ALGORITHMIC INVARIANT THEORY

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Tutorial at the Simons Institute Workshop on Geometric Complexity Theory
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Brief Personal History

My Masters Thesis ("Diplom", Darmstadt 1984) used classical invariants ("brackets") as a tool for geometric computations with convex polytopes.

At that time, I was inspired by Felix Klein’s Erlanger Programm (1872) which postulates that Geometry is Invariant Theory.
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In Fall 1987, during my first postdoc at the IMA in Minneapolis, I was the notetaker for Gian-Carlo Rota’s lectures Introduction to Invariant Theory in Superalgebras. This became our joint paper.

In Spring 1989, during my second postdoc at RISC-Linz, Austria, I taught a course on Algorithms in Invariant Theory. This was published as a book in the RISC series of Springer, Vienna.

During the year 1989-90, DIMACS at Rutgers ran a program on Computational Geometry. There I met Ketan Mulmuley....
Changing Coordinates

Fix a field $K$ of characteristic zero. Consider a matrix group $G$ inside the group $\text{GL}(n, K)$ of all invertible $n \times n$-matrices.

Every matrix $g = (g_{ij})$ in $G$ gives a linear change of coordinates on $K^n$. This transforms polynomials in $K[x_1, x_2, \ldots, x_n]$ via

$$x_i \mapsto g_{i1}x_1 + g_{i2}x_2 + \cdots + g_{in}x_n \quad \text{for} \quad i = 1, 2, \ldots, n.$$

An invariant of $G$ is a polynomial that is left unchanged by these transformations for all $g \in G$. These form the invariant ring

$$K[x_1, x_2, \ldots, x_n]^G \subset K[x_1, x_2, \ldots, x_n].$$
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Example

For the group $S_n$ of $n \times n$ permutation matrices, this is the ring of symmetric polynomials. For instance,

$$K[x_1, x_2, x_3]^{S_3} = K[x_1 + x_2 + x_3 , x_1x_2 + x_1x_3 + x_2x_3 , x_1x_2x_3] = K[x_1 + x_2 + x_3 , x_1^2 + x_2^2 + x_3^2 , x_1^3 + x_2^3 + x_3^3].$$
Rotating by 90 Degrees

The cyclic group

\[ G = \mathbb{Z}/4\mathbb{Z} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \]

has the invariant ring

\[ K[x, y]^G = \{ f \in K[x, y] : f(-y, x) = f(x, y) \} \]
\[ = K[x^2 + y^2, x^2y^2, x^3y - xy^3] \]

This is the coordinate ring of the quotient space \( K^2/\!/G \).

The three generators embed this surface into \( K^3 \) via

\[ K[x, y]^G \cong K[a, b, c]/\langle c^2 - a^2b + 4b^2 \rangle \]

Q: How can we be sure that there are no other invariants?
Scaling the Coordinates

The multiplicative group $G = K^*$ is known as the \textit{algebraic torus}. Consider its action on $S = K[x, y, z]$ via

$$x \mapsto t^{2}x, \quad y \mapsto t^{3}y, \quad z \mapsto t^{-7}z.$$ 

The invariant ring equals

$$S^G = K\{x^i y^j z^k : 2i + 3j = 7k\} = K[x^{7} z^{2}, x^{2} y z, x y^{4} z^{2}, y^{7} z^{3}]$$

**Big Question:** \textit{Is $S^G$ always finitely generated as a $K$-algebra?}

\textit{True if $G$ is an algebraic torus.}

Reason: Every semigroup of the form $\mathbb{N}^n \cap L$, where $L \subset \mathbb{Q}^n$ is a linear subspace, has a finite \textit{Hilbert basis}.

\textit{Also true if $G$ is a finite group.}
Averaging Polynomials

For a finite matrix group $G$, the Reynolds operator is the map

$$ S \to S^G, \quad p \mapsto p^* = \frac{1}{|G|} \sum_{g \in G} g(p) $$

Key Properties

(a) The Reynolds operator $\ast$ is a $K$-linear map.
(b) The Reynolds operator $\ast$ restricts to the identity on $S^G$.
(c) The Reynolds operator $\ast$ is an $S^G$-module homomorphism, i.e.

$$ (p \cdot q)^* = p \cdot q^* \quad \text{for all invariants} \quad p \in S^G. $$
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**Definition**

More generally, a matrix group $G$ is called reductive if it admits an operator $\ast : S \rightarrow S^G$ with these three properties.

