

Exact Minimax Estimation for Phase Synchronization

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Anderson Zhang

Phase Synchronization

Model $Y_{jk} = z_j \bar{z}_k + \sigma W_{jk} \in \mathbb{C} \quad 1 \leq j < k \leq n$

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**Parameter
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$$z_j \in \mathbb{C}_1 = \{z \in \mathbb{C} : |z| = 1\}$$

Phase Synchronization

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$\text{Im}(W_{jk}), \text{Re}(W_{jk}) \sim N(0, 1/2)$

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$$\ell(\hat{z}, z) = \min_{a \in \mathbb{C}_1} \frac{1}{n} \sum_{j=1}^n |\hat{z}_j a - z_j|^2$$

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SDP

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SDP

$$\max_{Z=Z^H \in \mathbb{R}^{n \times n}} \text{Tr}(Y Z) \quad \text{subject to } \text{diag}(Z) = I_n \text{ and } Z \succeq 0$$

Literature

MLE

GPM

SDP

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MLE

[Bandeira, Boumal & Singer 17]

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GPM

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Theorem [G & Zhang]. Assume $\sigma^2 = o(np)$ and $\frac{np}{\log n} \rightarrow \infty$. Then,

$$\inf_{\hat{z} \in \mathbb{C}_1^n} \sup_{z \in \mathbb{C}_1^n} \mathbb{E}_z \ell(\hat{z}, z) \geq (1 - o(1)) \frac{\sigma^2}{2np} .$$

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Moreover, the MLE, GPM initialized by the leading eigenvector of $A \circ Y$, and the leading eigenvector of SDP all achieve

$$\ell(\hat{z}, z) \leq (1 + o(1)) \frac{\sigma^2}{2np}$$

with high probability.

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Moreover, SDP achieves

$$\frac{1}{n^2} \|\hat{Z} - zz^H\|_F^2 \leq (1 + o(1)) \frac{\sigma^2}{np}$$

with high probability.

How to prove the results?

Statistical and Computational Guarantees of Lloyd's Algorithm and Its Variants

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¹*Yale University*

December 8, 2016

Abstract

Clustering is a fundamental problem in statistics and machine learning. Lloyd's algorithm, proposed in 1957, is still possibly the most widely used clustering algorithm in practice due to its simplicity and empirical performance. However, there has been little theoretical investigation on the statistical and computational guarantees of Lloyd's algorithm. This paper is an attempt to bridge this gap between practice and theory. We investigate the performance of Lloyd's algorithm on clustering sub-Gaussian mixtures. Under an appropriate initialization for labels or centers, we show that Lloyd's algorithm converges to an exponentially small clustering error after an order of $\log n$ iterations, where n is the sample size. The error rate is shown to be minimax optimal. For the two-mixture case, we only require the initializer to be slightly better than random guess.

In addition, we extend the Lloyd's algorithm and its analysis to community detection and crowdsourcing, two problems that have received a lot of attention recently in statistics and machine learning. Two variants of Lloyd's algorithm are proposed respectively for community detection and crowdsourcing. On the theoretical side, we provide statistical and computational guarantees of the two algorithms, and the results improve upon some previous signal-to-noise ratio conditions in literature for both problems. Experimental results on simulated and real data sets demonstrate competitive performance of our algorithms to the state-of-the-art methods.

1 Introduction

Lloyd's algorithm, proposed in 1957 by Stuart Lloyd at Bell Labs [40], is still one of the most popular clustering algorithms used by practitioners, with a wide range of applications from computer vision [3], to astronomy [45] and to biology [26]. Although considerable innovations have been made on developing new provable and efficient clustering algorithms in the past six decades, Lloyd's algorithm has been consistently listed as one of the top ten data mining algorithms in several recent surveys [55].

