# Geometric Complexity Theory and Matrix Multiplication (Tutorial)

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# Background and motivation

# Goals

- Tensor rank is a natural math. concept arising in various places.
- It is intimitely related to the computational complexity of evaluating bilinear maps, in particular to the multiplication of matrices.
- To determine the (asymptotic) complexity for multiplying matrices is a major open question in algebraic complexity theory.

- GCT was proposed for the permanent vs determinant problem by Mulmuley and Sohoni in 2001.
- In joint work with Christian Ikenmeyer, we further developed the ideas of GCT in the setting of tensors (STOC 11, STOC 13).
- We managed to prove lower bounds on the border rank of matrix multiplication by exhibiting representation theoretic "occurrence obstructions".
- Our bounds are not as good as Landsberg and Ottaviani's recent bounds ('11), but they have the same order of magnitude as Strassen and Lickteig's bounds ('83).

- This talk: set the ground. More details on Wednesday (Christian).

# Tensor rank

- Consider finite dimensional complex vector spaces $W_i$ for $i = 1, 2, 3$ and put $W := W_1 \otimes W_2 \otimes W_3$. Elements $w \in W$ are called tensors.
- The rank $R(w)$ of $w \in W$ is defined as the minimum $r \in \mathbb{N}$ s.t. there are $w_{1i}, \dots, w_{ri} \in W_i$, $i = 1, 2, 3$, with

$$w = \sum_{\rho=1}^{r} w_{\rho 1} \otimes w_{\rho 2} \otimes w_{\rho 3}.$$

- Special case $W_3 = \mathbb{C}$: $R(w)$ equals the rank of the corr. linear map $W_1^* \to W_2$. In this case we know everything about $R(w)$.
- General case much harder: comp. of $R(w)$ is NP-hard (Hastad).
- To $w \in W$ there corresponds a bilinear map $\varphi\colon W_1^* \times W_2^* \to W_3$. The nonscalar complexity $L(\varphi)$ is defined as the minimum number of nonscalar multiplications sufficient to evaluate the map $\varphi$ by an arithmetic circuit.
- Strassen: $L(\varphi) \leq R(w) \leq 2L(\varphi)$.

# Complexity of matrix multiplication: the records

- Consider the tensor $M(n) \in \mathbb{C}^{n\times n} \otimes \mathbb{C}^{n\times n} \otimes \mathbb{C}^{n\times n}$ of the matrix multiplication map

$$\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}, (A, B) \mapsto AB.$$

- Best known lower bound (Landsberg '12):

$$R(M(n)) \ \geq \ \mathbf{3}\, n^2 + o(n^2).$$

  (Before, $R(M(n)) \geq \mathbf{2.5}\, n^2 + o(n^2)$ due to Bläser's ('99).)
- Asymptotic upper bounds: the exponent $\omega$ of matrix multiplication is defined as

$$\omega := \lim_{n\to\infty} \log_n R(M(n)),$$

- Coppersmith & Winograd 1990: $\omega \leq \mathbf{2.376}$. Recent improvements by Davie & Stothers, Williams, Le Gall ('14):

$$\omega \leq \mathbf{2.3728639}.$$

# Border rank ...

- The border rank $\underline{R}(w)$ of a tensor $w \in W$ is defined as the minimum $r \in \mathbb{N}$ such that there exists a sequence $w_k \in W$ with $\lim_{k\to\infty} w_k = w$ and $R(w_k) \leq r$ for all $k$.
- $\underline{R}(w) \leq R(w)$
- Fact: $\omega = \lim_{n\to\infty} \log_n \underline{R}(M(n))$.
- Best known lower bound (Landsberg and Ottaviani '11)

$$\underline{R}(M(n)) \ \geq \ \mathbf{2}\, n^2 - n.$$

(Before, Lickteig '84: $\underline{R}(M(n)) \ \geq \ \mathbf{1.5}\, n^2 + 0.5n - 1$.)

