Coordination of for Large Collections of Dynamical Systems with Constraint Satisfaction Guarantees

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Research partly funded by


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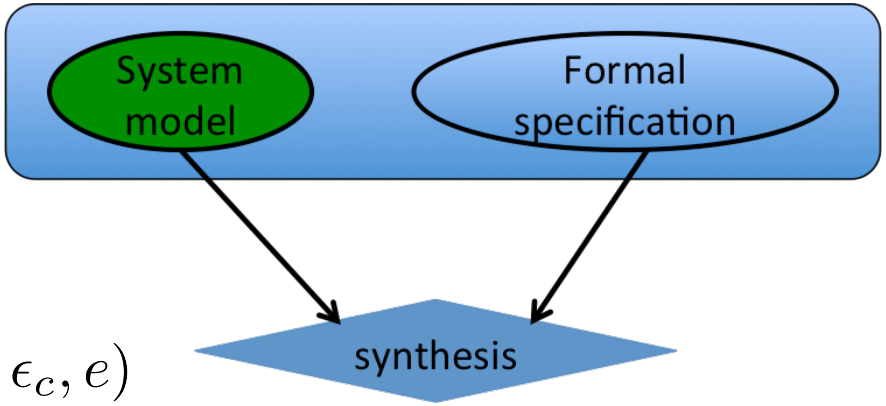
System models



Differential equations (continuous-time):

$$\dot{x} = f(x, u_c, u_d, \epsilon_c, e)$$

Or, difference equations (discrete-time):

$$x(k+1) = f(x(k), u_c(k), u_d(k), \epsilon_c(k), e(k))$$

$$\begin{aligned} x \in \mathcal{X} &: \text{ state} \\ u_c \in \mathcal{U}_c &: \text{ continuous control input} \\ u_d \in \mathcal{U}_d &: \text{ discrete control input} \\ \epsilon_c \in \mathcal{D}_c &: \text{ disturbance input} \\ e \in \mathcal{D}_d &: \text{ discrete uncontrollable input} \end{aligned}$$

$$\boxed{\mathcal{X} \subset \mathbb{R}^N}$$

Some characteristics:

- Hard constraints (on input and states)
- Infinite horizon specifications
- Hybrid (either the system or the controller or both)
- Robust/reactive

Landscape of current methods*

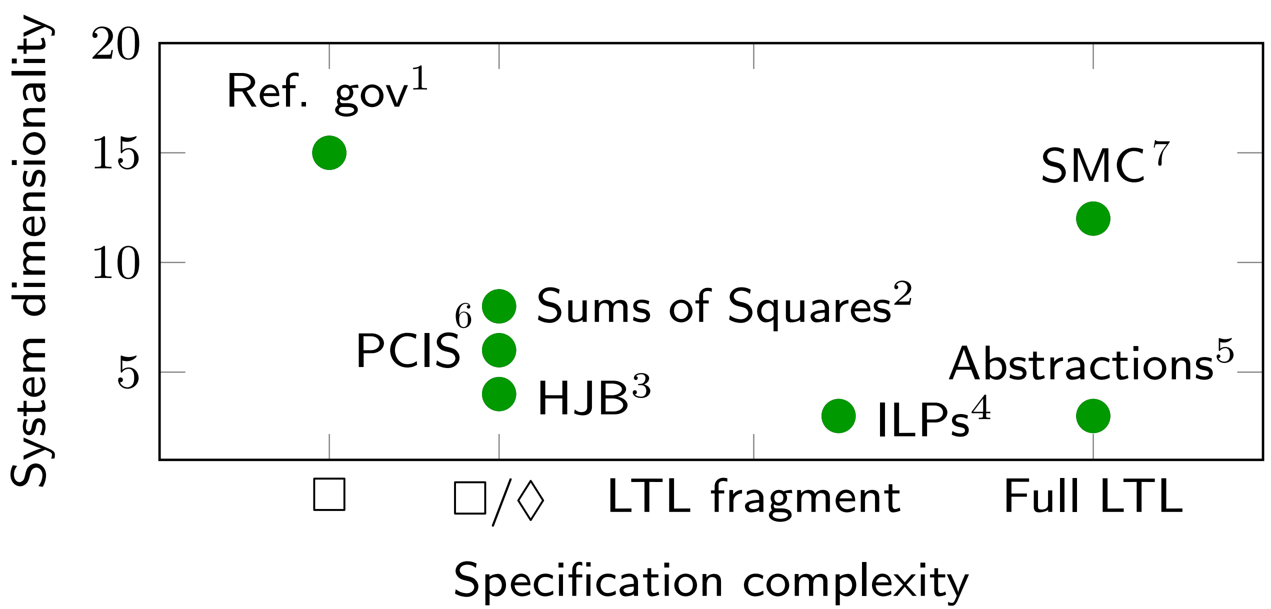
Many factors affecting scalability:

- State-space dimension
- Complexity of the dynamics
- Complexity of the specifications
- Strength of conclusions (complete vs. sound)
- Accuracy of the results (correct vs. approximate)
- Ability to handle uncertainty, non-determinism, (open-loop vs. closed-loop)
- etc.

