Linear growth of quantum circuit complexity

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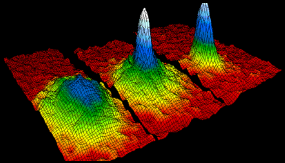
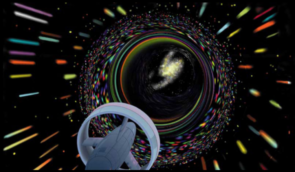
arXiv:2106.05305

Quantum complexity

Quantum (circuit) complexity is a standard concept in quantum information theory. Applications abound:

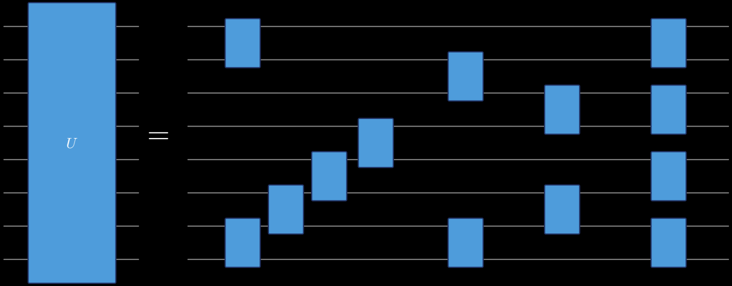
- Which operations are hard and which are easy?
- Classical analogue is one of the most pervasive objects in CS.
- Definition of topological phases of matter.
- Black holes in AdS/CFT.





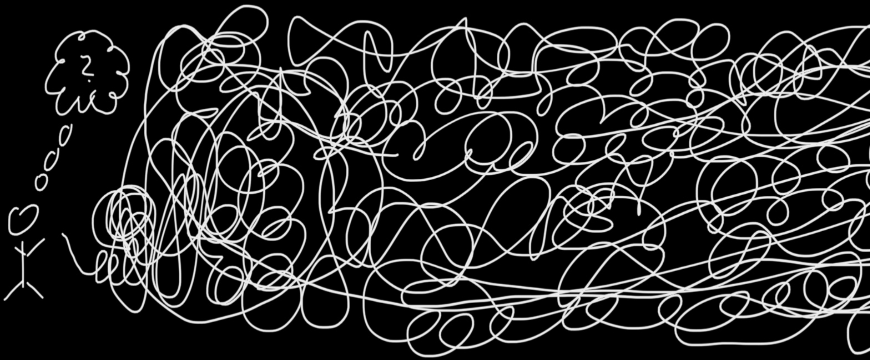
A traditional definition of complexity

How many 2-local gates are necessary to implement a unitary (or a state)?



Denote the number of gates in a minimal decomposition by $\mathcal{C}(U)$.

How complex is this thing I am looking at?



Notoriously hard to compute or even to bound!

Complexity growth: A universal phenomenon?

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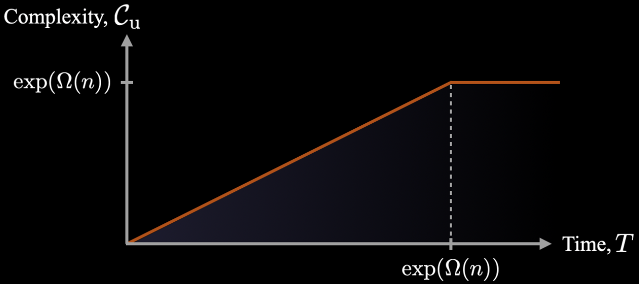
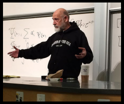


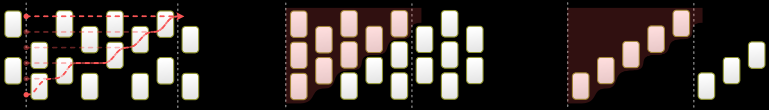


Linear growth beyond saturation of entanglement entropy.

[Brown, Susskind]

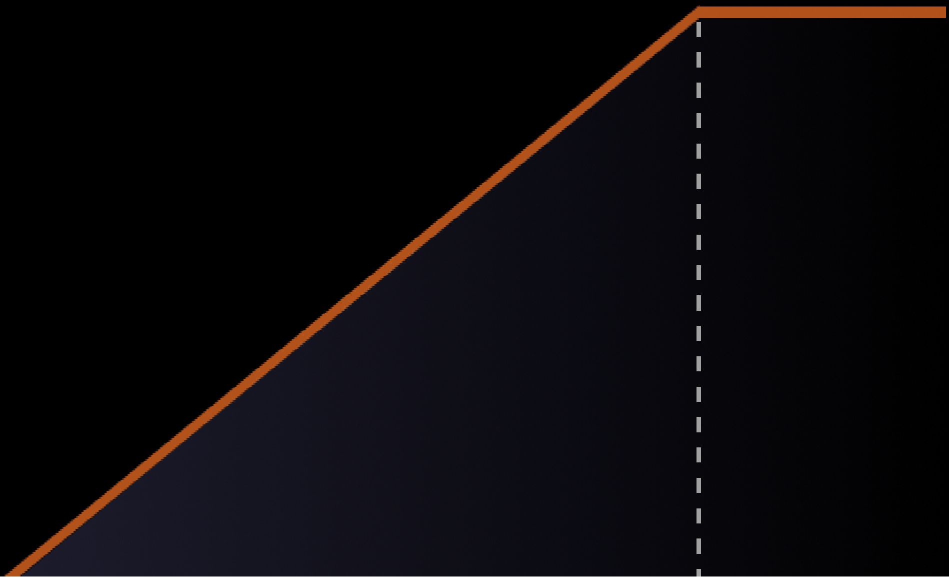
Random quantum circuits: A model for local dynamics