# ... as orbit closure problem

- The group
$$G := \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \times \mathrm{GL}(W_3) \tag{1}$$
acts on $W = W_1 \otimes W_2 \otimes W_3$ via
$$(g_1, g_2, g_3)(w_1 \otimes w_2 \otimes w_3) := g_1(w_1) \otimes g_2(w_2) \otimes g_3(w_3).$$
- Tensor $w \in W$ defines orbit $Gw$ and orbit closure $\overline{Gw}$. The same for euclidean topology and Zariski topology!
- Could interpret $Gw$, $\overline{Gw}$ as subsets of $\mathbb{P}(W)$ as both are cones.
- Let $r \in \mathbb{N}$, $r \leq \min_i \dim W_i$. Define $r$-th unit tensor in $W$:
$$\langle r \rangle := \sum_{\rho=1}^{r} e_{\rho 1} \otimes e_{\rho 2} \otimes e_{\rho 3},$$
where $e_{1i}, \ldots, e_{ri}$ are part of a basis of $W_i$.
- The $G$-orbit of $\langle r \rangle$ is a basis independent notion.
- Strassen (1987):
$$\underline{R}(w) \leq r \iff w \in \overline{G\langle r \rangle} \iff \overline{Gw} \subseteq \overline{G\langle r \rangle}.$$

# Basic ideas for lower bounds

# Orbit closure problem

- Reductive algebraic group $G$ acts linearly on vector space $W$ (eg. $G = \mathrm{GL}_m(\mathbb{C})$ or products thereof).
- $\mathcal{O}(W)$ ring of polynomial functions $W \to \mathbb{C}$.
- degree grading: $\mathcal{O}(W) = \oplus_{d\in\mathbb{N}}\mathcal{O}(W)_d$
- Vanishing ideal of $\overline{Gw}$ for $w \in W$

$$I(\overline{Gw}) := \{f \in \mathcal{O}(W) \mid \forall v \in \overline{Gw} \quad f(v) = 0\}.$$

- Elementary fact:

$$v \notin \overline{Gw} \Longleftrightarrow \exists f \in I(\overline{Gw}) \quad f(v) \neq 0.$$

- Such $f$ may serve as a witness for $v \notin \overline{Gw}$.
- In which degree $d$ to search for such $f$? $\mathcal{O}(W)_d$ has huge dimension even for small $d$!
- Representation theory allows for guided search for $f$.

# Representations in rings of regular functions

- The group $G$ acts on the ring $\mathcal{O}(W)$ of polynomial functions on $W$:

$$(gf)(w) := f(g^{-1}w), \quad f \in \mathcal{O}(W),\ w \in W.$$

- The vanishing ideal $I(\overline{Gw})$ is $G$-invariant.
- Representation theory: $I(\overline{Gw})$ splits into a direct sum of irreducible modules (as $G$ is reductive).
- The isomorphy types of irreducible $G$-modules in $\mathcal{O}(W)_d$ are determined by discrete data called highest weights $\underline{\lambda}$. Those are triples $\underline{\lambda}$ of partitions of $d$ (Schur, Young, Weyl).
- Irreducible $G$-modules in $\mathcal{O}(W)_d$ are generated their highest weight functions (unique up to scaling). They have a "weight" $\underline{\lambda}$.
- Recall:

$$v \notin \overline{Gw} \Longleftrightarrow \exists f \in I(\overline{Gw}) \quad f(Gv) \neq 0.$$

- One may take for $f$ a highest weight function! See Christian's talk.

# Strassen's resultant for 3-slice tensors

- $W = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^3 \simeq \oplus^3 \mathbb{C}^{m\times m}$, $m \geq 3$.
- Interpret $w \in W$ as a triple $(A, B, C)$ of $m \times m$ matrices.
- Strassen (1983):

$$f_m(A,B,C) := (\mathrm{det} A)^2 \, \mathrm{det}(BA^{-1}C - CA^{-1}B)$$

  is a semi-invariant: for $(g_1, g_2, g_3) \in \mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_3$, $w \in W$,

$$f_m((g_1 \otimes g_2 \otimes g_3)w) = (\mathrm{det} g_1 \, \mathrm{det} g_2)^3 \, (\mathrm{det} g_3)^m \, f(w).$$

- $f_m$ vanishes on the tensors of border rank$< 3m/2$.
- Semi-invariants are highest weight functions of rectangular weights $\underline{\lambda}$.
- Bläser's bound relied on Strassen's resultant.