* **disclaimer:** as any categorization, this is incomplete and inaccurate when done wrt few factors…

State-of-the-art in formal methods in control (incomplete list!)

- Hard state/input constraints, hybrid dynamics, complex specifications (e.g., temporal logics)
  - Belta, Dimarogonas, Fainekos, Girard, Liu, Pappas, Tabuada, Tumova, Wongpironsarn, Zamani…
- Applications (with “small” state-space dim.)
  - Robotics, building thermal management, adaptive cruise control, aircraft subsystems, traffic control



1 - Reference governors: Gilbert et al., CDC’94
2 - Korda et al., SIAM C&O’14
3 - Hamilton Jacobi Bellman: Tomlin et al., HSCCC’98
4 - (Mixed) integer linear programs, Wolff et al., ICRA’14
5 - Tools such as Pessoa, Tulip, SCOTS
6 - Polytopic controlled invariance: Bertsekas TAC’72
7 - Satisfiability modulo convex optimization: Shoukry et al., HSCC’17

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- “Medium”-scale systems
  - Monotonicity (Hafner & Del Vecchio 11, Coogan & Arcak 15)
  - Multi-scale abstractions for safety (Girard et al. 13)
  - Compositional synthesis (Nilsson & Ozay 16, Chen et al. 16, Kim et al. 15), incremental abstractions (Nilsson & Ozay 15)

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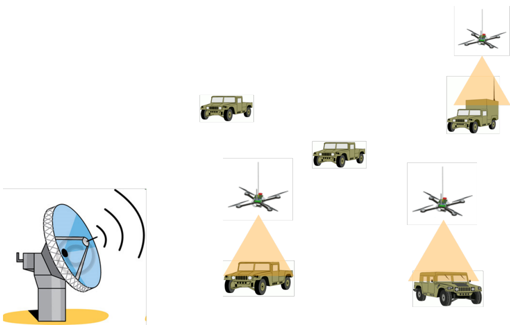
- "Large"-scale (but not synthesis)
  - Parametric verification of rectangular hybrid automata (Johnson & Mitra 12)
  - Abstractions of large collections of stochastic systems (Soudjani & Abate 15)

Recurring theme: structural properties

Large collections of systems

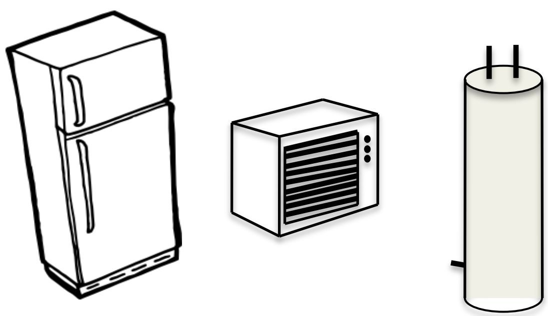
Example 1: Emergency response with a robotic swarm



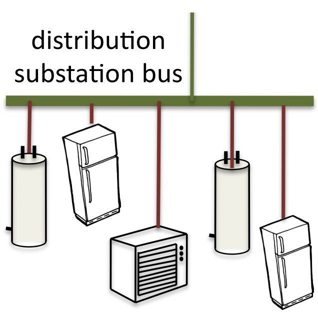
- Deploy a large collection of robots (e.g., quadrotors, ground vehicles) for search and rescue mission
- Plan trajectories by taking dynamic constraints into account
- Requirements:
  - <u>Sufficiently many</u> robots in certain areas at any given time
  - <u>Not too many</u> robots in certain regions (danger zones)
  - Collision avoidance
  - Charging/reporting constraints

Large collections of systems

Example 2: Coordination of thermostatically controlled loads (TCLs)



TCLs

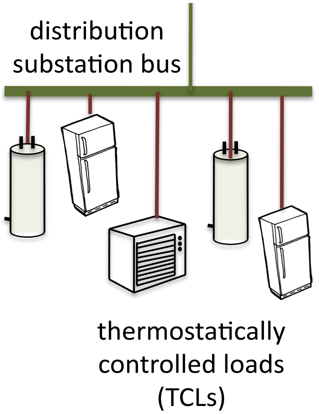


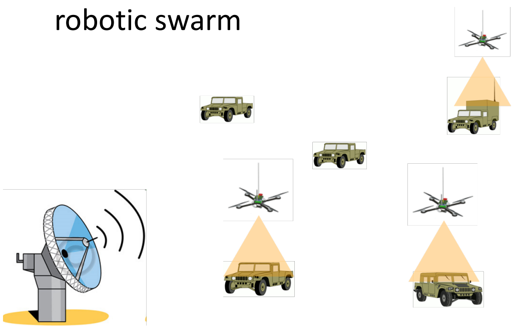
thermostatically
controlled loads
(TCLs)

- Thermostatically controlled loads (e.g., refrigerators, air conditioners, water heaters) for demand response
- Thermal dynamics can be controlled via ON/OFF switches
- Requirements:
  - Not too many TCLs ON at the same time (to avoid line overload)
  - Sufficiently many ON all the time (to utilize renewable energy)
  - Local temperature constraints (never out of desired temperature range)