**Remark**

*Finite matrix groups in characteristic zero are reductive.*
Finite Generation

Theorem (David Hilbert, 1890)
The invariant ring $S^G$ of a reductive group $G$ is finitely generated.

Proof.
By (a), the invariant ring $S^G$ is the $K$-vector space spanned by all symmetrized monomials $(x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n})^*$. Let $I_G$ be the ideal in $S$ generated by these invariants, for $(e_1, \ldots, e_n) \neq (0, \ldots, 0)$. By Hilbert’s Basis Theorem, the ideal $I_G$ is generated by a finite subset of these invariants, say, $I_G = \langle p_1, p_2, \ldots, p_m \rangle$. We claim that $S^G = K[p_1, p_2, \ldots, p_m]$. Suppose not, and pick $q \in S^G \setminus K[p_1, p_2, \ldots, p_m]$ of minimum degree. Since $q \in I_G$, we can write $q = f_1 p_1 + f_2 p_2 + \cdots + f_m p_m$, where $f_i \in S$ are homogeneous of strictly smaller degree. By (b) and (c), $q^* = q^* = f_1^* p_1 + f_2^* p_2 + \cdots + f_m^* p_m$. By minimality, each $f_i^*$ lies in $K[p_1, p_2, \ldots, p_m]$. Hence so does $q^*$.
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$$q = q^* = f_1^* \cdot p_1 + f_2^* \cdot p_2 + \cdots + f_m^* \cdot p_m.$$ 

By minimality, each $f_i^*$ lies in $K[p_1, \ldots, p_m]$. Hence so does $q$. \qed
Finite Groups

Let $G$ be finite and $\text{char}(K) = 0$.

Theorem (Emmy Noether, 1916)

*The invariant ring $S^G$ is generated by invariants of degree $\leq |G|$.**
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Theorem (Theodor Molien, 1897)

The Hilbert series of the invariant ring $S^G$ is the average of the inverted characteristic polynomials of all group elements, i.e.

$$\sum_{d=0}^{\infty} \dim_K(S^G_d) \cdot z^d = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(\text{Id} - z \cdot g)}.$$
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$$

**Example (Rotations by 90 Degrees)**

\[
\begin{vmatrix} 1-z & 0 \\ 0 & 1-z \end{vmatrix}^{-1} + \begin{vmatrix} 1+z & 0 \\ 0 & 1+z \end{vmatrix}^{-1} + \begin{vmatrix} 1 & z \\ -z & 1 \end{vmatrix}^{-1} + \begin{vmatrix} 1 & -z \\ z & 1 \end{vmatrix}^{-1} = \frac{1 - z^8}{(1 - z^2)^2 \cdot (1 - z^4)} = 1 + z^2 + 3z^4 + 3z^6 + \cdots
\]
Two Algorithms

Crude Algorithm
1. Compute the Molien series.
2. Produce invariants of low degree using the Reynolds operator.
3. Compute the Hilbert series of the current subalgebra of $S$.
4. If that Hilbert series equals the Molien series, we are done.
5. If not, increase the degree and go back to 2.
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**Derksen’s Algorithm (1999)**
1. Introduce three sets of variables: $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ for $K^n$, and $\mathbf{g} = (g_1, \ldots, g_r)$ for $G \subset \text{GL}(n, K)$.
2. Consider the ideal $J = \langle \mathbf{y} - \mathbf{g} \cdot \mathbf{x} \rangle + \langle \mathbf{g} \in G \rangle$ in $K[\mathbf{x}, \mathbf{y}, \mathbf{g}]$.
3. Compute generators $p_1, \ldots, p_m$ for $I_G = (J \cap K[\mathbf{x}, \mathbf{y}])|_{\mathbf{y}=0}$.
4. Output: The invariants $p_1^*, \ldots, p_m^*$ generate $K[\mathbf{x}]^G$.