# Splitting into irreducible representations

- The ring $\mathcal{O}(\overline{Gw})$ of regular functions on $\overline{Gw}$ consists of the restrictions of polynomial functions to $\overline{Gw}$.
- Have induced $G$-action and surjective $G$-equivariant restriction $\mathcal{O}(W) \to \mathcal{O}(\overline{Gw})$.
- $\mathcal{O}(\overline{Gw}) = \oplus_{d\in\mathbb{N}}\mathcal{O}(\overline{Gw})_d$ is graded, $\mathcal{O}(\overline{Gw})_d$ is a (f.d.) $G$-module.
- $G$ is reductive, so any (rational) $G$-module splits into irreducible $G$-modules.
- Let $V_{\underline{\lambda}}(G)$ denote the irreducible $G$-modules of highest weight $\underline{\lambda}$.
- The splitting into irreducibles can be written as

$$\mathcal{O}(\overline{Gw})_d = \bigoplus_{\underline{\lambda}} \mathrm{mult}_{\underline{\lambda}}(w)\, V_{\underline{\lambda}}(G)^*.$$

- We are interested in the multiplicities $\mathrm{mult}_{\underline{\lambda}}(w)$.

# The idea of comparing multiplicities

- Observation:

$$\overline{Gv} \subseteq \overline{Gw} \Longrightarrow \forall \underline{\lambda} \quad \mathrm{mult}_{\underline{\lambda}}(v) \leq \mathrm{mult}_{\underline{\lambda}}(w).$$

- Proof: Restriction of regular functions yields, for all degrees $d$, a surjective $G$-module morphism $\mathcal{O}(\overline{Gw})_d \to \mathcal{O}(\overline{Gv})_d$. Use Schur's lemma. □
- A representation theoretic obstruction consists of $\underline{\lambda}$ violating the above inequality of multiplicities.
- Christandl et al. '12: If $\dim \overline{Gv} < \dim \overline{Gw}$ and $\overline{Gv} \subseteq \overline{Gw}$, then $k \mapsto \mathrm{mult}_{k\underline{\lambda}}(w)$ grows at a faster rate than $k \mapsto \mathrm{mult}_{\underline{\lambda}}(kv)$.
- Therefore, asymptotic considerations of cannot help. This supports the following concept:

# Occurrence obstructions

- An occurrence obstruction consists of $\underline{\lambda}$ such that

$$\mathrm{mult}_{\underline{\lambda}}(w) = 0 \text{ and } \mathrm{mult}_{\underline{\lambda}}(v) > 0.$$

- Reformulation: $\mathrm{mult}_{\underline{\lambda}}(w) = 0$ means that all highest weight functions of weight $\underline{\lambda}$ vanish on $\overline{Gw}$. This is a very strong condition!
- Strassen's example is not an occurence obstruction: for $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^3$ there is another semi-invariant of the same weight, but which doesn't vanish on tensors of rank$\leq 5$.
- Warning: while it is true that, in principle, orbit closure problems $\overline{Gv} \not\subseteq \overline{Gw}$ can always be disproved using highest weight functions, it is nor clear that one can always do so with occurrence obstructions!
- But we will see at least one family of occurrence obstructions.

# Towards determining $\mathrm{mult}_{\underline{\lambda}}(w)$

# Decomposition of $\mathcal{O}(W)$ and Kronecker coefficients

- The space $W = W_1 \otimes W_1 \otimes W_3$ decomposes as

$$\mathcal{O}(W_1 \otimes W_2 \otimes W_3)_d = \bigoplus_{\underline{\lambda}} k(\underline{\lambda})\, V_{\underline{\lambda}}(G)^*;$$

  the sum being over the triples $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ of partitions of the same size $d$.
- Schur-Weyl duality: the multiplicities $k(\underline{\lambda})$ are the Kronecker coefficients.
- Characterization in terms of representations of the symmetric group $S_d$:

$$k(\underline{\lambda}) := \dim \left( [\lambda_1] \otimes [\lambda_2] \otimes [\lambda_3] \right)^{S_d} \qquad (2)$$

  Here $[\lambda_i]$ denotes the irreducible $S_d$-module labeled by $\lambda_i$.
- Ex. $W = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, $\underline{\lambda} = ((2,2),(2,2),(2,2))$. Then $k(\underline{\lambda}) = 1$. Hence there is semi-invariant $f$ (Cayley's hyperdeterminant) s.t.

$$f((g_1 \otimes g_2 \otimes g_3)w) = (\det g_1 \det g_2 \det g_3)^2 f(w).$$

# Monoids of representations

- The Kronecker monoids are defined as

$$K(m_1, m_2, m_3) \;:=\; \{\underline{\lambda} \mid \mathbb{N}, d \in \lambda_i \vdash_{m_i} d, k(\underline{\lambda}) > 0\}.$$

