Automata on Infinite Words

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$L \subseteq \Sigma^*$

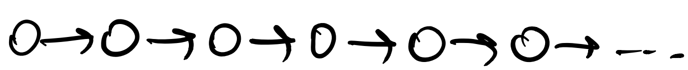
$L_1 = \{ w : |w| \text{ is even} \}$

$L_2 = \{ w : w \text{ contains } 001 \}$

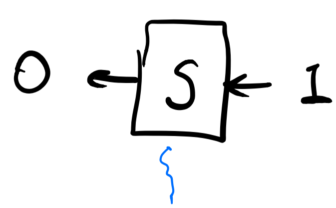
$L \subseteq \Sigma^\omega$

Why infinite words?

1962 The natural numbers



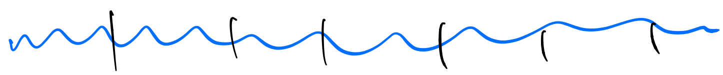
80s On-going behavior of non-terminating systems



× the system finds gcd(x,y)

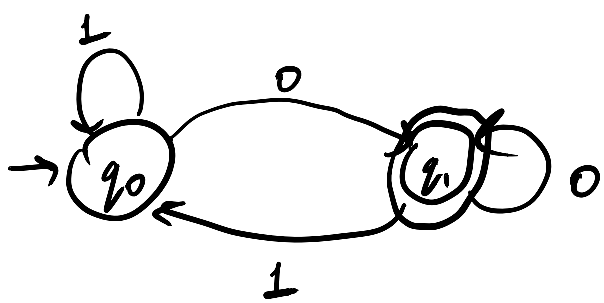
✓ whenever the button is pressed then eventually . . .

✓ infinitely often green lights



A beautiful theory

A



$\Sigma = \{0, 1\}$

$L(A) = (0+1)^* 0$

finite words

01011011000.....

$A = \langle \Sigma, Q, Q_0, \delta, \alpha \rangle$

$Q_0 \subseteq Q$ set of initial states

$\delta : Q \times \Sigma \to 2^Q$

α : an acceptance conditions

A run of A on $w = w_1 w_2 w_3 \ldots$

$r : \mathbb{N} \to Q$ such that

$r(0) \in Q_0$ and for every $i \geq 0$

$r(i+1) \in \delta(r(i), w_{i+1})$

Also $q_0 q_1 q_2 \ldots \in Q^\omega$ $q_i = r(i)$

Accaptance :

Büchi $\alpha \subseteq Q$

r is accepting if it visits α infinitely often.

$\mathrm{inf}(r) \subseteq Q$

$\{q : r \text{ visits } q \text{ i.o.}\} \neq \emptyset$

there are ∞ i $r(i)=q$

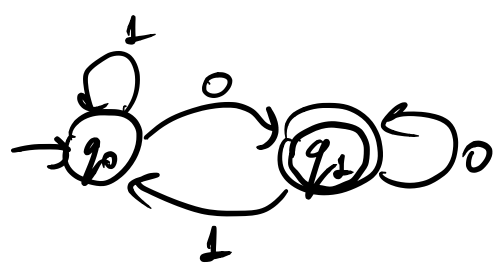
r is accepting

if $\mathrm{inf}(r) \cap \alpha \neq \emptyset$

$L(A) = \{ w :$ there is an accepting run of A on w $\}$

L is ω-regular $\Leftrightarrow$ there is a NBW A such that $L(A) = L$.

Examples:



$\Sigma = \{0, 1\}$ $\quad Q_0 = \{q_0\}$ $\quad \alpha = \{q_1\}$

$Q = \{q_0, q_1\}$ $\quad \delta(q_0, 1) = q_0$

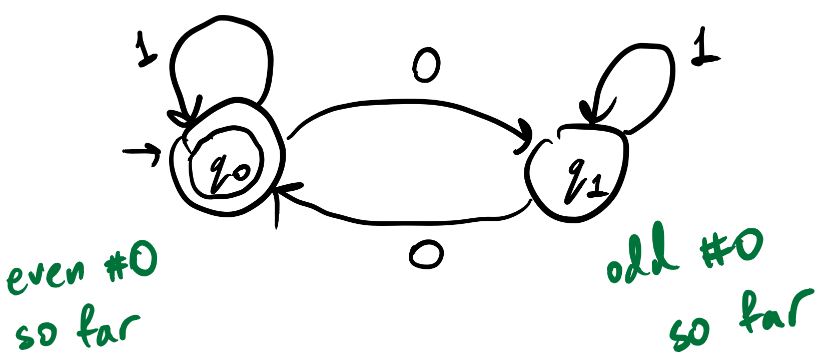
$L(A) = \{ w : w \text{ includes infinitely many } 0\text{s} \}$

$= (1^* 0)^\omega$

$L_2 = \{ w : w \text{ has a finite even number of } 0\text{s or infinitely many } 0\text{s} \}$