Mathieu, Koch, Callaway, IEEE Trans. on Power Systems

Common structural properties



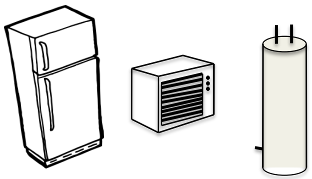


- Large number of systems, small number of classes
- Counting constraints: “how many in each mode?”, “how many in what region?”
- Identity of individual systems is not important

For simplicity, assume:

- dynamics are identical within each class
- (wlog) there is only one class

Mathematical formulation: TCLs



TCLs

The temperature θ in a room with a TCL has dynamics

$$\dot{\theta}_i = \begin{cases} f_{on}(\theta_i), & \text{if TCL is on} \\ f_{off}(\theta_i), & \text{if TCL is off} \end{cases}$$

Suppose we have a collection of rooms with TCL's $\{\theta_i\}_{i \in [N]}$.

- Customers: Want room temperature to be close to a desired temperature θ_i^{des}, but small deviations are allowed.

$$\|\theta_i - \theta_i^{des}\| \leq \Delta \tag{1}$$

- Utility company: Wants to control aggregate demand, i.e. the number of TCLs that are on

$$\sum_{i=1}^{N} \mathbb{1}_{\{\text{TCL } i \text{ is on}\}} \tag{2}$$

Goal: Find a switching (i.e., on/off) strategy that exploits the flexibility in (1) so that (2) can be controlled.

Mathematical formulation: General

- N identical switched system with M modes:

$$\dot{x}_i(t) = f_{\sigma_i(t)}(x_i(t)), \quad \sigma_i : \mathbb{R} \mapsto [M],$$

- Mode-specific unsafe sets: $\mathcal{U}_m$, $m \in [M]$
  - Equivalent to forced mode switches.
- Mode-counting bounds:

$$\underline{K}_m \leq \sum_{i=1}^{N} \mathbb{1}_m(\sigma_i(t)) \leq \overline{K}_m \tag{3}$$

Want to synthesize a switching strategy σ_i such that (3) satisfied over time.

Structural property: both the dynamics and the specification (counting constraints) are permutation invariant!

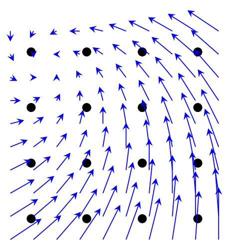
Solution overview

- Construct symbolic abstractions and aggregate dynamics and define “equivalent” problems on these structures
- (Analyze abstractions to understand fundamental limitations if any)
- An optimization-based solution approach
- Analysis of the solution approach

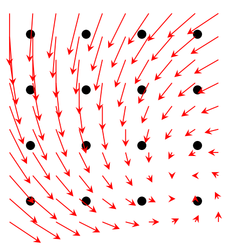
Solution overview

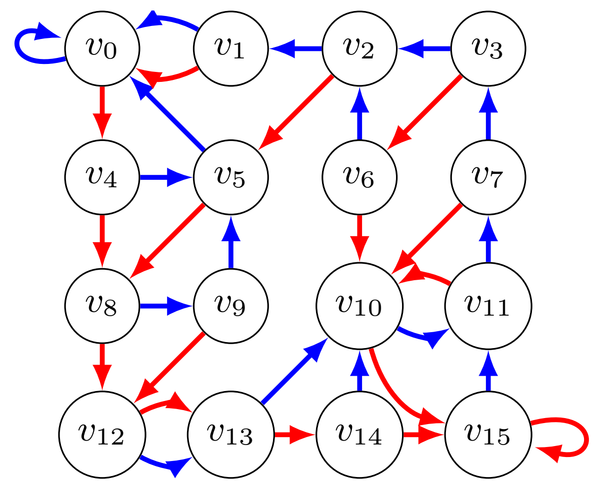
- Construct symbolic abstractions (i.e., a finite transition system)
  - ε-approximate bisimilar abstraction

$$\dot{x} = f(x, u_1)$$



$$\dot{x} = f(x, u_2)$$





  - for each path on the finite transition system, there is a piecewise constant input that generates a trajectory such that time-sampled trajectory remains ε-close to the discrete states

Abstraction of individual dynamics

- Assume dynamics are δ-GAS with $\mathcal{KL}$ functions β_m

$$\|\phi_t^m(x) - \phi_t^m(y)\|_\infty \leq \beta_m\left(\|x - y\|_\infty, t\right). \tag{4}$$

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- With discretization in time (τ) and space (η), an ϵ-approximate bisimilar model is obtained if $\beta_m(\epsilon, \tau) + \frac{\eta}{2} \leq \epsilon$.