- For $w \in W$ we consider the monoid of representations of $w$

$$S(w) := \{\underline{\lambda} \mid \mathrm{mult}_{\underline{\lambda}}(w) > 0\}.$$

- General principles (finiteness of ring of $U$-invariants) imply that these monoids are finitely generated.
- The surjective morphism $\mathcal{O}(W) \to \mathcal{O}(\overline{Gw})$ implies $S(w) \subseteq K(m_1, m_2, m_3)$.
- $S(w) = K(m_1, m_2, m_3)$ for almost all $w \in W$.
- The real cone generated by $K(m_1, m_2, m_3)$ is polyhedral. It is complicated, but understood to a certain extent, see Ressayre's talk.
- Occurrence obstructions $\underline{\lambda}$ are contained in $K(m_1, m_2, m_3) \setminus S(w)$.

# Inheritance

- What happens to $\mathrm{mult}_{\underline{\lambda}}(w)$ when we enlarge the ambient space $W$ and the group $G$ of symmetries?

  $W_i \subseteq W_i'$, $W' := W_1' \otimes W_2' \otimes W_3'$, $G' := \mathrm{GL}(W_1') \times \mathrm{GL}(W_2') \times \mathrm{GL}(W_3')$.

- NOTHING!
- Can interpret a highest $G$-weight $\underline{\lambda}$ with nonnegative entries as a highest $G'$-weight $\underline{\lambda}$ (appending zeros to partitions $\lambda_i$).
- Inheritance Theorem (GCT2, Weyman)
  Let $w \in W$ and $\underline{\lambda}$ be a highest $G'$-weight.
  1. If $V_{\underline{\lambda}}(G')^*$ occurs in $\mathcal{O}(\overline{G'w})$, then $\underline{\lambda}$ is a highest $G$-weight, i.e., $\lambda_i$ has at most $\dim W_i$ parts.
  2. If $\underline{\lambda}$ is a highest $G$-weight, then

  $$\mathrm{mult}(V_{\underline{\lambda}}(G)^*, \mathcal{O}(\overline{Gw})) = \mathrm{mult}(V_{\underline{\lambda}}(G')^*, \mathcal{O}(\overline{G'w})).$$

- Proof based on method of $U$-invariants.

# Coordinate rings of orbits

- Orbits are considerably easier to understand than orbit closures.
- $\mathcal{O}(Gw)$ denotes the ring of functions that can be locally written as the quotient of two polynomial functions. (This way, $Gw$ becomes an algebraic variety.)
- $\mathcal{O}(\overline{Gw})$ is a subring of $\mathcal{O}(Gw)$.
- Since the inclusion $\mathcal{O}(\overline{Gw}) \hookrightarrow \mathcal{O}(Gw)$ is $G$-equivariant, we get

$$\mathrm{mult}_{\underline{\lambda}}(w) := \mathrm{mult}_{\underline{\lambda}}(\mathcal{O}(\overline{Gw})) \ \leq \ \mathrm{mult}_{\underline{\lambda}}(\mathcal{O}(Gw)).$$

- We have (quite complicated) formulas for the multiplicities on the right hand side.
- But currently, we have no systematic way to compute $\mathrm{mult}_{\underline{\lambda}}(w)$.
- In our example of an occurence obstruction we even have $\mathrm{mult}_{\underline{\lambda}}(\mathcal{O}(Gw)) = 0$, hence $\mathrm{mult}_{\underline{\lambda}}(w) = 0$.

# Peter-Weyl Theorem

- The stabilizer group $\mathrm{stab}(w) := \{g \in G \mid gw = w\}$ describes the symmetries of $w \in W$.
- Space of $\mathrm{stab}(w)$-invariants in $V_{\underline{\lambda}}(G)$:

$$V_{\underline{\lambda}}(G)^{\mathrm{stab}(w)} := \left\{v \in V_{\underline{\lambda}}(G) \mid \forall g \in \mathrm{stab}(w)\; gv = v\right\}$$

- Theorem. For any $\underline{\lambda}$

$$\mathrm{mult}_{\underline{\lambda}}(\mathcal{O}(Gw)) = \dim V_{\underline{\lambda}}(G)^{\mathrm{stab}(w)}.$$

(Consequence of alg. Peter-Weyl Thm. on decomposition of $\mathcal{O}(G)$.)

