Homework for "Solving Polynomials"

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1. Use the resultant (eliminate x) to count the number of real solutions for the following systems:

   - $f(x,y) = \begin{bmatrix} 2xy + 3x + 4y - 1 \\ y^2 + 5x + 2y - 3 \end{bmatrix}$;
   - $f(x,y) = \begin{bmatrix} 5y^2 - 2xy + 3x + 4y - 1 \\ y^2 + 5x + 2y - 3 \end{bmatrix}$.

   (For a cubic polynomial $g(z) = az^3 + bz^2 + cz + d$, one option is the use the cubic discriminant
   $$\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$
   Recall that $g = 0$ has three real solutions if $\Delta > 0$ and one real solution if $\Delta < 0$.)

2. Compute the 2-homogeneous Bézout count for eigenvalue-eigenvector problem, i.e., for $A \in \mathbb{C}^{n\times n}$:
   $$f(x,\lambda) = Ax - \lambda x$$
   where $x \in \mathbb{P}^{n-1}$ and $\lambda \in \mathbb{C}$.

3. Describe the irreducible components of the 2×2 adjacent minors of a 3×3 matrix. That is, for
   $$A = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix},$$
   we want to solve the system
   $$f(x_1,\ldots,x_9) = \begin{bmatrix} x_1x_5 - x_2x_4 \\ x_2x_6 - x_3x_5 \\ x_4x_8 - x_5x_7 \\ x_5x_9 - x_6x_8 \end{bmatrix}.$$
   (These are the four minors dependent on x_5.)

4. Compare linear and quadratic convergence by performing 5 Newton iterations with $x_0 = 2$ for

   - $f(x) = (x-1)^2$;
   - $f(x) = x^2 - 1$.

5. Test the divergence of Newton's method for Griewank-Osborne's system
   $$f(x,y) = \begin{bmatrix} 29/16 \cdot x^3 - 2xy \\ y - x^2 \end{bmatrix}$$
   by starting at several points of your choice in $\mathbb{C}^2 \setminus \{x = 0\}$.