Random quantum circuits: Draw gates iid from the Haar measure on $\mathrm{SU}(4)$ and contract along a fixed arrangement/architecture.



[Brandão, Chemissany, Hunter-Jones, Kueng, Preskill]

Why should Brown & Susskind's conjecture be true?



Why should Brown & Susskind's conjecture be true?

- Linear growth: The unitary group is big and circuits should be expected to generate ever new unitaries.
- Complexity as entanglement beyond entropies. [Nahum, Ruhman, Vijay, Haah]
- Wormhole growth paradox in AdS/CFT.

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- Saturation: Every unitary can be implemented with $O(4^n)$ many gates.
- Most unitaries are very complex: $\mathrm{SU}(2^n)$ is of dimension $4^n - 1$, while a circuit with R gates is described by at most $\dim \mathrm{SU}(4) * R$ parameters.

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Proof idea: Can this argument be refined?

Linear growth

Theorem

U = random quantum circuit in architecture A.

T=# of disjoint backward lightcones in A.

$$\mathcal{C}(U) \geq \frac{T}{9} - \frac{n}{3}\,,$$

with unit probability, until the number of gates grows to $T \geq 4^n - 1$.

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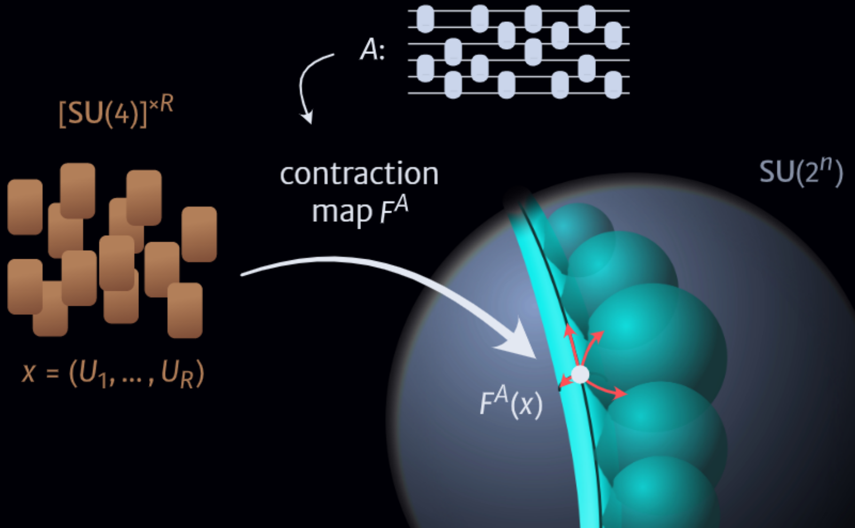
Brickwork circuits:

$$\mathcal{C}(U) \geq \frac{\#\text{ of gates}}{9n^2} - \frac{n}{3} \, .$$

The circuit as a map

We view the contraction as a smooth map with an image $\mathcal{U}(A)$:



A dimension for $\mathcal{U}(A)$

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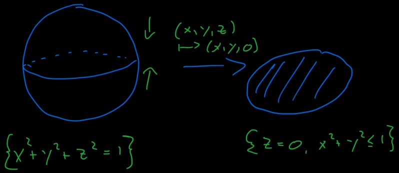
- Tarski-Seidenberg principle: The image of a polynomial map is semialgebraic.
- A semialgebraic set is the solution to a set of polynomial equations and inequalities.
- A finite union of manifolds. $\Longrightarrow$ Highest dimension in this decomposition.



Comparing dimensions works!

Using tools from differential topology and algebraic geometry:

Lemma

If $\dim \mathcal{U}(A') < \dim \mathcal{U}(A)$, *then* $\mathcal{U}(A')$ *has probability* 0 *for random quantum circuits in architecture* A.

- Every unitary with circuit complexity $\leq R'$ in some $\mathcal{U}(A')$ with $\#\text{gates}(A') \leq R'$.
- $\dim \mathcal{U}(A') \leq O(R')$.
- Suffices to lower bound $\dim \mathcal{U}(A)$.