**Torus Action Example:**
\[
\langle u - t^2x, v - t^3y, w - s^7z, st - 1 \rangle \cap K[x, y, z, u, v, w]|_{u=v=w=0}
\]
Classical Invariant Theory

We fix a polynomial representation of the special linear group:

\[ \text{SL}(d, K) \xrightarrow{\rho} G \subset \text{GL}(V) \quad \text{where} \quad V \cong K^n. \]

**Fact:** The matrix group \( G \) is reductive.
Classical Invariant Theory

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**Fact**: The matrix group \( G \) is reductive.

**Example**

Let \( d = 2, n = 4 \) and consider the adjoint representation where \( g \in \text{SL}(2, K) \) acts on matrix space \( V = K^{2\times2} \) via \( g \mapsto g \cdot x \cdot g^{-1} \).

Explicitly, this is the quadratic representation given by

\[
\rho(g) = \begin{pmatrix}
g_{11}g_{22} & -g_{11}g_{21} & g_{12}g_{22} & -g_{12}g_{21} \\
-g_{11}g_{12} & g_{11}^2 & -g_{12}^2 & g_{11}g_{12} \\
g_{21}g_{22} & -g_{21}^2 & g_{22}^2 & -g_{21}g_{22} \\
-g_{12}g_{21} & g_{11}g_{21} & -g_{12}g_{22} & g_{11}g_{22}
\end{pmatrix}
\]

The vectorization of the \( 2 \times 2 \)-matrix \( g \cdot x \cdot g^{-1} \) equals the \( 4 \times 4 \)-matrix \( \rho(g) \) times the vectorization of \( x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \).
Orbits

The invariant ring for the adjoint action on $2 \times 2$-matrices $\mathbf{x}$ is

$$\mathbb{C}[\mathbf{x}]^{\text{SL}(2, \mathbb{C})} = \mathbb{C}[\text{trace}(\mathbf{x}), \text{det}(\mathbf{x})].$$

The invariants are constant along orbits and their closures.

Example

The orbit of $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ is closed. It is the variety defined by the ideal

$$\langle \text{trace}(\mathbf{x}) - 9, \text{det}(\mathbf{x}) + 1 \rangle = \langle \text{trace}(\mathbf{x}) - 9, \text{trace}(\mathbf{x}^2) - 83 \rangle.$$

Question: Are all orbits closed? Do the invariants separate orbits?
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Question: Are all orbits closed? Do the invariants separate orbits?

Answer: Not quite. The nullcone $V(\langle \text{trace}(\mathbf{x}), \text{det}(\mathbf{x}) \rangle)$ contains many orbits (of nilpotent matrices) that cannot be separated.

Recall the Jordan canonical form, and consider the orbits of

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \ldots$$
Let $n = dm$ and $V = K^{d \times m}$ be the space of $d \times m$-matrices. Our group $G = \text{SL}(d, K)$ acts on $V$ by left multiplication.

**First Fundamental Theorem**

$K[V]^G$ is generated by the $\binom{m}{d}$ maximal minors of $x = (x_{ij})$.

**Second Fundamental Theorem**

The relations among these generators, which are denoted by $[i_1i_2 \cdots i_d]$, are generated by the quadratic Plücker relations.

**Example**

For $d = 2$, $m = 4$, the generators are $[ij] = x_{1i} \cdots x_{2j} - x_{1j} \cdots x_{2i}$ and the ideal of relations is $\langle [12][34] - [13][24] + [14][23] \rangle$.

**Example**

For $d = 3$, $m = 6$, our matrix $x$ represents six points in $\mathbb{P}^2$. These lie on a conic if and only if $[123][145][246][356] = [124][135][236][456]$. 
Algebraic Geometry

Let \( n = \binom{d+m-1}{m-1} \) and consider the action of \( G = \text{SL}(d, K) \) on \( V = S^d K^m = \{ \text{homog. polynomials of degree } d \text{ in } m \text{ variables} \} \).

The invariant ring \( K[V]^G \) is finitely generated. Its generators express geometric properties of hypersurfaces of degree \( d \) in \( \mathbb{P}^{m-1} \).

This is the point of departure for \textbf{Geometric Invariant Theory}.