# Example: generic tensor

- Thm. (?) Let $m \geq 3$. The stabilizer of almost all $w \in (\mathbb{C}^m)^{\otimes 3}$ is trivial: it equals $\{(a\,\mathrm{id}, b\,\mathrm{id}, c\,\mathrm{id}) \mid a, b, c \in \mathbb{C}^\times, abc = 1\}$.
- This implies via Peter-Weyl that

$$\{\underline{\lambda} \mid \mathrm{mult}_{\underline{\lambda}}(\mathcal{O}(Gw)) > 0\}$$

  is very large: it consists of all triples of partitions of the same size.
- By contrast, for generic $w$,

$$S(w) = K(m_1, m_2, m_3)$$

  is much smaller.
- The example of generic tensors impressively shows that monoids of representations for orbit and orbit closure can differ considerably!

# Example: stabilizer of unit tensor

- group $G := \mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_m$, unit tensor

$$\langle m \rangle := \sum_{\rho=1}^{m} e_\rho \otimes e_\rho \otimes e_\rho \in (\mathbb{C}^m)^{\otimes 3}$$

- Recall $\overline{G\langle m \rangle} = \{w \in (\mathbb{C}^m)^{\otimes 3} \mid \underline{R}(w) \leq m\}$.
- What is $H := \mathrm{stab}(\langle m \rangle)$?
- The torus

$$T := \{(\mathrm{diag}(a), \mathrm{diag}(b), \mathrm{diag}(c)) \in G_m \mid \forall \rho \; a_\rho b_\rho c_\rho = 1\}$$

  is contained in $H$.
- Symmetric group $S_m$ is embedded in $G$ via $\pi \mapsto (P_\pi, P_\pi, P_\pi)$ (simultaneous permutation of standard bases). Clearly, $S_m \leq H$.
- Proposition. $\mathrm{stab}(\langle m \rangle)$ is the semidirect product of $T$ and $S_m$.
- $\langle m \rangle$ is uniquely determined by its stabilizer $H$ (up to a scalar).

# Orbit versus orbit closure

# Stability

- Consider the subgroup $G_s := \mathrm{SL}(W_1) \times \mathrm{SL}(W_2) \times \mathrm{SL}(W_3)$.
- We call $w \in W$ polystable if $G_s w$ is closed (and $w \neq 0$).
- Polystability can be shown with the Hilbert-Mumford criterion. The unit tensors $\langle m \rangle$ are polystable.
- Essential: It turns out that if $w$ is polystable, then there is a close connection between $\mathcal{O}(\overline{Gw})$ and $\mathcal{O}(Gw)$.

# The period of tensors

- We obtain a group homomorphism $\det\colon G \to \mathbb{C}^\times$ by composing the representation $D\colon G \to \mathrm{GL}(W)$ with the determinant:

$$\det(g) := \det(D(g)).$$

- Specifically,

$$\det(g_1, g_2, g_3) = (\det g_1)^{m_2 m_3} \cdot (\det g_2)^{m_1 m_3} \cdot (\det g_3)^{m_1 m_2}.$$

- Let $w \in W$ be polystable and assume that $\det(stab(w)) = \mu_a$ is the group of $a$-th roots of unity. We call $a$ the period of $w$.
- $\langle m \rangle$ has period 1 if $m$ is even and period 2 otherwise.
  Proof. $\det(P_\pi, P_\pi, P_\pi) = (\mathrm{sgn}\pi)^{3m^2} = \mathrm{sgn}\pi$. □

# The determinant of tensors

- If $w \in W$ is polystable and has period $a$, then the map
$$\det_w^a \colon Gw \to \mathbb{C}^\times, gw \mapsto \det(g)^a$$
is a well-defined morphism of algebraic varieties. (Recall $\det(stab(w)) = \mu_a$.) Warning: $det_w$ is undefined if $a > 1$.
- Lemma. The extension of $\det_w^a$ to the boundary of $Gw$ by zero yields a function $\overline{Gw} \to \mathbb{C}$ that is continuous in the $\mathbb{C}$-topology.
- However, this extension may not need to be regular. In this case, $\overline{Gw}$ is not normal.
- Consider the exponent monoid $E_w$
$$E(w) := \{e \in \mathbb{N} \mid (\det_w^a)^e \mid \text{ has a regular extension to } \overline{Gw}\}.$$
- Thm. The group generated by $E(w)$ equals $\mathbb{Z}$. Moreover, $\exists e_0 \in \mathbb{N}\, \forall e \geq e_0\;\, e \in E(w)$.