Structured Linear Models

$$Y = \mathcal{X}_z(B) + w$$

Structured Linear Models

structure

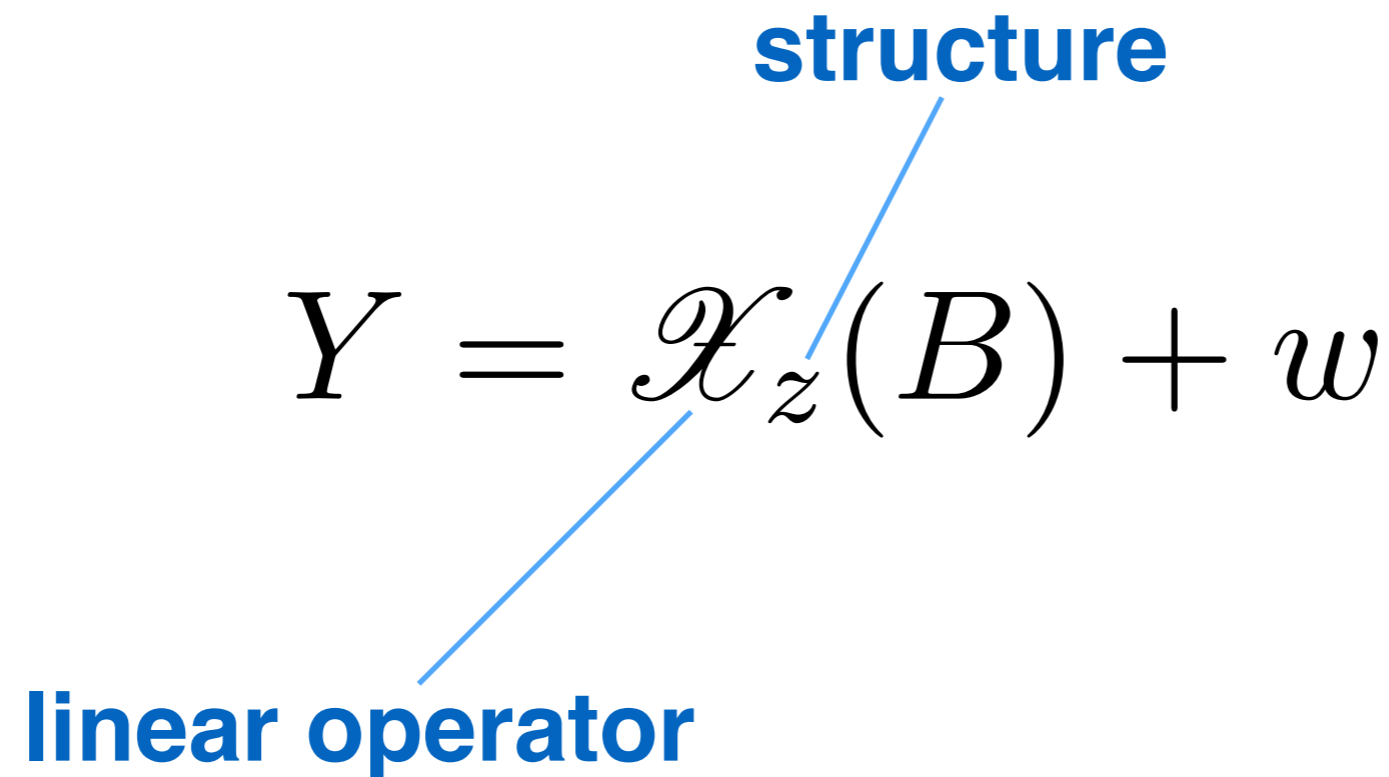
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Structured Linear Models

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parameter

The diagram illustrates the components of the structured linear model equation $Y = \mathcal{K}_z(B) + w$. Three blue lines point from labels to parts of the equation: 'structure' points to the \mathcal{K}_z operator, 'linear operator' points to the \mathcal{K}_z operator, and 'parameter' points to the B matrix.