Pola, Girard, Tabuada 08

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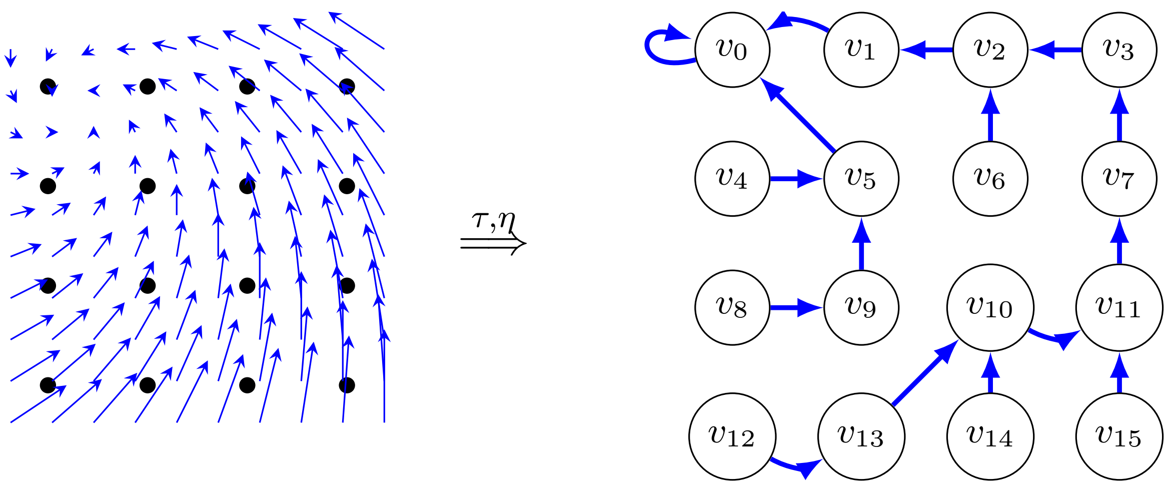
  - Mode 1 abstraction



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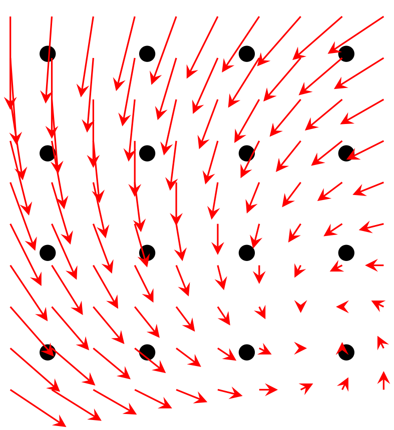
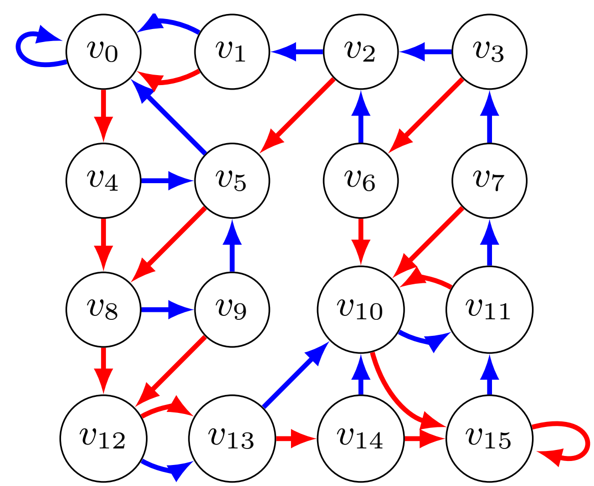
    - Mode 2 abstraction



$$\xRightarrow{\tau,\eta}$$



mode-transition graph $G = (V, E)$

Some observations

- For a homogeneous collection, each system will have an identical mode-transition graph
- Transition graphs are deterministic

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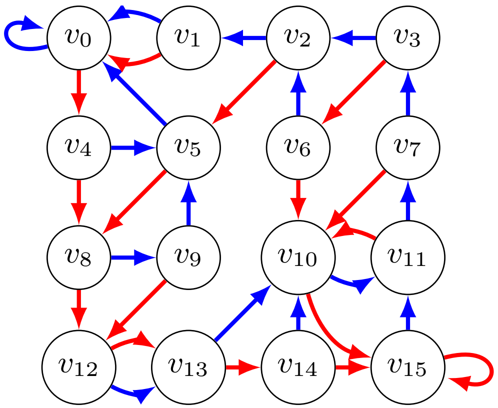
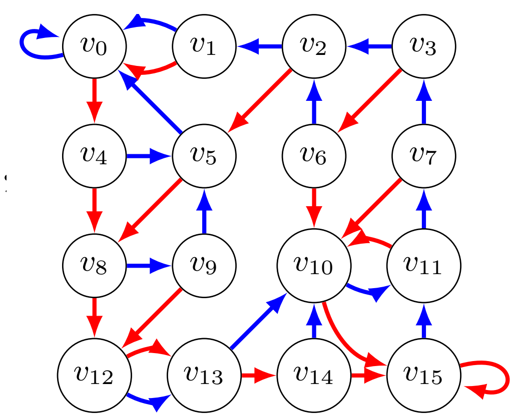


Some observations

- For a homogeneous collection, each system will have an identical mode-transition graph
- Transition graphs are deterministic
- Consider mild heterogeneity

mode-transition graph $G = (V, E)$



$$\dot{x}_i(t) = f_{\sigma_i(t)}(x_i(t), d_i(t)), \quad \sigma_i : \mathbb{R} \mapsto [M],$$

where $d_i \in \mathcal{D}$ (bounded parametric uncertainty or disturbance). If $f_m(x,d)$ is L_m-Lipschitz in x, and

$$||f_m(x,d) - f_m(x,0)|| \leq \delta_m \quad \text{for all } \; d_i \in \mathcal{D},$$

then, with discretization in time (τ) and space (η), an ϵ-approximate bisimilar model is obtained if $\beta_m(\epsilon, \tau) + \frac{\delta_m}{L_m}(e^{L_m \tau} - 1) + \frac{\eta}{2} \leq \epsilon$.