# Fundamental invariant of tensors

- We call $e(w) := \min E(w) \setminus \{0\}$ the regularity of $w$.
- So the regularity $e(w)$ is the smallest $e > 0$ such that $(\det_w^a)^e$ is regular.
- We call

$$\Phi_w := (\det_w^a)^{e(w)} \text{ the fundamental invariant of } w \ .$$

- The zero set of $\Phi_w$ in $\overline{Gw}$ is the boundary of $Gw$.
- Theorem. Under the above assumptions, $\mathcal{O}(Gw)$ is the localization of $\mathcal{O}(\overline{Gw})$ with respect to $\Phi_w$:

$$\mathcal{O}(Gw) = \left\{ \frac{f}{\Phi_w^s} \mid f \in \mathcal{O}(\overline{Gw}), s \in \mathbb{N} \right\}.$$

- Hence any $h \in \mathcal{O}(Gw)$, when multiplied with a sufficiently high power of $\Phi_w$, has a regular extension to $W$.

# Nonnormality of orbit closures

- Proposition. (compare Kumar for determinant orbit )
If $w \in (\mathbb{C}^m)^{\otimes 3}$ has period $a < \sqrt{m}$, then $e(w) > 1$ and hence $\overline{Gw}$ is not normal.
- Proof. $\det_w^a$ is a semi-invariant of weight $(m \times a, m \times a, m \times a)$.
Rules for Kronecker coeff. yield
$k(m \times a, m \times a, m \times a) = k(m \times a, a \times m, a \times m) = 0$ if $m > a^2$. □
- Since the unit tensor $\langle m \rangle$ has period $a \leq 2$, we obtain $e(\langle m \rangle) > 1$, provided $m > 2^2$. So the orbit closure of $\langle m \rangle$ is not normal in this case.
- Proposition. $e(\langle m \rangle) = 1$ for $m \leq 4$.
- Problem. Determine the regularity of unit tensors!
- Problem. Write $\Phi_{\langle m \rangle}$ explicitly as a quotient of two highest weight functions in $\mathcal{O}(W)$. (Such representations must exist.)

# Representations for orbit of unit tensors

# Representations for orbit of unit tensor

- Recall: $\mathrm{mult}_{\underline{\lambda}}(\mathcal{O}(Gw)) = \dim V_{\underline{\lambda}}(G)^{\mathsf{stab}(w)}$.
- Recall: stabilizer of unit tensor $\langle m \rangle \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ consists of simultaneous permutations of standard bases and $(\mathrm{diag}(a), \mathrm{diag}(b), \mathrm{diag}(c))$ such that $a_i b_i c_i = 1$.
- Let $V_{\lambda_i} = \bigoplus_{\alpha \in \mathbb{Z}^n} V_{\lambda_i}^{\alpha}$ be the decomposition into weight spaces of the irreducible $\mathrm{GL}_m$-module $V_{\lambda_i}$ for $\lambda_i \vdash_m d$.
- The group $S_m$ operates on $\mathbb{Z}^m$ by permutation. Let $\mathsf{stab}(\alpha) \subseteq S_m$ denote the stabilizer of $\alpha \in \mathbb{Z}^m$.
- Theorem (Branching Formula). If $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ with partitions $\lambda_i$ of the same size $d$,

$$\mathrm{mult}_{\underline{\lambda}}(\mathcal{O}(\mathrm{GL}_m^3 \langle m \rangle)) = \sum_{\alpha} \dim \left( V_{\lambda_1}^{\alpha} \otimes V_{\lambda_2}^{\alpha} \otimes V_{\lambda_3}^{\alpha} \right)^{\mathsf{stab}(\alpha)},$$

where the sum is over all partitions $\alpha \vdash_m d$ such that $\alpha \preccurlyeq \lambda_i$ for $i = 1, 2, 3$ in the dominance order.

# An small example

- Branching Formula:

$$\mathrm{mult}_{\underline{\lambda}}(\mathcal{O}(\mathrm{GL}_m^3\langle m\rangle)) = \sum_{\alpha \preccurlyeq \lambda_i} \dim \left(V_{\lambda_1}^{\alpha} \otimes V_{\lambda_2}^{\alpha} \otimes V_{\lambda_3}^{\alpha}\right)^{\mathrm{stab}(\alpha)}$$

- We are interested in those $\underline{\lambda}$ where this zero: all the summands have to vanish, which is rarely the case.
- Regular partitions $\alpha$ are those where $\mathrm{stab}(\alpha) = \{\mathrm{id}\}$, i.e., its components are pairwise distinct. Those $\alpha$ always contribute.
- The above sum can only vanish if there is no regular $\alpha \vdash_m d$ such that $\alpha \preccurlyeq \lambda_i$ for $i = 1, 2, 3$.