Structured Linear Models

$$Y = \mathcal{K}_z(B) + w$$

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parameter

$$B \in \mathcal{B}_z$$

The diagram illustrates the components of the structured linear model equation $Y = \mathcal{K}_z(B) + w$. The term \mathcal{K}_z is identified as the **linear operator**, and the term B is identified as the **parameter**. The parameter B is constrained to belong to the set \mathcal{B}_z , as indicated by the equation $B \in \mathcal{B}_z$ below it. The word **structure** is positioned above the equation, indicating the overall form of the model.

Structured Linear Models

Clustering

$$Y_i \sim N(\theta_{z_i}, I_d)$$

Ranking

$$Y_{ij} \sim N(\beta(z_i - z_j), 1)$$

Regression

$$Y_i \sim N(X_i^T \beta, 1)$$

Structured Linear Models

Clustering

$$Y_i \sim N(\theta_{z_i}, I_d)$$

$$\mathcal{X}_z : [\theta_1, \dots, \theta_k] \mapsto [\theta_{z_1}, \dots, \theta_{z_n}]$$

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Iterative Algorithm

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some local statistic T_j

Iterative Algorithm

T_j

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unknown, but $B \in \mathcal{B}_z$

Iterative Algorithm

$$\mathbb{E}T_j = \mu_j(B, z_j)$$

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Conditions

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loss function

$$\ell(z, z^*) = \sum_j \|\mu_j(B^*, z_j) - \mu_j(B^*, z_j^*)\|^2$$

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$$\begin{aligned} \ell(z, z^*) &= \sum_j \|\mu_j(B^*, z_j) - \mu_j(B^*, z_j^*)\|^2 \\ &\geq \Delta_{\min}^2 \sum_j \mathbf{1}_{\{z_j \neq z_j^*\}} \end{aligned}$$

Conditions

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error conditions

$$\frac{\text{diff} \left(\mu_j(B(z), a), \mu_j(B(z^*), a) \right)}{\ell(z, z^*)} = o_{\mathbb{P}}(1)$$

$$\text{diff} \left(\mu_j(\hat{B}(z^*), a), \mu_j(B^*, a) \right) = o_{\mathbb{P}}(1)$$

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error conditions

$$\max_{\{z: \ell(z, z^*) \leq \tau\}} \frac{\text{diff} \left(\mu_j(B(z), a), \mu_j(B(z^*), a) \right)}{\ell(z, z^*)} = o_{\mathbb{P}}(1)$$

$$\text{diff} \left(\mu_j(\hat{B}(z^*), a), \mu_j(B^*, a) \right) = o_{\mathbb{P}}(1)$$

Convergence

Theorem [G & Zhang]. Assume

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and the conditions hold with the same τ .

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Then, we have

$$\ell(z^{(t)}, z^*) \leq [\text{ideal error}] + \frac{1}{2} \ell(z^{(t-1)}, z^*)$$

for all $t \geq 1$.

Iterative Algorithm

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$$\left\{ \begin{array}{l} z_j^{(t)} = \operatorname{argmin}_a \|T_j - \mu_j(\hat{B}(z^{(t-1)}), a)\|^2 \\ \hat{B}(z) = \operatorname{argmin}_{B \in \mathcal{B}_z} \|Y - \mathcal{X}_z(B)\|^2 \end{array} \right.$$

Clustering

model

$$Y_i \sim N(\theta_{z_i}, I_d)$$

Clustering

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specialization

$$T_i = Y_i$$

Clustering

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$$T_i = Y_i \quad \mu_i(B, a) = \theta_a$$

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algorithm

$$\left\{ \begin{array}{l} z_i^{(t)} = \operatorname{argmin}_{a \in [k]} \|Y_i - \hat{\theta}_a(z^{(t-1)})\|^2 \\ \hat{\theta}_a(z) = \frac{\sum_{i=1}^n \mathbf{1}_{\{z_i=a\}} Y_i}{\sum_{i=1}^n \mathbf{1}_{\{z_i=a\}}} \end{array} \right.$$

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[Lloyd 57]

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Initialize by spectral clustering, and then

$$\frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{z_i^{(t)} \neq z_i^*\}} \leq \exp\left(-\frac{(1 + o(1)) \min_{a \neq b} \|\theta_a^* - \theta_b^*\|^2}{8}\right) + 2^{-t}$$

for all $t \geq 1$ with high probability.