The dimension as a proxy for complexity

$\mathcal{C}(U)$

$\dim \mathcal{U}(A)$

Dimensions and the rank

- $\dim \mathcal{U}(A)$ equals the maximal rank of F^A.

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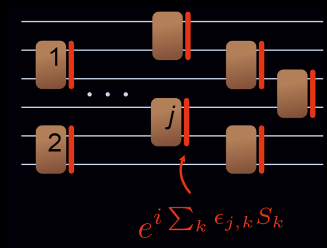
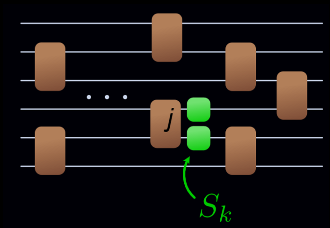
- $\dim \mathcal{U}(A)$ equals the maximal rank of F^A.
- It suffices to find a single circuit such that the Jacobian of F^A has high rank.
- $\text{Jacobian}_x(F^A) = \begin{pmatrix} \partial_1 F^A|_x & \partial_2 F^A|_x & \dots & \partial_{\dim \mathrm{SU}(4)^R} F^A|_x \end{pmatrix}$
- rank at x = rank of $\text{Jacobian}_x(F^A)$

Dimensions and the rank

Partial derivatives of F^A:





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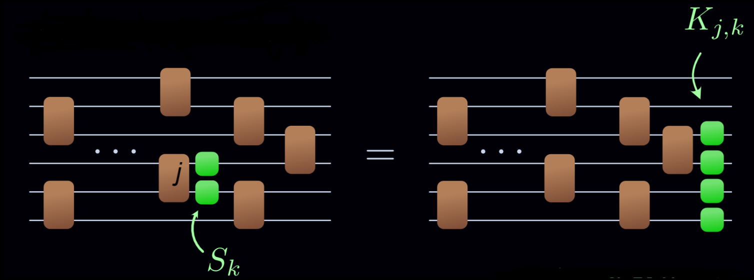
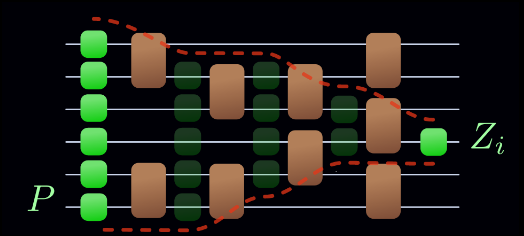


An inductively defined Clifford circuit

Every Pauli operator can be reduced to Z in a causal slice:



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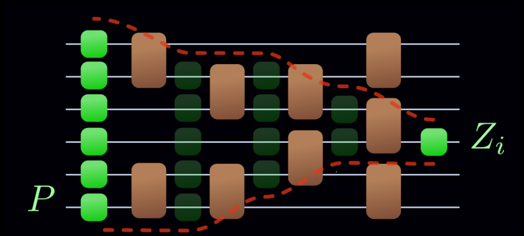
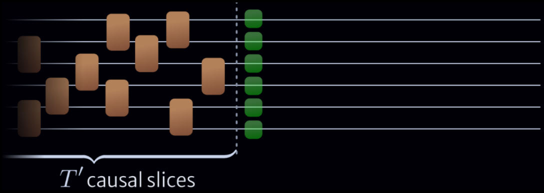


Choosing Pauli strings and Clifford circuits inductively:



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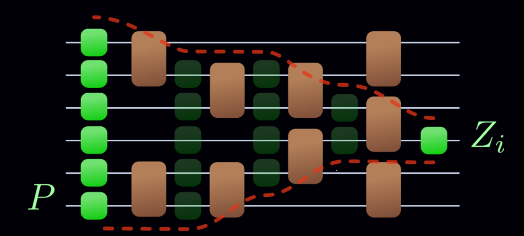

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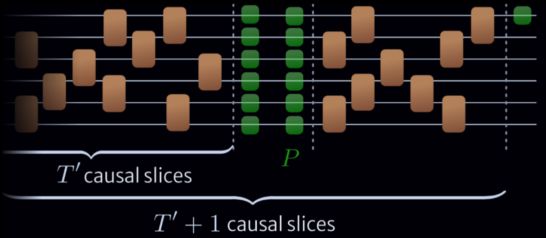

Open problems

More operational version of the result?

- Partial result: Error tolerance for uncontrollably small error.
- How "winded" is $\mathcal{U}(A')$ in $\mathcal{U}(A)$?
- Distinguishability from the completely depolarizing channel: t-designs in depth $O(nt)$? [Brandão, Chemissany, Hunter-Jones, Kueng, Preskill][Brandão, Harrow, Horodecki][Hunter-Jones]

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Time evolution of time independent Hamiltonians? Thermofield double state?

- Lots of evidence:[Brown, Susskind], [Aaronson, Susskind], [Brandão Bohdanowicz],[Balasubramanian et.al.][Susskind, Stanford]
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