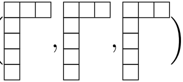
- Example: For $\underline{\lambda} = (\ydiagram{3,3,3}, \ydiagram{3,3,3}, \ydiagram{3,3,3})$, one calculates

$$\mathrm{mult}_{\underline{\lambda}}(\mathcal{O}(\mathrm{GL}_4^3\langle 4\rangle)) = 0.$$

Moreover, $k(\underline{\lambda}) = 1$. Consequence: $\underline{\lambda} \in K(4,4,4) \setminus S(\langle 4\rangle)$.
A generic $w \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ satisfies $\underline{R}(w) > 4$ (which is optimal).

# A family of occurrence obstructions

- Consider the sequence of triples $\underline{\lambda}$ consisting of three times the hook partition with a foot of length $\kappa + 1$ and a leg of length $2\kappa + 1$. E.g., for $\kappa = 2$,

$\underline{\lambda} = ($  $).$

- A nontrivial application of the branching formula implies $\mathrm{mult}_{\underline{\lambda}}(\mathcal{O}(\mathrm{GL}^3_{3\kappa}\langle 3\kappa \rangle)) = 0$.
- This relies on a criterion due to Rosas, telling us when the Kronecker coefficients of three hooks is positive (in which case it equals 1).
- One can show that $k(\underline{\lambda}) = 1$. As a consequence, a generic $w \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, $m = 2\kappa + 1$, satisfies

$$\underline{R}(w) > 3\kappa = \frac{3}{2}(2\kappa + 1) - \frac{3}{2} = \frac{3}{2}m - \frac{3}{2}.$$

# Application to matrix multiplication

- We have another, more insightful proof showing why

$$\mathrm{mult}_{\underline{\lambda}}(\mathcal{O}(\overline{\mathrm{GL}_{3\kappa}^3 \langle 3\kappa \rangle})) = 0.$$

- This statement ist weaker, since it refers to orbit closure.
- The other argument relies on the explicit construction of highest weight functions via "obstruction designs"; see Christian's talk.
- These occurence obstructions also give lower bounds for matrix multiplication tensors, since we can show that the three hook $\underline{\lambda}$ from above occurs in $\mathcal{O}(\overline{\mathrm{GL}_{n^2}^3 M(n)})$, where $n^2 = 2\kappa + 1$ and $\underline{\lambda}$ as above.
- This gives for odd $n$,

$$\underline{R}(M(n)) > \frac{3}{2}\, n^2 - \frac{3}{2}.$$

# Fundamental open problem

- For finding occurence obstructions for border rank, we need a way to determine when $\underline{\lambda}$ does not occur in the **orbit closure** of $\langle m \rangle$!
- The branching formula gives this information for the **orbit** of $\langle m \rangle$. Requiring that $\underline{\lambda}$ does not occur in orbits is an unnecessarily strong requirement.
- Previous insights imply: highest weight functions of weight $\underline{\lambda}$ on the orbit of $\langle m \rangle$ are of the form

$$\frac{f}{\Phi^{s}_{\langle m \rangle}},$$

where $f$ is a globally defined highest weight function on $(\mathbb{C}^m)^{\otimes 3}$ having weight

$$(m \times as, m \times as, m \times as) + \underline{\lambda}.$$

$\Phi_{\langle m \rangle}$ is the fundamental invariant of $\langle m \rangle$, $a \in \{1, 2\}$ is its period, and $s \in \mathbb{N}$.