Example

Let \( d = m = 2, n = 3 \), so \( V \) is the 3-dim’l space of 

\[
    f(t_0, t_1) = x_1 \cdot t_0^2 + x_2 \cdot t_0 t_1 + x_3 \cdot t_1^2
\]

\textcolor{brown}{\textbf{Pop Quiz: Can you}} write down the \( 3 \times 3 \)-matrix \( \rho(g) \)? \textcolor{brown}{Do now.}

Check: The invariant ring is generated by the discriminant

\[
    K[x_1, x_2, x_3]^G = K[x_2^2 - 4x_1x_3].
\]
Plane Cubics

The case $d = m = 3$ corresponds to cubic curves in the plane $\mathbb{P}^2$. A ternary cubic has $n = 10$ coefficients:

$$x_1 t_0^3 + x_2 t_1^3 + x_3 t_2^3 + x_4 t_0^2 t_1 + x_5 t_2^2 t_1 + x_6 t_0 t_1^2 + x_7 t_0 t_2^2 + x_8 t_1^2 t_2 + x_9 t_1 t_2^2 + x_{10} t_0 t_1 t_2$$

The invariant ring $K[V]^G$ is a subring of $K[V] = K[x_1, x_2, \ldots, x_{10}]$. It is generated by two classical invariants:

- a quartic $S$ with 26 terms;  
  \hspace{1cm} ← the Aronhold invariant
- a sextic $T$ with 103 terms.
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- a quartic $S$ with 26 terms; ← the Aronhold invariant
- a sextic $T$ with 103 terms.

Another important invariant is the discriminant $\Delta = T^2 - 64S^3$ which has 2040 terms of degree 12. It vanishes if and only if the cubic curve is singular. If $\Delta \neq 0$ then the cubic is an elliptic curve. Number theorists love the j-invariant:

$$j = \frac{S^3}{\Delta}$$

This serves as the coordinate on the moduli space

$$V//G = \text{Proj}(K[V]^G) = \text{Proj}(K[S, T])$$
Old and New

Theorem (Cayley-Bacharach)

Let $P_1, \ldots, P_8$ be eight distinct points in the plane, no three on a line, and no six on a conic. There exists a unique ninth point $P_9$ such that every cubic curve through $P_1, \ldots, P_8$ also contains $P_9$.

My paper with Qingchun Ren and Jürgen Richter-Gebert (May 2014) gives an explicit formula (in brackets) for $P_9$ in terms of $P_1, P_2, \ldots, P_8$. 
Hilbert’s 14th Problem

Given any matrix group $G$, is the invariant ring $K[V]^G$ always finitely generated?

Does Hilbert’s 1890 Theorem extend to non-reductive groups?

Note: Subalgebras of a polynomial ring need not be finitely generated, e.g.

$$K[x, xy, xy^2, xy^3, \ldots] \subset K[x, y].$$
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*A negative answer was given by Masayoshi Nagata in 1959.*

We shall describe Nagata’s counterexample, following the exposition in

*SAGBI bases of Cox-Nagata Rings* (with Z. Xu, JEMS 2010)
Additive Groups

Fix $n = 2m$. The group $(K^m, +)$ is not reductive. It acts on $K[x, y] = K[x_1, \ldots, x_m, y_1, \ldots, y_m]$ via

\[
\begin{align*}
x_i & \mapsto x_i & \text{and} \\
y_i & \mapsto y_i + u_i x_i & \text{for } u \in K^m.
\end{align*}
\]

Let $d \leq m$ and fix a generic $d \times m$-matrix $U$. Let $G = \text{rowspace}(U) \subset K^m$. The additive group $(G, +) \simeq (K^d, +)$ acts on $K[x, y]$ by the rule above.
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Among the invariants are \( x_1, \ldots, x_m \) and the maximal minors of

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\begin{pmatrix}
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Theorem

The ring \( K[x, y]^G \) is not finitely generated when \( m = d + 3 \geq 9 \).
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Proof.

Blow up \( m = 5, 6, 7, 8, 9, \ldots \) general points in the plane \( \mathbb{P}^2 \) and you will discover the Weyl groups \( D_5, E_6, E_7, E_8, E_9, \ldots \).
Conclusion

Invariant theory is timeless, relevant and fun.

Reinhard Laubenbacher and I had lots of fun when translating and editing the notes from Hilbert’s course (Summer Semester 1897 at Göttingen)