Aggregate dynamics on graph

Let $V = \{v_1, \ldots v_K\}$ denote the nodes of mode-transition graph $G = (V, E)$. Introduce the states $w_k^{m_1}$ and $r_k^{m_1,m_2}$.

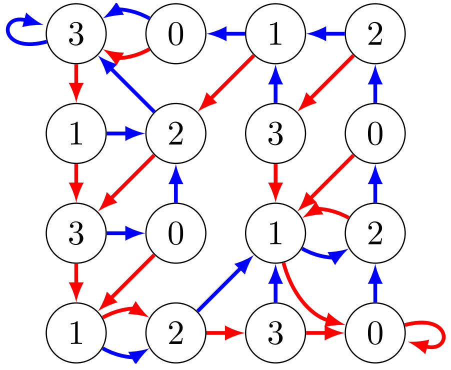
- w_i^m represents **number of systems in mode** m **at** v_k.
- $r_k^{m_1,m_2}$ represents **number of systems at** v_k **that switch from** m_1 **to** m_2**.**

- The dynamics become

$$(w_k^{m_1})^+ = \sum_{j \in \mathcal{N}_k^{m_1}} \left(w_j^{m_1} + \sum_{m_2} r_j^{m_2,m_1} - r_j^{m_1,m_2} \right),$$

- Constrained control actions:

$$0 \leq \sum_{m_2} r_k^{m_1,m_2} \leq w_k^{m_1},$$



- Compact description: $\mathbf{w}^+ = A\mathbf{w} + B\mathbf{r}$

Equivalent problem on aggregate dynamics

Theorem 1:

Consider aggregate dynamics $\Sigma_G : \mathbf{w}^+ = A\mathbf{w} + B\mathbf{r}$ with safety and mode-counting constraints:

$$w_k^m(t) = 0 \quad \forall k \in U_m, \tag{5}$$

$$\underline{K}_m \leq \sum_{i \in [N]} w_i^m(t) \leq \overline{K}_m. \tag{6}$$

Then,

- if $\exists$ sequence of control inputs $\mathbf{r}^\omega$ for Σ_G that enforce (5) and (6) with $U_m + B_\epsilon$, then $\exists$ a solution to the original problem.
- if $\nexists$ a sequence of control input $\mathbf{r}^\omega$ for Σ_G that enforces (5) and (6) with $U_m - B_\epsilon$, then no solution to the original problem.

We will focus on aggregate dynamics. We need infinite horizon strategies!

Solution strategy: from a given initial state, steer the system, while respecting the constraints, to a **nice state** from which a periodic input suffices.

Controllability-like conditions

Solution strategy: **from** a given **initial state**, **steer the system**, while respecting the constraints, **to** a **nice state** from which a periodic input suffices.

- Let's put the mode-counting constraints aside.
- Are there any fundamental limitations on what states can be reached from an initial condition?

$$\Sigma_G : \mathbf{w}^+ = A\mathbf{w} + B\mathbf{r}$$

with local safety and input constraints

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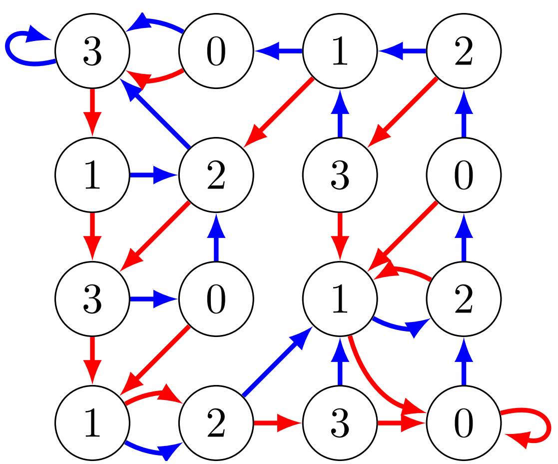
Definition: The period n of a strongly connected graph is the greatest common divisor of the lengths of its cycles.

Theorem 2: If the connected components of mode-transition graph has period n=1, any state is reachable from any other state (within the connected component). If n>1, then the reachable states live on a affine subspace arrangement with n affine subspaces.

Solution strategy

Solution strategy: from a given initial state, steer the system, while respecting the constraints, to a **nice state** from which a periodic input suffices.