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minimax rate

Clustering

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minimax rate

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can deal with structure
parameters in two positions

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linear in z (instead of quadratic)

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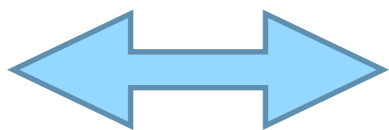
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with missing
data

Phase Synchronization

$$z_i^{(t)} = \frac{[(A \circ Y)z^{(t-1)}]_i}{|[(A \circ Y)z^{(t-1)}]_i|}$$

Phase Synchronization

$$z^{(t)} = f(z^{(t-1)}) \quad \longleftrightarrow \quad z_i^{(t)} = \frac{[(A \circ Y)z^{(t-1)}]_i}{|[(A \circ Y)z^{(t-1)}]_i|}$$

Phase Synchronization

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Lemma. Assume $\sigma^2 = o(np)$ and $p \gg \frac{\log n}{n}$. For any $\gamma = o(1)$, we have

$$\mathbb{P} \left(\ell(f(z), z^*) \leq \delta \ell(z, z^*) + (1 + \delta) \frac{\sigma^2}{2np} \text{ for any } z \in \mathbb{C}_1^n \text{ s.t. } \ell(z, z^*) \leq \gamma \right) \geq 1 - \delta$$

for some $\delta = o(1)$.

Phase Synchronization

Corollary. Assume $\sigma^2 = o(np)$ and $p \gg \frac{\log n}{n}$.
Initialized by PCA, the power method satisfies

$$\ell(\hat{z}, z) \leq (1 + o(1)) \frac{\sigma^2}{2np}$$

with high probability after $\log\left(\frac{np}{\sigma^2}\right)$ iterations.

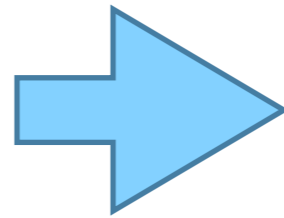
Phase Synchronization

MLE $\max_{z \in \mathbb{C}_1^n} z^H (A \circ Y) z$

Phase Synchronization

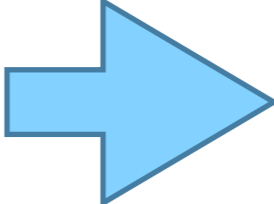
MLE

$$\max_{z \in \mathbb{C}_1^n} z^H (A \circ Y) z$$



$$\hat{z} = f(\hat{z})$$

Phase Synchronization

MLE $\max_{z \in \mathbb{C}_1^n} z^H (A \circ Y) z$  $\hat{z} = f(\hat{z})$

Corollary. Assume $\sigma^2 = o(np)$ and $p \gg \frac{\log n}{n}$.
The MLE satisfies

$$\ell(\hat{z}, z) \leq (1 + o(1)) \frac{\sigma^2}{2np}$$

with high probability.

SDP: A Non-Convex View

SDP: A Non-Convex View

MLE

$$\begin{aligned} & \max \operatorname{Tr}((A \circ Y)zz^H) \\ & \text{s.t. } |z_j| = 1 \text{ for all } j \in [n] \end{aligned}$$

SDP: A Non-Convex View

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GPM

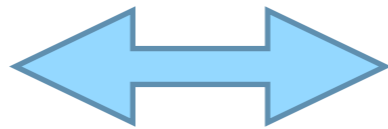
$$z_j^{(t)} = \frac{\sum_{k \in [n] \setminus \{j\}} A_{jk} Y_{jk} z_k^{(t-1)}}{\left| \sum_{k \in [n] \setminus \{j\}} A_{jk} Y_{jk} z_k^{(t-1)} \right|}$$