$0 1 0 1^\omega \in L_2$ $\quad$ $0 1^\omega \notin L_2$

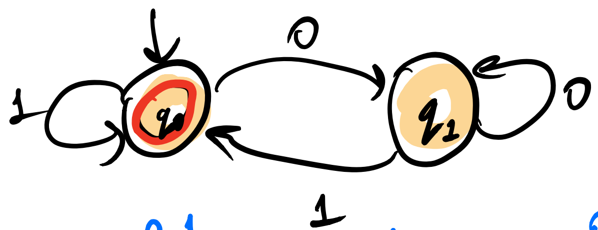
$(0 1)^\omega \in L_2$



$L_3 = \{ w : \infty 0 \to \infty 1 \}$

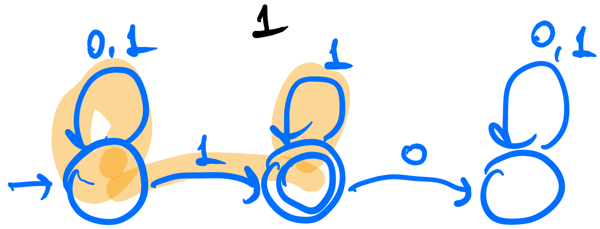
$\Sigma_1 = \{0, 1\}$ $\quad$ $\Sigma_2 = \{0, 1, 2\}$

$0^\omega \notin L_3 \quad \not\ni (20)^\omega$



$\infty 0$ DBW

$\infty 1$ $(01)^\omega$



$\neg \infty 0$

$(0+1)^* \cdot 1^\omega$

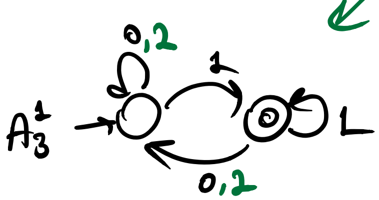
011001111101011^ω

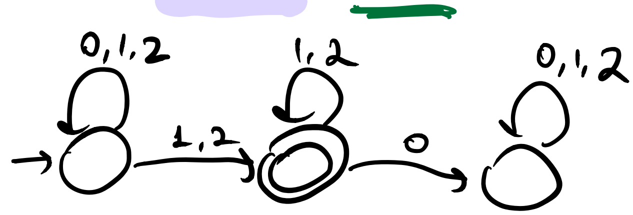
$DFW \to \overline{DFW}$ $\quad \overline{F} = Q \setminus F$

$L_3 = \infty 0 \to \infty 1$ $\quad \Sigma_2 = \{0, 1, 2\}$

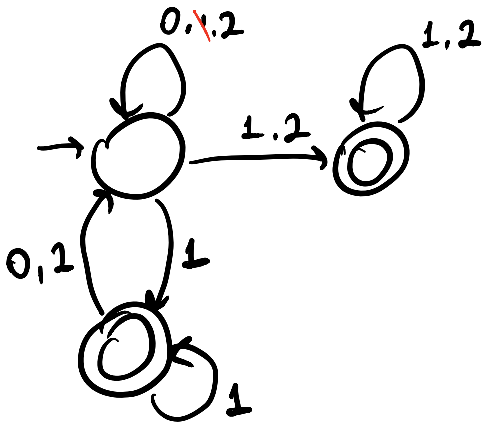
$\Sigma_1 = \{0, 1\}$ $\quad 2^\omega \in L_3$

$L_3 = \infty 1$ $\quad L_3 = \neg \infty 0 \vee \infty 1$



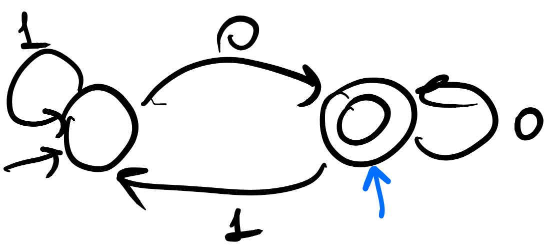


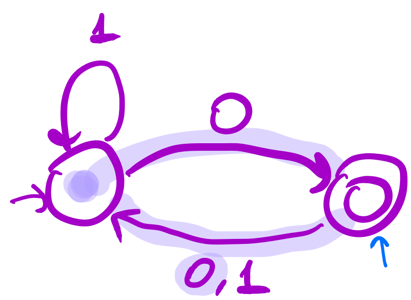
A_3^2



$\infty 0 \rightarrow \infty 1$

- There need not be a single minimal DBW for a language.



