, where $n^2 = 2\kappa + 1$ and $\lambda$ as above.

- This gives for odd $n$,
  \[ R(M(n)) > \frac{3}{2} n^2 - \frac{3}{2}. \]
Fundamental open problem

- For finding occurrence obstructions for border rank, we need a way to determine when $\lambda$ does not occur in the orbit closure of $\langle m \rangle$.
- The branching formula gives this information for the orbit of $\langle m \rangle$. Requiring that $\lambda$ does not occur in orbits is an unnecessarily strong requirement.
- Previous insights imply: highest weight functions of weight $\lambda$ on the orbit of $\langle m \rangle$ are of the form

$$\frac{f}{\Phi_{\langle m \rangle}^s},$$

where $f$ is a globally defined highest weight function on $(\mathbb{C}^m)^{\otimes 3}$ having weight

$$(m \times as, m \times as, m \times as) + \lambda.$$

$\Phi_{\langle m \rangle}$ is the fundamental invariant of $\langle m \rangle$, $a \in \{1, 2\}$ is its period, and $s \in \mathbb{N}$. 
Representations for orbit of matrix multiplication
Invariant description

- Fix vector spaces $U_i$ of dimension $n_i$ for $i = 1, 2, 3$.
- The contraction

$$U_1^* \otimes U_2 \otimes U_2^* \otimes U_3 \otimes U_3^* \otimes U_1 \rightarrow \mathbb{C},$$

$$\ell_1 \otimes u_2 \otimes \ell_2 \otimes u_3 \otimes \ell_3 \otimes u_1 \mapsto \ell_1(u_1) \ell_2(u_2) \ell_3(u_3).$$

defines a tensor

$$M_U \in (U_1 \otimes U_2^*) \otimes (U_2 \otimes U_3^*) \otimes (U_3 \otimes U_1^*).$$

- $M_U$ is exactly the structural tensor of matrix multiplication:

$$\text{Hom}(U_1, U_2) \times \text{Hom}(U_2, U_3) \rightarrow \text{Hom}(U_1, U_3), (\varphi, \psi) \mapsto \psi \circ \varphi.$$
The stabilizer $H$ of $M_U$ is a subgroup of

$$G := \text{GL}(U_1 \otimes U_2^*) \times \text{GL}(U_2 \otimes U_3^*) \times \text{GL}(U_3 \otimes U_1^*).$$

Put $S := \text{GL}(U_1) \times \text{GL}(U_2) \times \text{GL}(U_3)$ and consider the morphism

$$\Phi: S \to G, \ (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_1 \otimes (\alpha_2^{-1})^*, \alpha_2 \otimes (\alpha_3^{-1})^*, \alpha_3 \otimes (\alpha_1^{-1})^*)$$

with kernel $\mathbb{C}^\times (\text{id}, \text{id}, \text{id}) \simeq \mathbb{C}^\times$.

$\text{im}\Phi \subseteq H$: use $(\alpha_1^{-1})^*(\ell_1)(\alpha_1(u_1)) = \ell_1(\alpha_1^{-1}(\alpha_1(u_1))) = \ell_1(u_1)$.

Theorem (de Groote 1978, case $n_1 = n_2 = n_3$). The stabilizer $H \subseteq G$ of $M_U$ equals the image of $\Phi$. In particular, $H \simeq S/\mathbb{C}^\times$.

Moreover: the stabilizer characterizes $M_U$. 

Stabilizer of matrix multiplication
Representations: Kronecker coefficients again

- Let $\lambda_{12}$, $\lambda_{23}$, and $\lambda_{31}$ be highest weights for $\text{GL}(U_1 \otimes U_2^*)$, $\text{GL}(U_2 \otimes U_3^*)$, and $\text{GL}(U_3 \otimes U_1^*)$, respectively. Recall $n_i = \dim U_i$. Consider the irreducible $G$-module

$$V_{\lambda} := V_{\lambda_{12}} \otimes V_{\lambda_{23}} \otimes V_{\lambda_{31}}.$$ 

- **Theorem.** If $\lambda_{12}, \lambda_{23}, \lambda_{31}$ are partitions of the same size $d$, then

$$\dim(V_{\lambda})^H = \sum_{\mu_{1} \vdash n_1 d, \mu_{2} \vdash n_2 d, \mu_{3} \vdash n_3 d} k(\lambda_{12}, \mu_{1}, \mu_{2}) \cdot k(\lambda_{23}, \mu_{2}, \mu_{3}) \cdot k(\lambda_{31}, \mu_{3}, \mu_{1}).$$

- Using this, one can show that the triple hook weights $\lambda$ from before occur for orbits of matrix multiplication.

- However, for the lower bound on matrix multiplication, one would need to show that they even occur for the closure. This cannot be deduced from the theorem; yet it provides useful indications where to search.
References

The details can be found in the following papers by Bürgisser and Ikenmeyer.


Currently the best place to read more about this is Christian Ikenmeyer’s PhD thesis:

Geometric Complexity Theory, Tensor Rank, and Littlewood-Richardson Coefficients

Thank you!