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fixed point



GPM

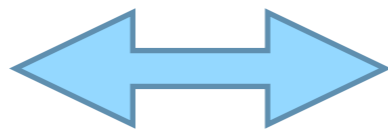
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SDP

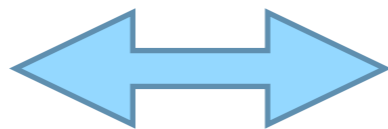
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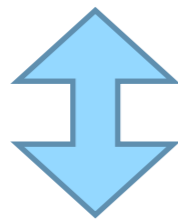


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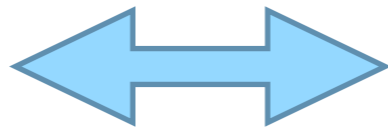
$$Z = V^H V.$$

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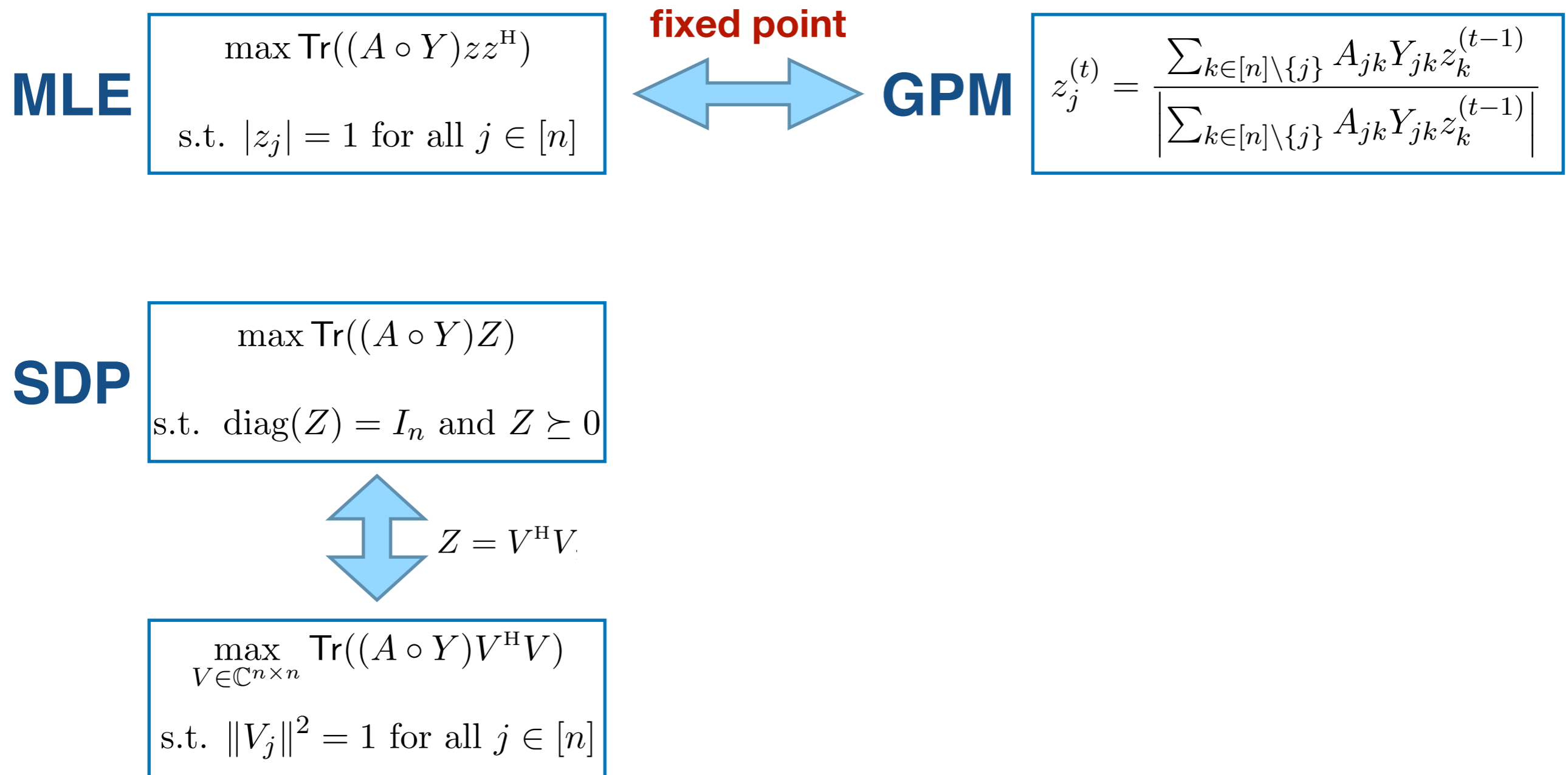
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SDP: A Non-Convex View