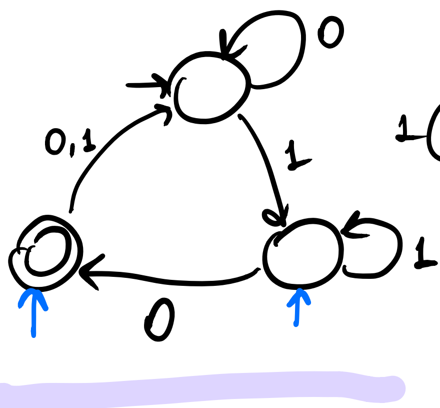


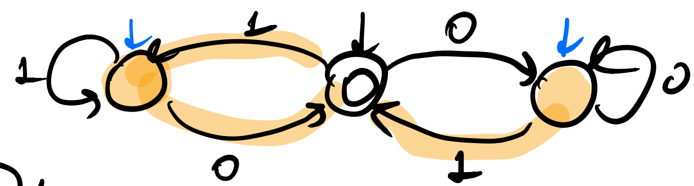
$\infty 0$ $\qquad\qquad\qquad\qquad$ $\infty 0$

four 2-state DFWs

different, minimal

$\Sigma = \{0,1\}$ $\infty 0 \wedge \infty 1$





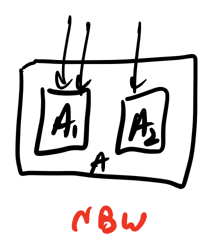
Closure properties of NBWs / DBWs ω-regular languages

Union :

If L_1 is ω-regular A_1

L_2 is ω-regular A_2

then $L_1 \cup L_2$ is ω-regular A



Intersection:

A_1 for L_1

A_2 for L_2

A for $L_1 \cap L_2$

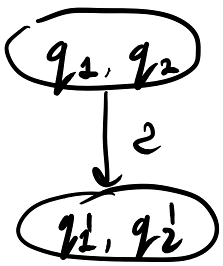
$A_i = \langle \Sigma, Q_i, Q_i^0, \delta_i, \alpha_i \rangle$

The product construction (for NFW)

$A = \langle \Sigma, Q_1 \times Q_2, \ldots$

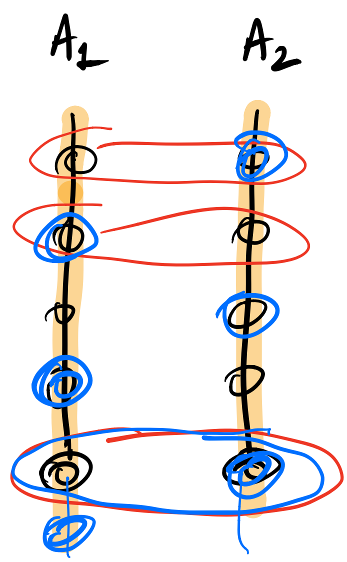
union for DBWs

$\alpha = (\alpha_1 \times Q_2) \cup (Q_1 \times \alpha_2)$



$\delta(\langle q_1, q_2 \rangle, \sigma) =$

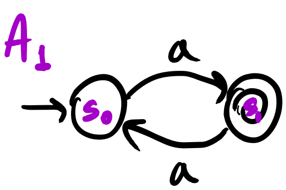
$\delta_1(q_1, \sigma) \times \delta_2(q_2, \sigma)$

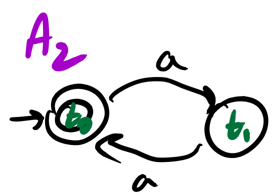


$\alpha = \alpha_1 \times \alpha_2$

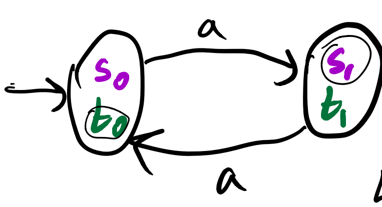
For NBW

$\Sigma = \{a\}$





$L(A_1) = L(A_2) = \{a^\omega\}$



$\alpha = \alpha_1 \times \alpha_2$

$L(A) = \emptyset$

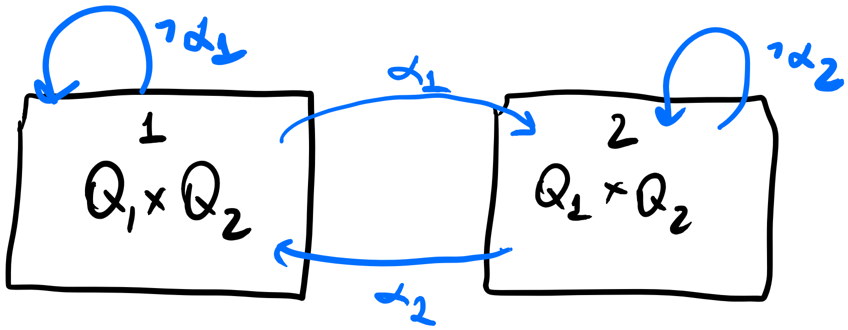
$\Sigma = \{a\}$

$L \subseteq \Sigma^*$ $\quad S \subseteq \mathbb{N}$

$L \subseteq \Sigma^\omega$ $\quad a^\omega$

$\emptyset$, $\{a^\omega\}$

Construct an NBW for $L_1 \cap L_2$:



$\overset{D}{NBW}$

$2 \cdot n_1 \cdot n_2$

preserves determinism

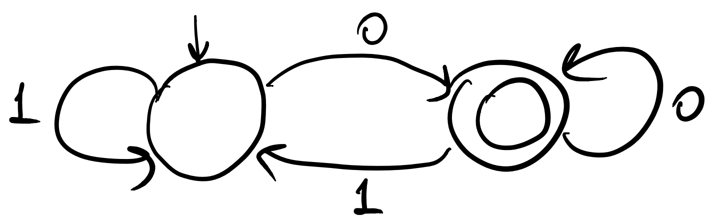
$$\alpha = \alpha_1 \times Q_2 \times \{1\} =$$

$$= \{\langle q_1, q_2, 1\rangle : q_1 \in \alpha_1\}$$

DBW: DBWs A_1 A_2

DBW

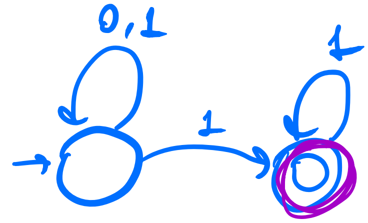
$\cup$ $\cap$ complementation



DBW

$\infty 0$

DBW $\rightarrow$ $\overline{NBW}$



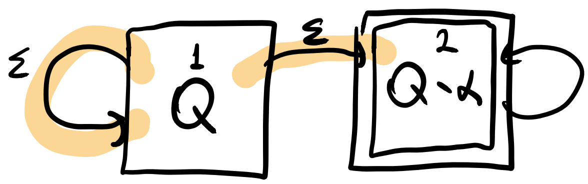
NBW

$\neg\infty 0$

$A = \langle \Sigma, Q, q_0, \delta, \alpha \rangle$ DBW

$A' = \langle \Sigma, Q', q_0', \delta', \alpha' \rangle$

A':



some run ← $\boxed{\text{the}}$ run of A sees only f.m. α's to reject.

for NBW we need all runs of

$Q' = (Q \times \{1\}) \cup ((Q \setminus \alpha) \times \{2\})$ $\quad q_0' = \langle q_0, 1 \rangle$

$$\delta'(\langle q, 1\rangle, \sigma) = \begin{cases} \{\langle s,1\rangle, \langle s,2\rangle\} & \text{if } s \notin \alpha \\ \{\langle s,1\rangle\} & \text{if } s \in \alpha \end{cases} \qquad \delta(q,\sigma) = s$$

$$\delta(\langle q, 2\rangle, \sigma) = \begin{cases} \{\langle s,2\rangle\} & \text{if } s \notin \alpha \\ \emptyset & \text{if } s \in \alpha \end{cases}$$

DBW $\rightarrow$ $\overline{\text{NBW}}$

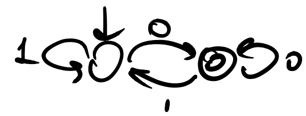
n $\qquad$ $2n$

① DBW $\xrightarrow{?}$ $\overline{\text{DBW}}$

② NBW $\xrightarrow{?}$ $\overline{\text{NBW}}$

- DBWs are not closed under complementation.

$L = \infty 0$ There is a DBW for L



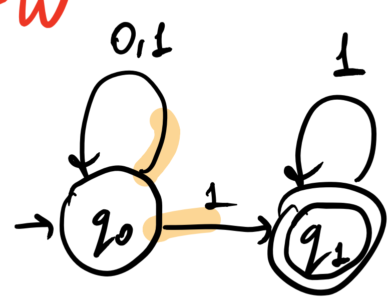
$\boxed{\text{Landweber 69:}}$ There is no DBW for $\bar{L}$

"only f.m 0s". $\neg\infty 0$ $\quad \Sigma^\omega \setminus L$

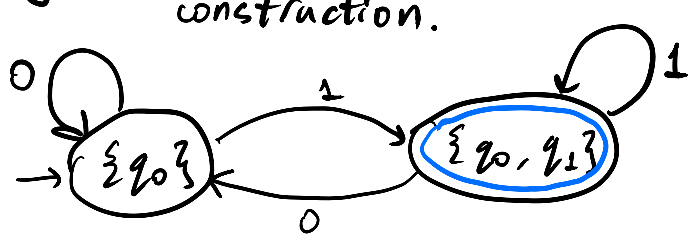
NBW > DBW

NFW = DFW

NBW for $\neg\infty 0$:



Apply the subset construction.



$\infty 1$

$(01)^\omega \notin \neg\infty 0$

$\in \infty 1$

S after reading w: $\forall s \in S$, A has a run on w that reaches s.

No DBW for "only finitely many 0s".

- Assume there is. A, n states.

