A Particle Method for Degenerate Diffusion, and Applications to Robotic Swarming

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Outline

- Intro – the PDE
- Robotic swarms
- Numerical method
- Proof ideas
- Back to swarms
- Conclusion

Introduction

- Fix $\Omega \subset \mathbb{R}^d$ and $\bar{\rho} : \Omega \to \mathbb{R}^+$, both nice.
- We focus on:

$$\begin{cases} \partial_t \rho - \operatorname{div}\left(\rho \nabla \left(\dfrac{\rho}{\bar{\rho}}\right)\right) = 0 \text{ in } \Omega \times (0,\infty), \\ \partial_\nu \left(\frac{\rho}{\bar{\rho}}\right) = 0 \text{ on } (\partial\Omega) \times (0,\infty). \end{cases} \tag{D}$$

- If $\bar{\rho} \equiv 1$, then the diffusion term becomes,

$$\operatorname{div}(\rho \nabla \rho) = \frac{1}{2} \operatorname{div}(\nabla \rho^2) = \frac{1}{2} \Delta(\rho^2).$$

  So this PDE is an inhomogeneous porous medium equation.
- Observation: as $t \to \infty$, the solutions $\rho(x,t)$ converges to $\bar{\rho}(x)$.

Application

Seek to pollinate a field of flowers using robotic bees.

- Each bee:
  - can sense whether it is over a flower;
  - can sense whether there are other bees nearby;
  - can change its velocity based on this info.
- Bees cannot communicate with each other or with a central hub.

Modeling:

- $\Omega \subset \mathbb{R}^2$ represents the field.
- $\bar{\rho} : \Omega \to \mathbb{R}^+$ represents the distribution of flowers in Ω.
- Note: $\bar{\rho}$ is not known to the bees.

Goal:

- Get the bees to come to the flowers, i.e.

$$\text{density of swarm} \approx \bar{\rho}.$$

Numerical method

$$\partial_t \rho - \operatorname{div}\left(\rho \nabla \left(\frac{\rho}{\bar{\rho}}\right)\right) = 0 \text{ in } \Omega \times (0, \infty). \tag{D}$$

- Solutions to (D) are Wasserstein gradient flows of the energy:

$$\mathcal{E}[\rho] = \frac{1}{2} \int_\Omega \frac{\rho}{\bar{\rho}} \, d\rho.$$

  (under an additional assumption on $\bar{\rho}$.)

- Main idea:

  approximate the energy $\mathcal{E}$,

  in a way that ensures "particles remain particles."

- This idea originates from Carillo, Craig, Patacchini, 2019 ($\bar{\rho} \equiv 1$).

Approximating the energy

$$\mathcal{E}[\rho] = \frac{1}{2}\int_\Omega \frac{\rho}{\bar{\rho}}\, d\rho = \frac{1}{2}\int_\Omega \frac{\rho^2}{\bar{\rho}}\, dx.$$

- Let ζ be a mollifier "like the Gaussian."
- We define,

$$\mathcal{E}_\varepsilon[\rho] = \frac{1}{2}\int_\Omega \frac{(\rho * \zeta_\varepsilon)^2}{\bar{\rho}}\, dx.$$

- Wasserstein GF of $\mathcal{E}_\varepsilon$ are solutions of the continuity equation,

$$\partial_t \rho - \text{div}\left(\rho \nabla \frac{\partial \mathcal{E}_\varepsilon}{\partial \rho}\right) = 0,$$

where

$$\frac{\partial \mathcal{E}_\varepsilon}{\partial \rho} = \zeta_\varepsilon * \left(\frac{\zeta_\varepsilon * \rho}{\bar{\rho}}\right).$$

- Key point: solution to this equation with particle initial data remains so.

The approximate continuity equation

Lemma: Let $\rho_{N,\varepsilon}$ denote the (distributional) solution to the continuity equation

$$\partial_t \rho - \operatorname{div}\left(\rho \nabla \left(\zeta_\varepsilon * \left(\frac{\zeta_\varepsilon * \rho}{\bar{\rho}}\right)\right)\right) = 0,$$

with initial data $\sum_{i=1}^N m_i \delta_{X_i^0}$. We have, for $t > 0$,

$$\rho_{N,\varepsilon}(x,t) = \sum_{i=1}^N m_i \delta_{X_i(t)},$$

where the X_i evolve via the system of ODEs associated to the continuity equation:

$$\frac{d}{dt} X_j(t) = -V_\varepsilon^{(j)}(X_1(t), ..., X_N(t)), \quad X_j(0) = X_j^0,$$

where V_ε is given by,

$$V_\varepsilon^{(i)}(y_1, ..., y_N) = \sum_{j=1}^N m_j \int_{\mathbb{R}^d} \nabla \zeta_\varepsilon(y_i - z) \zeta_\varepsilon(z - y_j) \frac{1}{\bar{\rho}(z)}\, dz.$$