- **Prefix:** for a fixed horizon T, given initial state, we will steer the state at time T to "**nice**" cycles
- **Suffix:** let individual systems circulate in the cycles



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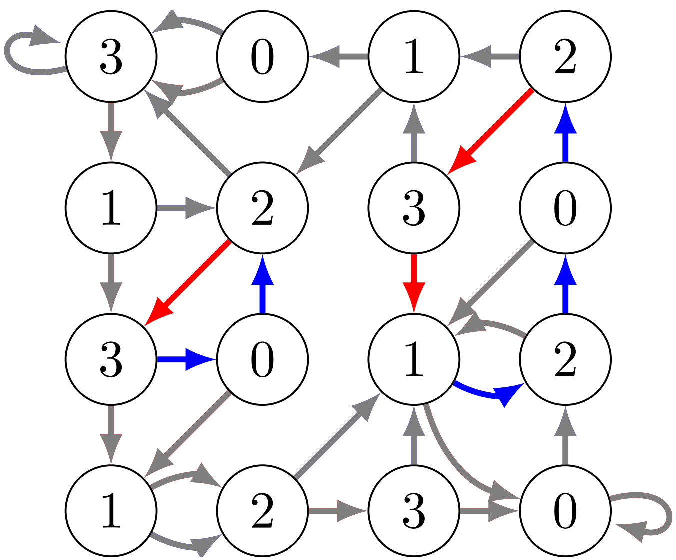


Mode-counting constraints

$$\underline{\Psi}^m(C,\alpha) \geq \underline{K}_m, \quad \overline{\Psi}^m(C,\alpha) \leq \overline{K}_m,$$

can be represented as linear constraints

$$\underline{K}_m\mathbf{1} \leq Y_C^m\alpha \leq \overline{K}_m\mathbf{1}$$

Y_c^m is a circulant matrix.

Solution via linear programming

For cycles $C_1, \ldots, C_J$, required mode-counts K_m, horizon T

find $\alpha_1, \ldots, \alpha_J$ cycle assignments,

$\mathbf{r}(0), \ldots, \mathbf{r}(T-1)$,

$\mathbf{w}(0), \ldots, \mathbf{w}(T)$,

s.t. $\underline{K}_m \le \sum w_i^m(t) \le \overline{K}_m, \quad 0 \le t \le T-1$ mode-counting during prefix

$\underline{K}_m$ ode-counting during suffix

$\sum_j$ oundary conditions between

$\Lambda($ efix and suffix

Feasibility problem with linear constraints:

- integrality constraints on the inputs (ILP)
- relaxing integrality (LP)

Number of constraints and variables are independent of the number of systems N!

$\mathbf{w}(t+1) = A\mathbf{w}(t) + B\mathbf{r}(t), \quad t = 0, \ldots, T-1,$ system dynamics

$\Lambda(\mathbf{w}(0)) = \lambda_0,$

$\sum_{m_2} r_j^{m_1,m_2} = w_j^{m_1}$ for all $j \in \bigcup_{i \in U_{m_1}} \mathcal{N}_i^{m_1},$

local safety constraints

$r_j^{m_2,m_1} = 0$ for all $m_2 \in [M], j \in U_{m_1},$

control constraints.

Analysis

- **Integer solutions (ILP)**
  - **Completeness of prefix-suffix solutions:** There exists a finite T and some maximal cycle length L such that ILP with all cycles with length less than L provides a complete solution to the original problem
  - From any feasible ILP solution, we can extract a solution to the original problem
- **Non-integer solutions (LP):**
  - Enough to consider simple cycles
  - Gives certificates for non-existence of solutions
- **Rounding a non-integer solution:**
  - A non-integer solution over the cycles can be rounded to an integer feasible solution with mode counting loss at most

$$\underline{\Psi}^m(C, \alpha_{int}) \leq \underline{\Psi}^m(C, \alpha_{avg}) + \frac{|C|}{4}$$

Intuition behind cycles: TCLs

$$\dot{\theta}_i = -a(\theta_i - \theta_a) - bP_m$$

θ :room temperature

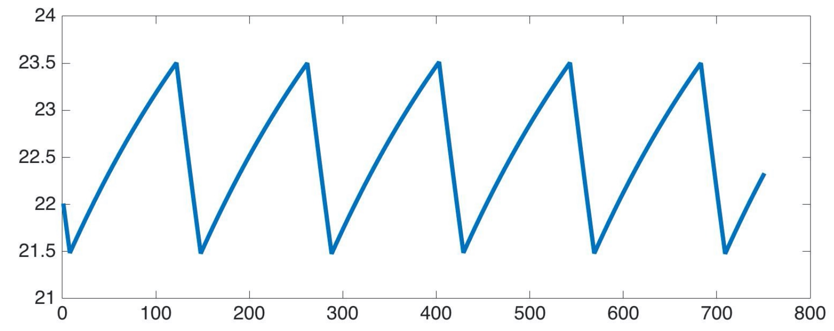
θ_a :ambient temperature

$P_m = 0$ when OFF

$P_m = 5.6$ when ON

local safety
$\theta_i \in [21.5, 23.5]$

For an individual system if only local ON/OFF control is used (no demand response for extra switching), the temperature evolves as follows:



Parameters from Mathieu, Koch, Callaway, IEEE Trans. on Power Systems, 2013

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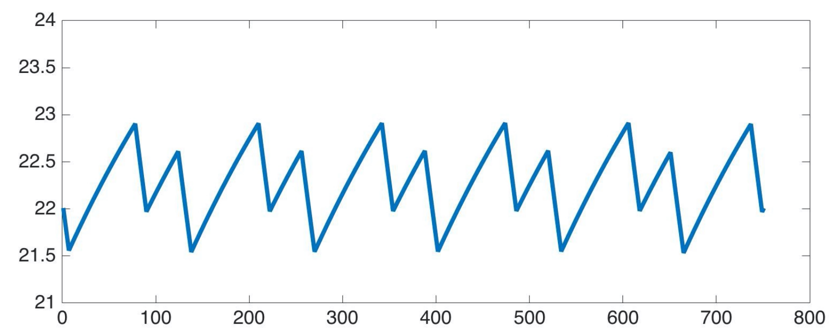

Roughly, cycles are defining new "bands" within the dead-band allowed by the local safety constraints. That is, we are changing the duty cycle.