# Representations for orbit of matrix multiplication

# Invariant description

- Fix vector spaces $U_i$ of dimension $n_i$ for $i = 1, 2, 3$.
- The contraction

$$\begin{aligned} U_1^* \otimes U_2 \otimes U_2^* \otimes U_3 \otimes U_3^* \otimes U_1 &\to \mathbb{C}, \\ \ell_1 \otimes u_2 \otimes \ell_2 \otimes u_3 \otimes \ell_3 \otimes u_1 &\mapsto \ell_1(u_1)\,\ell_2(u_2)\,\ell_3(u_3). \end{aligned}$$

defines a tensor

$$\mathrm{M}_{\underline{U}} \in (U_1 \otimes U_2^*) \otimes (U_2 \otimes U_3^*) \otimes (U_3 \otimes U_1^*).$$

- $\mathrm{M}_{\underline{U}}$ is exactly the structural tensor of matrix multiplication:

$$\mathrm{Hom}(U_1, U_2) \times \mathrm{Hom}(U_2, U_3) \to \mathrm{Hom}(U_1, U_3),\ (\varphi, \psi) \mapsto \psi \circ \varphi.$$

# Stabilizer of matrix multiplication

- The stabilizer $\mathcal{H}$ of $\mathrm{M}_{\underline{U}}$ is a subgroup of

$$\mathcal{G} := \mathrm{GL}(U_1 \otimes U_2^*) \times \mathrm{GL}(U_2 \otimes U_3^*) \times \mathrm{GL}(U_3 \otimes U_1^*).$$

- Put $\mathcal{S} := \mathrm{GL}(U_1) \times \mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$ and consider the morphism

$$\Phi \colon \mathcal{S} \to \mathcal{G},\ (\alpha_1, \alpha_2, \alpha_3) \mapsto \left(\alpha_1 \otimes (\alpha_2^{-1})^*, \alpha_2 \otimes (\alpha_3^{-1})^*, \alpha_3 \otimes (\alpha_1^{-1})^*\right)$$

  with kernel $\mathbb{C}^\times(\mathrm{id}, \mathrm{id}, \mathrm{id}) \simeq \mathbb{C}^\times$.
- $\mathrm{im}\Phi \subseteq \mathcal{H}$: use $(\alpha_1^{-1})^*(\ell_1)(\alpha_1(u_1)) = \ell_1(\alpha_1^{-1}(\alpha_1(u_1))) = \ell_1(u_1)$.
- Theorem (de Groote 1978, case $n_1 = n_2 = n_3$). The stabilizer $\mathcal{H} \subseteq \mathcal{G}$ of $\mathrm{M}_{\underline{U}}$ equals the image of $\Phi$. In particular, $\mathcal{H} \simeq \mathcal{S}/\mathbb{C}^\times$.
- Moreover: the stabilizer characterizes $\mathrm{M}_{\underline{U}}$.

# Representations: Kronecker coefficients again

- Let $\lambda_{12}$, $\lambda_{23}$, and $\lambda_{31}$ be highest weights for $\mathrm{GL}(U_1 \otimes U_2^*)$, $\mathrm{GL}(U_2 \otimes U_3^*)$, and $\mathrm{GL}(U_3 \otimes U_1^*)$, respectively. Recall $n_i = \dim U_i$. Consider the irreducible $\mathcal{G}$-module

$$V_{\underline{\lambda}} := V_{\lambda_{12}} \otimes V_{\lambda_{23}} \otimes V_{\lambda_{31}}.$$

- Theorem. If $\lambda_{12}, \lambda_{23}, \lambda_{31}$ are partitions of the same size $d$, then

$$\dim(V_{\underline{\lambda}})^{\mathcal{H}} = \sum_{\mu_1 \vdash_{n_1} d, \mu_2 \vdash_{n_2} d, \mu_3 \vdash_{n_3} d} k(\lambda_{12}, \mu_1, \mu_2) \cdot k(\lambda_{23}, \mu_2, \mu_3) \cdot k(\lambda_{31}, \mu_3, \mu_1).$$

- Using this, one can show that the triple hook weights $\underline{\lambda}$ from before occur for orbits of matrix multiplication.
- However, for the lower bound on matrix multiplication, one would need to show that they even occur for the closure. This cannot be deduced from the theorem; yet it provides useful indications where to search.

# References

The details can be found in the following papers by Bürgisser and Ikenmeyer.


- Geometric complexity theory and tensor rank (STOC 2011).
  See arXiv:1011.1350 for full proofs.
- Explicit lower bounds via geometric complexity theory (STOC 2013).
  arXiv:1210.8368
- Geometric complexity theory: symmetries and representations.
  Journal version with more results and full proofs in preparation.


Currently the best place to read more about this is Christian Ikenmeyer's PhD thesis:


Geometric Complexity Theory, Tensor Rank,
and Littlewood-Richardson Coefficients

PhD thesis, Paderborn University, Germany, 2012.


# Thank you!