Numerical method

- Discretize initial data into particles: $\rho_0 \approx \sum_{i=1}^N m_i \delta_{X_i^0}$.
- Let the particles evolve via the ODE system:

$$\frac{d}{dt} X_j(t) = -V_\varepsilon^{(j)}(X_1(t), ..., X_N(t)), \quad X_j(0) = X_j^0.$$

- Let

$$\rho_{N,\varepsilon}(x,t) = \sum_{i=1}^N m_i \delta_{X_i(t)}.$$

- Consider “blobs” over each particle:

$$\tilde{\rho}_{N,\varepsilon}(x,t) = \sum_{i=1}^N m_i \zeta_\varepsilon(x - X_i(t)) = (\zeta_\varepsilon * \rho_{N,\varepsilon})(x,t).$$

- This $\tilde{\rho}_{N,\varepsilon}$ is our approximate solution.

Numerical method

Videos.

Rigorous results

Craig, Elamvazhuthi, Haberland, T, 2021
Assumptions on $\bar{\rho}$:

1. $\bar{\rho} \in C^1(\Omega)$ is bounded from above and away from zero on Ω;
2. $\bar{\rho}$ log-concave, i.e $\bar{\rho}(x) = e^{V(x)}$ for some concave V.

Also some assumptions on initial data.

Theorem

Let ρ solve the PDE (D). Then as $\varepsilon \to 0$ and $N \to \infty$,

$$\rho_{N,\varepsilon} \to \rho \text{ and } \tilde{\rho}_{N,\varepsilon} \to \rho \text{ narrowly.}$$

Follows from:

Theorem

Let ρ_ε be the Wasserstein gradient flow of $\mathcal{E}_\varepsilon$ with initial data ρ_0. Then the ρ_ε narrowly converge to ρ, where ρ is the gradient flow of $\mathcal{E}$ with initial data ρ_0.

Elements of the proof

- When $\bar{\rho}$ is log-concave, the energy $\mathcal{E}$ is a standard "internal energy", so we already know GFs for $\mathcal{E}$ are well-defined (Ambrosio, Gigli, Savaré).
- Convexity of $\mathcal{E}_\varepsilon$:
  - $\mathcal{E}_\varepsilon$ is λ_ε-convex along generalized geodesics in Wasserstein space (where $\lambda_\varepsilon < 0$ and $\lambda_\varepsilon \to -\infty$ as $\varepsilon \to 0$).
  - Does not require log-concavity assumption.
- So, GF for $\mathcal{E}_\varepsilon$ is well-defined.
- Γ-convergence of $\mathcal{E}_\varepsilon$ to $\mathcal{E}$.
- Obtain an estimate on an "H^1-like norm" for GFs of $\mathcal{E}_\varepsilon$.
  - formally: play with the ε-PDE.
  - rigorously: flow interchange method of Matthes, McCann, Savaré.
  - Does not require log-concavity assumption.
- Prove that metric slopes for $\mathcal{E}_\varepsilon$ converge to those for $\mathcal{E}$.
- Conclude via a general result of Serfaty.

Back to robots

Recall: goal was to get the bees to come to the flowers, i.e.

$$\text{density of swarm} \approx \bar{\rho}.$$

- Let X_i^0 be the initial bee locations.
- Define $X_i(t)$ to be solutions of that same ODE system.
- Let

$$\tilde{\rho}_{N,\varepsilon}(x,t) = \sum_{i=1}^{N} m_i \zeta_\varepsilon(x - X_i(t)).$$

- (Here ε represents "pollination radius.")

So, for "small ε", "large N", and "large t",

$$\tilde{\rho}_{N,\varepsilon}(x,t) \approx \rho(x,t) \approx \bar{\rho}(x).$$

Great!

Future work

- Rate of convergence
  - Numerical observation: convergence is of rate $1/N$ (or better!)
- Remove log-concave assumption
  - Numerical observation: everything still works
- Compact mollifier
- More general diffusion

Thank you!