Parameters from Mathieu, Koch, Callaway, IEEE Trans. on Power Systems, 2013

Results on TCLs

N = 10000 units

10000-D state-space with 2^{10000} modes!

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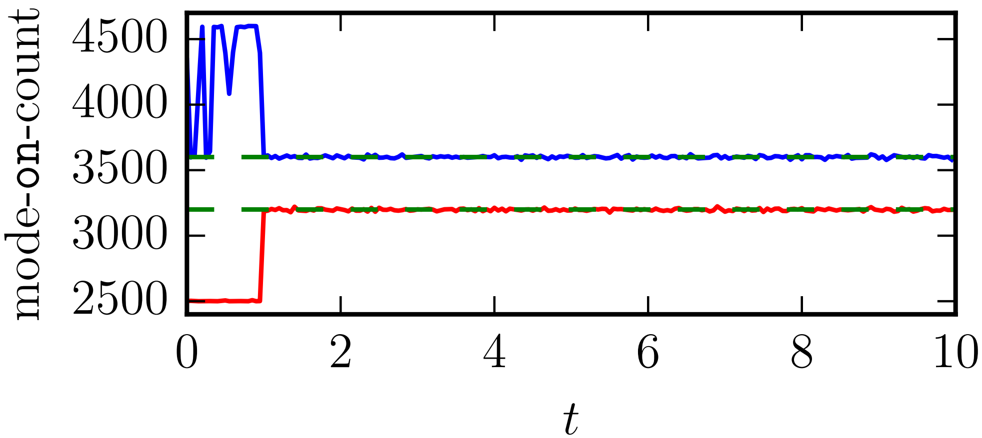
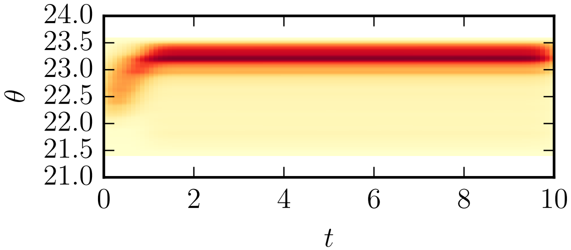
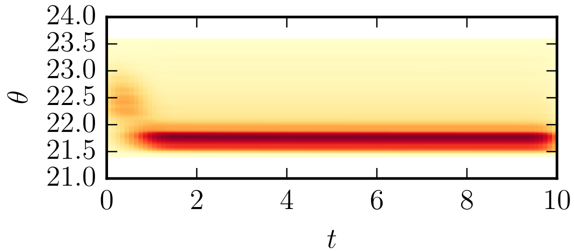
Two different runs with different mode-counting constraints (also stricter constraints at the suffix)



Lower mode-count:



Higher mode-count:



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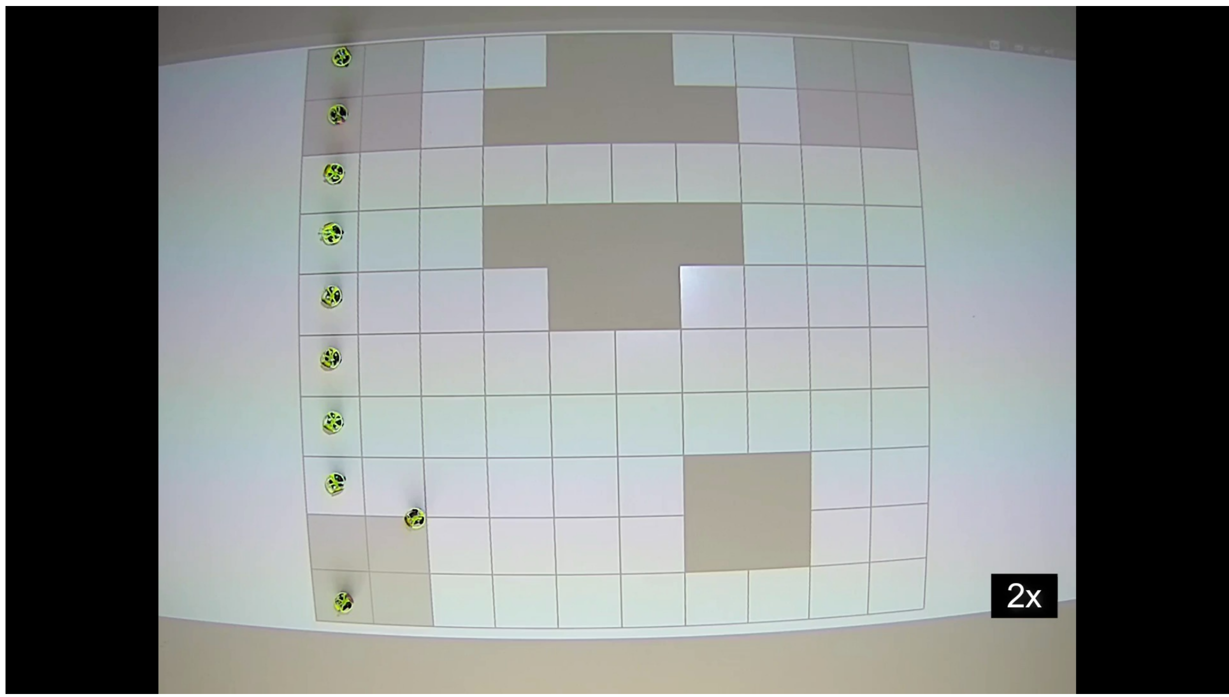
Beyond mode counting