$w_1 = 1^\omega \in \neg\infty 0 \rightarrow A$ accepts 1^ω

$A = \langle \Sigma, Q, q_0, \delta, \alpha \rangle$

$1\ 1\ 1\ 1\ 1 \ldots$

i_1

$\delta(q_0, 1^{i_1}) \in \alpha$

$w_2 = 1^{i_1} 0 1^\omega \in \neg\infty 0 \rightarrow A$ accpets w_2

$1^{i_1} 0\ 1\ 1\ 1\ 1 \ldots$

i_1 i_2

$\delta(q_0, 1^{i_1} 0 1^{i_2}) \in \alpha$

$w_3 = 1^{i_1} 0 1^{i_2} 0 1^\omega$ α

i_3

For every $k > 1$

$w_k = 1^{i_1} 0 1^{i_2} 0 1^{i_3} 0 1^{i_4} \ldots 0 1^{i_k} 0 1^\omega$

$k > n$ $\exists_{1 \le j_1, j_2 \le k}$ $j_1 \neq j_2$

$j_1 < j_2$

$$\delta(q_0, w_{j_1}) = \delta(q_0, w_{j_2}) \overset{= q}{\in} \alpha$$

Consider: $w = 1^{i_1} 0 \cdots 1^{i_{j_1}} (0 1^{i_{j_1+1}} \cdots 0 1^{i_{j_2}})^\omega$

$\uparrow q$ $\uparrow q$

not empty
at least one 0

① A accepts w
(the run of A on w visits q i.o.)

② $w \in \infty 0 \notin L(A)$

$j_1 < j_2$

$\rightarrow$ no DBW A exist.

NBW > DBW

① Characterize DBW

② Stronger conditions

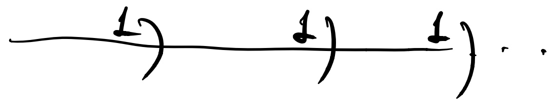
Landweber 69

For $R \subseteq \Sigma^*$, we define

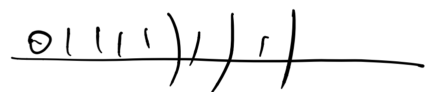
$\Sigma^\omega \supseteq \lim(R) = \{w :$ w has i.m. prefixes $\}$

in R

$R = (0+1)^* 1$

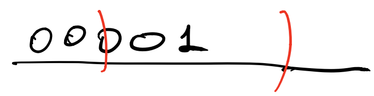


$\lim(R) = \infty 1$



$R = (0 \cdot 1^+)$ $\{0 1^\omega\}$

$\lim(R)$

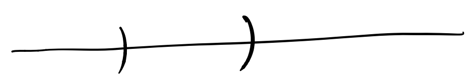


$R = (0^* 1)$

$\lim(R) = \emptyset$

$\lim^{-1}(L)$ need not be unique

$\infty 1$ $R = (0+1)^* 1 (0+1)^* 1$ $((0+1)^* 1)^i$



$R = (0+1)^* 1 (0+1)^i$

- For every $L \subseteq \Sigma^\omega$

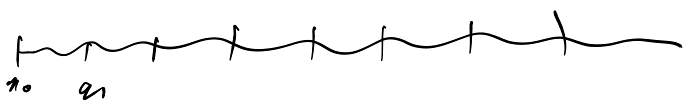
$L \in DBW \iff \exists R \subseteq \Sigma^*$ s.t.

$L = \lim(R)$

proof: For a deterministic automaton A

$$L_B(A) = \lim(L_F(A))$$

$\downarrow$ A as a DBW $\qquad$ $\downarrow$ A as a DFW



$L \in DBW \rightarrow A \rightarrow R = L_F(A)$

$\exists R \rightarrow A \rightarrow$ A when viewed as a DBW recognizes L.

stronger acceptace conditions.