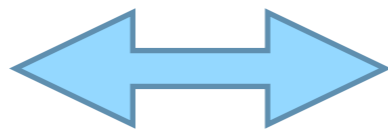


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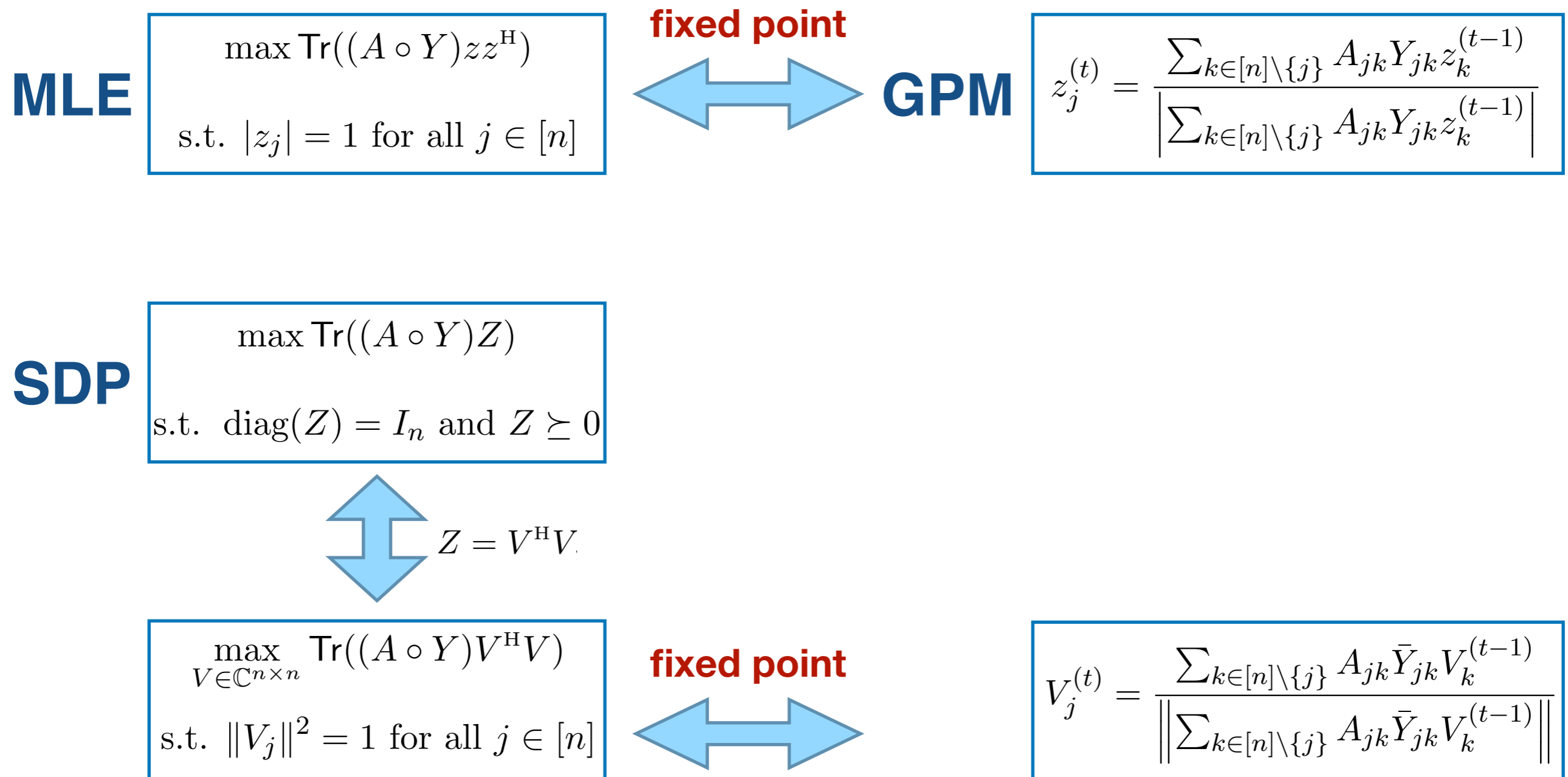
$$\begin{aligned} & \max \operatorname{Tr}((A \circ Y) Z) \\ & \text{s.t. } \operatorname{diag}(Z) = I_n \text{ and } Z \succeq 0 \end{aligned}$$