- Counting the agents in a region of state-space
- Time-evolution of counting constraints (counting LTL)
- Possible to encode asynchrony as well

$$\varphi ::= \mathit{True} \mid cp \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi_1 \, \mathcal{U} \, \varphi_2,$$

cLTL : cp = [atom prop., count]

cLTL+ : tcp = [LTL formula, count]



With Yunus Emre Sahin & Petter Nilsson ICCPS17, CDC17, TRO20

Summary: structure for scalability

- A control synthesis method for large collections of systems with **counting constraints**
  - ex| dy
  - wc
  - wi
  - ex

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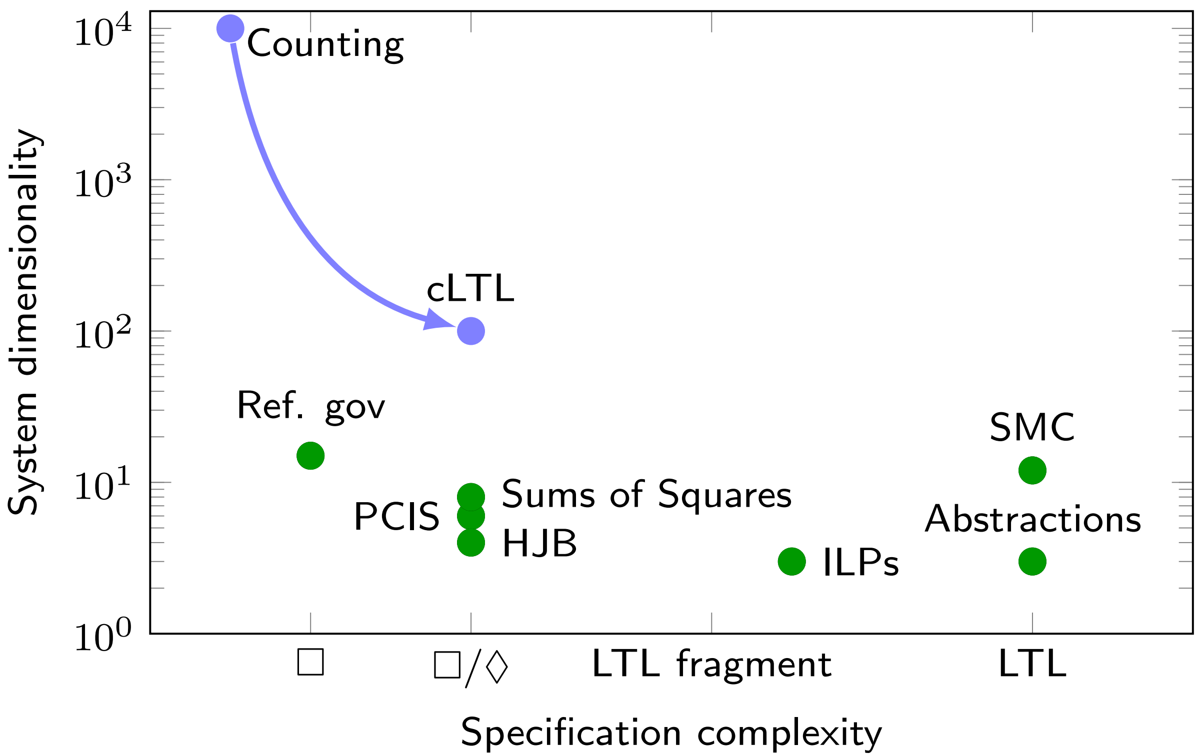

Preprints and more information available @ http://web.eecs.umich.edu/~necmiye/

Summary: structure for scalability

- A control synthesis method for large collections of systems with **counting constraints**
  - exploits the symmetry (permutation invariance) in the dynamics and in specifications
  - works across scales (10 to 10K or more systems)
  - with potential applications in different domains
  - extensions to asynchrony, counting temporal logic
- Current work
  - partial information
  - non-deterministic abstractions (for not incrementally stable systems), asynchronous switching
  - other types of symmetries that can be exploited
  - other approaches for scalability: decomposition, contracts

Preprints and more information available @ http://web.eecs.umich.edu/~necmiye/

References

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