$\inf(r) \cap \alpha \neq \emptyset$ Büchi

D[illegible]W = NBW > DBW

$\downarrow$

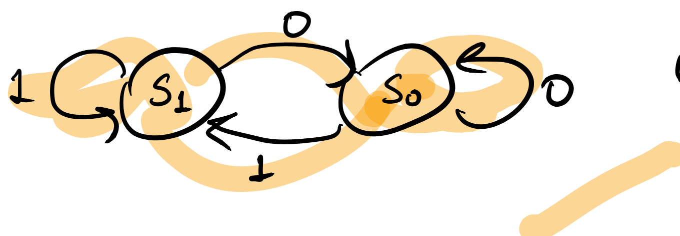
all ω-regular languages

① Generalized Büchi

$\alpha = \{\alpha_1, \alpha_2, \ldots \alpha_k\}$

r is accepting iff

$inf(r) \cap \alpha_i \neq \emptyset$ for all $1 \leq i \leq k$



$\infty 0 \wedge \infty 1$

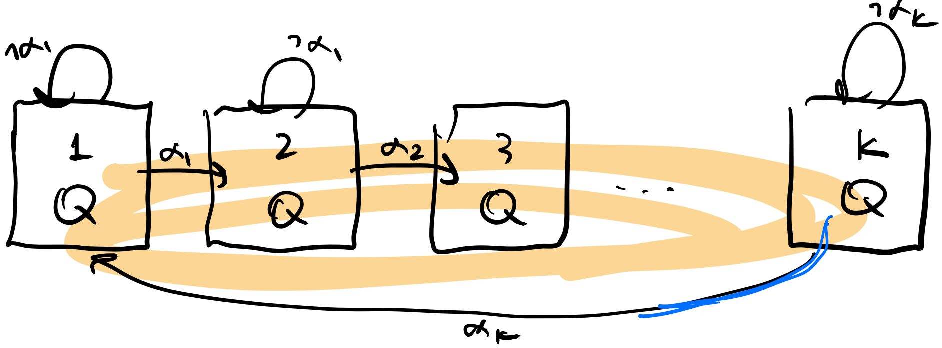
$\alpha = \{ \{s_0\}, \{s_1\} \}$

DGBW → DBW
NGBW → NBW

no added expressive power

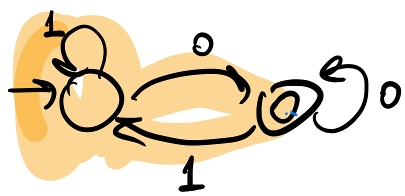
NGBW A $\alpha = \{\alpha_1, \ldots \alpha_k\}$

→ NBW A'



$$Q' = Q \times \{1, \dots k\} \qquad \alpha' = \alpha_k \times \{k\}$$

no DBW

A 

$\infty 0$

no DBW for $\neg \infty 0$.

② co-Büchi

r is accepting iff $\inf(r) \cap \alpha = \emptyset$

only f.m. visits in α.

$$L_c(A) = \neg \infty 0$$

A as a DCW

$$L_c(A) = \overline{L_B(A)}$$

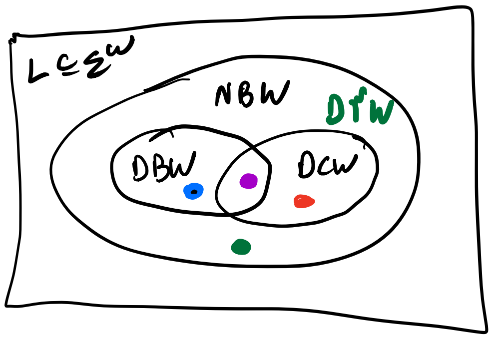
↓

r is accepting if $\inf(r) \cap \alpha = \emptyset$

$$DCW = \overline{DBW}$$

no DCW for $\infty 0$

③ Rabin, Streett, parity



$\infty 0$ $\quad (1^*0)^\omega$

$\neg\infty 0$ $\quad (0+1)^*1^\omega$

$0\cdot(0+1)^\omega$

$\infty 0 \wedge \neg\infty 1$ $\quad \Sigma = \{0,1,2\}$

- Rabin $\quad \alpha = \{\langle L_1, R_1\rangle, \langle L_2, R_2\rangle .. \langle L_k, R_k\rangle\}$

$L_i, R_i \subseteq Q$ $\quad$ index of α

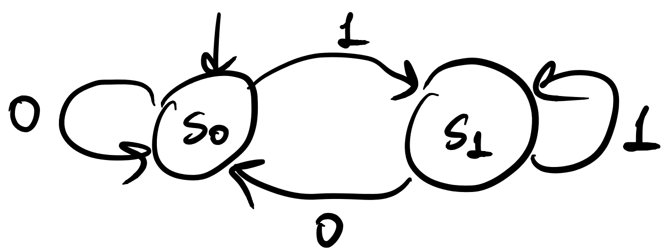
$\alpha \in 2^{2^Q \times 2^Q}$

r is accepting iff $\exists\ 1 \le i \le k$
$inf(r) \cap L_i \neq \emptyset$ and $\quad \infty L_i$
$inf(r) \cap R_i = \emptyset$ $\quad \neg\infty R_i$

$\alpha_1 = \{\langle \{s_1\}, \emptyset\rangle\}$

$\downarrow$

$\infty 1$



Buchi α $\rightarrow$ Rabin $\{\langle \alpha, \emptyset \rangle\}$

$\alpha_2 = \{\langle \{s_0, \cancel{s_1}\}, \{s_1\} \rangle$ $\quad \neg\infty 1$

co-Buchi α $\rightarrow$ Rabin $\{\langle Q, \alpha \rangle\}$

$\langle Q \setminus \alpha, \alpha \rangle$

- Streett $\quad$ dual to Rabin

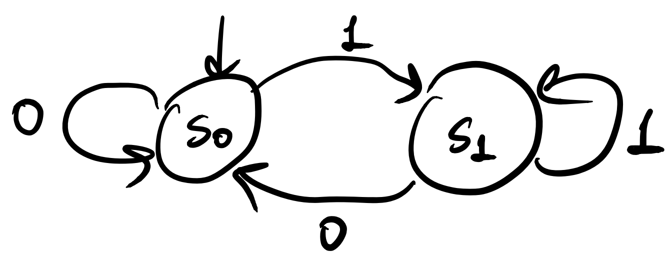
$\alpha = \{\langle L_1, R_1 \rangle .. \quad \langle L_k, R_k \rangle\}$