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$$V_j^{(t)} = \frac{\sum_{k \in [n] \setminus \{j\}} A_{jk} \bar{Y}_{jk} V_k^{(t-1)}}{\left\| \sum_{k \in [n] \setminus \{j\}} A_{jk} \bar{Y}_{jk} V_k^{(t-1)} \right\|}$$

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SDP: A Non-Convex View

$$V^{(t)} = f(V^{(t-1)}) \quad \longleftrightarrow \quad V_j^{(t)} = \frac{\sum_{k \in [n] \setminus \{j\}} A_{jk} \bar{Y}_{jk} V_k^{(t-1)}}{\left\| \sum_{k \in [n] \setminus \{j\}} A_{jk} \bar{Y}_{jk} V_k^{(t-1)} \right\|}$$

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$$\ell(\hat{V}, z^*) = \min_{a \in \mathbb{C}^n: \|a\|^2=1} \frac{1}{n} \sum_{j=1}^n \|\hat{V}_j - \bar{z}_j^* a\|^2$$

Phase Synchronization

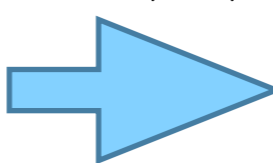
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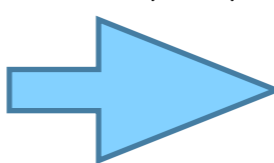
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Corollary. Assume $\sigma^2 = o(np)$ and $p \gg \frac{\log n}{n}$.

The SDP satisfies

$$\ell(\hat{V}, z) \leq (1 + o(1)) \frac{\sigma^2}{2np},$$

$$\frac{1}{n^2} \|\hat{Z} - zz^H\|_F^2 \leq (1 + o(1)) \frac{\sigma^2}{np},$$

$$\ell(\hat{z}, z) \leq (1 + o(1)) \frac{\sigma^2}{2np}$$

with high probability.

