TAUT, TFNP and SAT

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Fundamental theories

$T_2 \subset I\Sigma_1 \subset PA \subset SOA \subset ZFC \subset$

ZFC+strongly inaccessible cardinal $\subset$ ZFC+measurable cardinal ...

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Related question:
Is the hierarchy of subtheories $T_2^1 \subseteq T_2^1 \subseteq \ldots$ of Bounded Arithmetic strictly increasing?

Example: proof complexity of algorithms

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If FACTORING is hard, then S_2^1 does not prove the soundness of any polynomial time algorithm for PRIMALITY.

We are not able to prove the soundness of AKS algorithm in any fragment of T_2.

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- Can we formalize every algorithm so that its soundness is provable in PA (or some other fixed theory)?

Syntactic and semantic classes

The rule of thumb:

- syntactic classes have complete problems
- semantic classes do not

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The reason why a semantic class $\mathcal{C}$ does not have complete problems is:

1. we need a proof of the defining condition to show $P \in \mathcal{C}$,
2. there is no single theory T that is able to prove it for all $P \in \mathcal{C}$.

TAUT

TAUT$=_{df}$ DNF tautologies

Proof systems

1. complete (can always be made syntactic)
2. sound (semantic)

Polynomial simulations

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Conjecture (TAUT conjecture)

Equivalent formulations

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Proposition

TAUT *conjecture* $\rightarrow$ EXP$\neq$NEXP.

DisjNP

DisjNP$=_{df}$ $\{(A, B) \mid A, B \in \mathrm{NP} \wedge A \cap B = \emptyset\}$

Polynomial reductions (Turing or many-one)

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1. *There is no complete disjoint* **NP** *pair.*
2. *There is no consistent theory that proves the disjointness of all disjoint* **NP** *pairs.*

The canonical pair of a proof system P

Definition (R'94)

$A_P = \{(\phi, 0^n) |\ \phi \in CNF \wedge \exists\ P$-refutation of ϕ of length $\leq n\}$;

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Fact

If P simulates Q, then (A_Q, SAT^*) *is reducible to* (A_P, SAT^*).

Corollary (KMT'03)

DisjNP *conjecture* $\Rightarrow$ TAUT *conjecture.*

TFNP

TFNP = Total Function NP

Definition

A TFNP problem is given by a binary relation R in **P** and a polynomial bound r such that

$$\forall a \exists b \; |b| \leq r(|a|) \wedge R(a, b).$$

The task is, for a given a, to find b such that $|b| \leq r(|a|) \wedge R(a, b)$.

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Reduction R to R'

- many-one: $R'(f(a), b) \rightarrow\ R(a, g(a, b))$,
- or Turing: $R(a, g^{\text{oracle}R'}(a))$

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Evidence?

- The set of provably total computable functions increases with the strength of the theories.
- The well-known characterizations of provably total TFNP problems in fragments of bounded arithmetic suggest that these sets also increase.

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"X" means: "X does not have a complete problem"

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Where is SAT?

SAT

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But

- the standard proof system for SAT = satisfying assignments
- the standard proof system is polynomially bounded
- yet, some proof systems for SAT are not polynomially bounded

Example

Define a proof system $P^{FACTORING}$ *for* SAT *by defining a proof of* $\phi(\bar{x})$ *to be either*

1. *a satisfying assignment* $\bar{a}$*, or*
2. n *if* n *is a non-prime and* $\phi(\bar{x})$ *expresses the fact that* $\bar{x}$ *is a proper divisor of* n*.*

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Fact

If FACTORING *is hard, then the standard proof system does not polynomially simulate this system.*

Some natural proof systems for SAT

Observation

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- G is a sequent calculus for quantified propositional tautologies.
- G_i is G restricted to Σ_i^q sequents.
- G_i^* is the tree-like version of G_i.
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Theorem (witnessing for G_1^*, Cook 2002)

Given a G_1^*-proof of $\exists\bar{y}.\phi(\bar{x},\bar{y})$ and an assignment $\bar{x} := \bar{a}$, one can construct in polynomial time $\bar{b}$ such that $\phi(\bar{a},\bar{b})$ is true.

Proof of Proposition.

Given a proof of $\exists\bar{y}.\phi(\bar{y})$ we get in polynomial time $\bar{b}$ that satisfies $\phi(\bar{y})$. $\square$

Theorem

If there is an optimal proof system for SAT*, then there exists a complete problem in* TFNP*.*

Proof.

Given a TFNP problem R, we define a proof system P^R for SAT:

- same construction as with NONPRIME, i.e., a is a proof of satisfiability of $R(a, y)$.

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Given a proof system Q for SAT, define a TFNP problem R^P:

- $R^Q(x, y)$ iff
    1. $x = (\phi, v)$, v is a Q-proof of ϕ, and y is a satisfying assignment for ϕ;
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Soundness of P implies that R^P is total.

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- If P^R is reducible to Q, then R is reducible to R^Q.

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Soundness of P implies that R^P is total.

- If P^R is reducible to Q, then R is reducible to R^Q.

Hence if Q is an optimal proof system for SAT, then R^Q is complete in TFNP.

□

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If b satisfies $R^Q((\phi, v), b)$, then b satisfies ϕ, hence it is a proper divisor of n.

Hence we can compute a proper divisor of n using an oracle for solutions of R^Q.

Conjecture (SAT conjecture)

SAT *does not have an optimal proof system.*

Corollary

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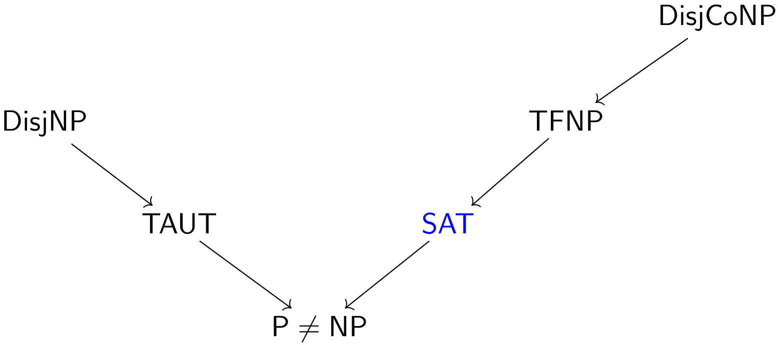

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Relativizations

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- DisjCoNP $\not\Rightarrow$ TAUT [Khaniki'19]
- DisjNP $\not\Rightarrow$ SAT [Dose'20]

Thank You