r is accepting iff $\forall\ 1 \leq i \leq k$

$\inf(r) \cap L_i = \emptyset$ or $\qquad \neg\infty L_1 \vee \infty R_i$

$\inf(r) \cap R_i \neq \emptyset$ $\qquad \infty L_i \rightarrow \infty R_i$

$\infty 1$



$\{\langle Q, \{s_1\} \rangle\}$

Buchi α $\qquad \{\langle Q, \alpha \rangle\}$ $\qquad \infty 0 \wedge \infty 1$

GB $\{\alpha_1 .. \alpha_k\}$ $\qquad \{\langle Q, \alpha_1 \rangle .. \langle Q, \alpha_k \rangle\}$

co-Buchi α $\qquad \{\langle \alpha, \emptyset \rangle\}$

$\{\langle Q, \{s_0\} \rangle, \langle Q, \{s_1\} \rangle\}$

③ parity $\alpha : Q \to \{0, 1, \dots, k\}$

r is accepting iff the minimal color that r visits i.o. is even

$\min\{i : \inf(r) \cap \alpha^{-1}(i) \neq \emptyset\}$ is even.

α_i

$\alpha = \{\alpha_0, \alpha_1, \dots \alpha_k\}$

- parity as Rabin

$\alpha'_{\text{Rabin}} = \{\langle \alpha_0, \emptyset \rangle, \langle \alpha_2, \alpha_1 \cup \alpha_0 \rangle, \langle \alpha_4, \alpha_3 \cup \alpha_2 \cup \alpha_1 \cup \alpha_0 \rangle$

- parity as Streett

$\alpha' = \{\langle \alpha_1, \alpha_0 \rangle, \langle \alpha_3, \alpha_0 \cup \alpha_1 \cup \alpha_2 \rangle, \dots \}$

- Büchi as parity

$\alpha_{\text{Buchi}} \to \{\overset{0}{\alpha}, \overset{1}{Q \setminus \alpha}\}$

1. Expressive power and succinctness
2. complexity of decision problems
3. Determinization and complementatio

NRW → NBW

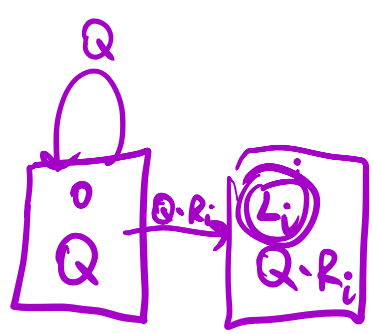
$A = \langle \Sigma, Q, Q_0, \delta, \alpha \rangle$

$\alpha = \{\langle L_1, R_1 \rangle \ldots \langle L_k, R_k \rangle\}$

$\alpha_i = \{\langle L_i, R_i \rangle\}$ $\quad \exists \; 1 \le i \le k$

A_i NRW[1] with α_i

$L(A) = \bigcup_{1 \le i \le k} A_i$



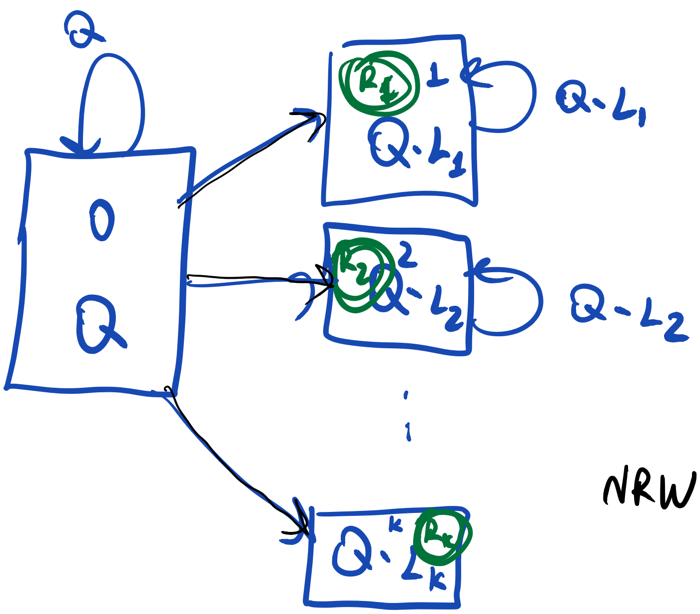
$A_i \rightarrow A'_i$

NRW[1] NBW

$\langle L_i, R_i \rangle$

$(Q \times \{0\}) \cup ((Q \times R_i) \times \{i\})$

$\alpha'_i = L_i \times \{i\}$



$NRW(n,k)$

$\to NBW(n \cdot (k+1))$

$$\underline{NSW \to NBW}$$

A A'

$\forall i \quad \infty L_i \to \infty R_i$

A' guess a subset $I \in \{1..k\}$

s.t. $i \in I \leftrightarrow$ the run visits

L_i i.o.