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normalized leading eigenvector

$$\frac{1}{n^2} \|\hat{Z} - zz^H\|_F^2 \leq (1 + o(1)) \frac{\sigma^2}{np},$$

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Lower Bound

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**proof of
lower bound**

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$$\inf_{\hat{z} \in \mathbb{C}_1^n} \sup_{z \in \mathbb{C}_1^n} \mathbb{E}_z \min_{\theta \in \mathbb{R}} \sum_{j=1}^n |\hat{z}_j e^{i\theta} - z_j|^2$$

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**proof of
lower bound**


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not separable

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$$\geq \frac{1}{2n} \inf_{\hat{z} \in \mathbb{C}_1^n} \sup_{z \in \mathbb{C}_1^n} \mathbb{E}_z \|\hat{z}\hat{z}^H - zz^H\|_F^2$$

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separable

$$\geq \frac{1}{2n} \inf_{\hat{z}} \sum_{1 \leq j \neq k \leq n} \left(\int \prod_{l=1}^n \pi(z_l) \right) \mathbb{E}_z |\hat{z}_j \bar{\hat{z}}_k - z_j \bar{z}_k|^2 dz$$

Lower Bound

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$$\geq \frac{1}{2n} \sum_{1 \leq j \neq k \leq n} \mathbb{E}_{z_{-(j,k)} \sim \pi} \inf_{\hat{T}} \int \pi(z_j) \pi(z_k) \mathbb{E}_z |\hat{T} - z_j \bar{z}_k|^2 dz_j dz_k$$

Lower Bound

**proof of
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van Tree's inequality

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$$= (1 - o(1)) \frac{\sigma^2}{2p}$$

Z2 Synchronization

Model

$$Y_{jk} = z_j z_k + \sigma W_{jk}$$

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**Parameter
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**Missing
data**

$$A_{jk} \sim \text{Bernoulli}(p)$$

Z2 Synchronization

MLE

$$\max_{z \in \{-1, 1\}^n} z^T (A \circ Y) z$$

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SDP

$$\max_{Z=Z^T \in \mathbb{R}^{n \times n}} \text{Tr}((A \circ Y)Z) \quad \text{subject to} \quad \text{diag}(Z) = I_n \text{ and } Z \succeq 0$$

Z2 Synchronization

Theorem [G & Zhang]. Assume $\sigma^2 = o(np)$ and $\frac{np}{\log n} \rightarrow \infty$. Then,

$$\inf_{\hat{z} \in \{-1,1\}^n} \sup_{z \in \{-1,1\}^n} \mathbb{E}_z \ell(\hat{z}, z) \geq \exp\left(- (1 + o(1)) \frac{np}{2\sigma^2}\right).$$

Moreover, the MLE, GPM initialized by the leading eigenvector of $A \circ Y$, and the leading eigenvector of SDP all achieve

$$\ell(\hat{z}, z) \leq \exp\left(- (1 - o(1)) \frac{np}{2\sigma^2}\right)$$

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“p=1” by [Fei & Chen 20]

Comparison



Comparison

phase sync

$$\begin{aligned} \max \quad & \text{Tr}((A \circ Y)Z) \\ \text{s.t.} \quad & \text{diag}(Z) = I_n \text{ and } Z \succeq 0 \end{aligned}$$

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$$(1 + o(1)) \frac{\sigma^2}{2np}$$

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$$\exp\left(-\left(1 - o(1)\right) \frac{np}{2\sigma^2}\right)$$

Comparison

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**extra term
in complex
space**

Rotation Synchronization

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Model $Y_{ij} = Z_i Z_j^T + \sigma W_{ij} \in \mathbb{R}^{d \times d} \quad Z_i \in \text{SO}(d)$

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Theorem [G & Zhang]. Assume $\sigma^2 = o(np)$ and $\frac{np}{\log n} \rightarrow \infty$. Then,

$$\inf_{\hat{Z} \in \text{SO}(d)^n} \sup_{Z \in \text{SO}(d)^n} \mathbb{E}_n \ell(\hat{Z}, Z) \geq (1 - o(1)) \frac{(d-1)d\sigma^2}{2np}.$$

Moreover, both MLE and GPM achieve

$$\ell(\hat{Z}, Z) \leq (1 + o(1)) \frac{(d-1)d\sigma^2}{2np}$$

with high probability.

Thank You