Does not work $\quad A' = \bigcap_{1 \le i \le k} A_i$

$$NSW(n,k) \xrightarrow[n \cdot 2^k]{n \cdot 2^k} NBW(n \cdot 2^k)$$

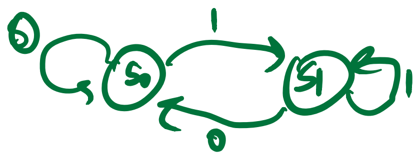
$n + 2^k \cdot n \cdot k$

DRW $\rightarrow$ DBW when exists.

no blow up

Typeness: Rabin is Büchi-type:

if L is in DBW, and
A is a DRW for L, then there is
a DBW for L, on the structure of A.



$\infty 1$

Streett is not Büchi type

$\{\langle Q, \{s_0\}\rangle, \langle Q, \{s_1\}\rangle\}$ DSW

$\underline{\infty 0 \wedge \infty 1}$ $\in$ DBW

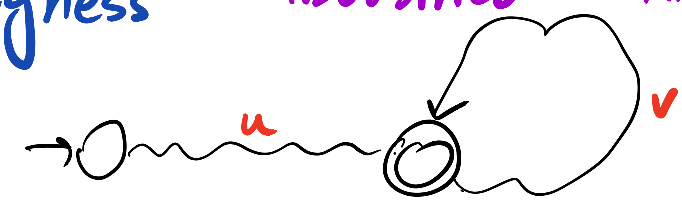
- Decision procedures

- Emptyness : Given A

  Is $L(A) = \emptyset$?

- nonemptyness NLOGSPACE linear time

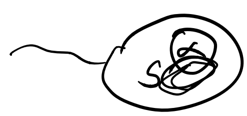
OBW
NBW



A is not empty iff there is $q \in \alpha$ such that q is reachable from Q_0 and from it self. $u \cdot v^{\omega} \in L(A)$

- is there a maximal strongly connected component S (not trivial) such that S is reachable from Q_0

$S \cap \alpha \neq \emptyset$



- nonemptyness for NRW ? NL
  NSW PTIME

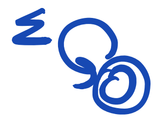
- $NRW(n,k) \to NBW(n \cdot k)$

  S $n \cdot 2^k$

- universality $\quad L(A) = \Sigma^\omega$

PSPACE-complete

- containment $\quad \underline{L(A_1)} \subseteq \underline{L(A_2)}$

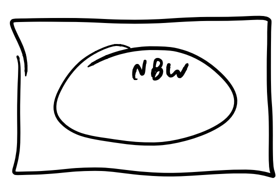
A is universal $\leftrightarrow$  $\subseteq L(A)$

$$A \subseteq B \quad \leftrightarrow \quad A \cap \overline{B} = \emptyset$$

$$L(A_1 \times \overline{A_2}) = \emptyset$$

product $\qquad$ complementation

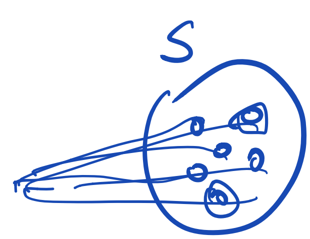
③ Determinization 4



$$NBW \longrightarrow D\begin{smallmatrix}R\\S\\P\end{smallmatrix}W$$

$$NFW \longrightarrow DFW$$

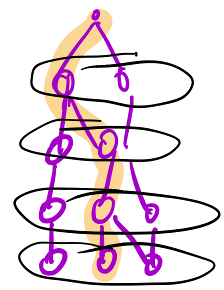
$$Q \qquad\qquad 2^Q$$

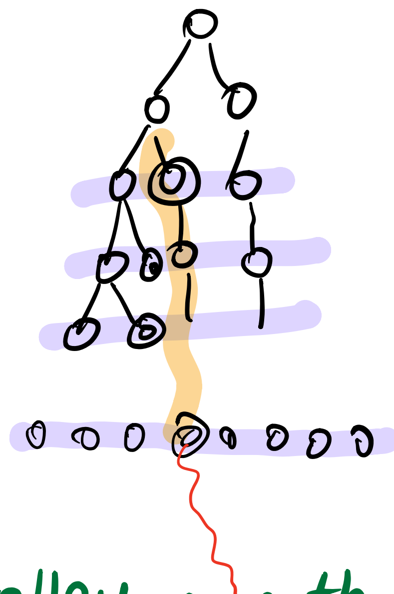


NBW $\to$ DRW   Safra 1988

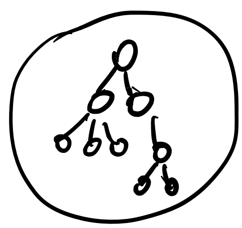
$2^{O(n \log n)}$

S + additional information





Safra tree



S

∞ $\neg\infty$

$\langle L, R \rangle$

R: follow a path

L: this path has i.m. visits in α.

NBW $\to$ DPW   Piterman 2006