Sharp isoperimetric inequality for affine quermassintegrals

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joint work with Amir Yehudayoff (Technion)

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Classical Quermassintegrals

Let K convex body in $\mathbb{R}^n$ (convex, compact, $\text{int}(K) \neq \emptyset$).

$$\text{Steiner } \sim 1850: \quad |K + tB_2^n| = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} W_k(K).$$

Definition: $W_k(K)$ is called the k-th quermassintegral of K.
By Kubota's formula:

$$W_k(K) = \frac{|B_2^n|}{|B_2^k|} \int_{G_{n,k}} |P_F K| \sigma(dF).$$

E.g. $W_{n-1}(K) = n\mathcal{H}^{n-1}(\partial K)$, $W_1(K) = |B_2^n| \cdot \text{half-mean-width}(K)$.

Fix $k \in \{1, \ldots, n-1\}$ ($W_n(K) = |K|$, $W_0(K) = |B_2^n|$).
Classical (Alexandrov): $W_k(K) \geq W_k(B_K)$ with equality iff $K = B_K$.
Here and throughout, B_K is a Euclidean ball with $|B_K| = |K|$.
$k = n-1$ is isoperimetric inequality, $k = 1$ is Urysohn's inequality.

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Grinberg: $\Phi_k(\cdot)$ is invariant under volume-preserving affine transformations (or simply "affine invariant").

Affine quermassintegrals are central pillars of *affine* convex geometry.

Conjecture (Lutwak '88)

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  $$\text{General } K: \quad \text{Lutwak's Conj} \quad \Leftarrow |K||K^{\circ,s}| \leq |B_2^n|^2.$$

  Equality in Blaschke–Santaló inq: B–S, Saint-Raymond, Petty.

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- Definition (Lutwak): Dual affine quermassintegrals:

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Grinberg (following Fursternberg-Tzkoni): $\tilde{\Phi}_k(\cdot)$ affine-invariant.
Busemann–Strauss, Grinberg, Gardner:

$$\tilde{\Phi}_k(K) \leq \tilde{\Phi}_k(B_K) \text{ with equality iff } K \text{ ellipsoid } (k > 1).$$

- Upper-bounds on $\Phi_k(K)$:
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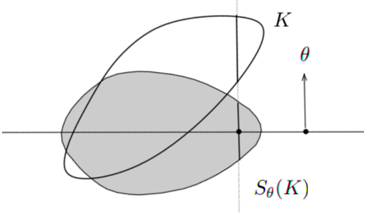
Thm 1: $\Phi_k(K) \geq \Phi_k(B_K)$, equality iff K is ellipsoid.

Thm 2: $\Phi_k(K) \geq \Phi_k(S_u K)$ $\forall u \in \mathbb{S}^{n-1}$, equality iff K is ellipsoid.



$S_u K$ - Steiner symmetral of K in direction of u

(graphic by Thaicia Stona)

Thm 3: Among all convex bodies in $\mathbb{R}^n$ of given volume, ellipsoids are the only local minimizers of Φ_k (w.r.t. Hausdorff topology).

Cases $k \in \{1, n-1\}$ of Thm 2's inq previously known; equality case:

- $k = 1$ - Meyer–Pajor, Lutwak–Zhang, Meyer–Reisner.
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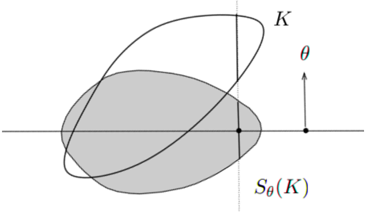

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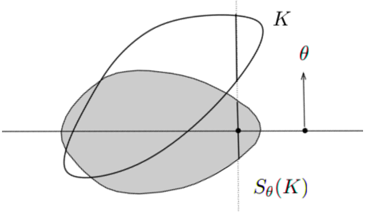

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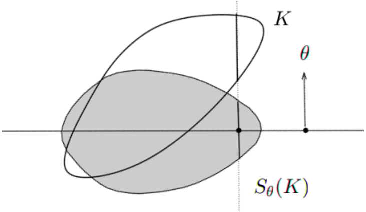

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(graphic by Thaicia Stona)

Thm 3: Among all convex bodies in $\mathbb{R}^n$ of given volume, ellipsoids are the only local minimizers of Φ_k (w.r.t. Hausdorff topology).

Cases $k \in \{1, n-1\}$ of Thm 2's inq previously known; equality case:

- $k = 1$ - Meyer–Pajor, Lutwak–Zhang, Meyer–Reisner.
- $k = n - 1$ - appears new.

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- $k = 1$ - established recently by Meyer–Reisner (for $|K^{\circ,s}|$).
- $k = n - 1$ - appears new.

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$$\int_{G_{n,n-2}} |P_F K|^{-q} \sigma_{n,n-1}(dF) = \int_{G_{n,n-1}} \int_{G_{E,n-2}} |P_F P_E K|^{-q} \sigma_{E,n-2}(dF) \sigma_{n,n-1}(dE)$$

For $q \leq n-1$, by Petty's projection inq applied to $P_E K$ and then to K:

$$\leq c_n^1 \int_{G_{n,n-1}} |P_E K|^{-q \frac{n-2}{n-1}} \sigma_{n,n-1}(dE) \leq c_n^1 c_n^2 |K|^{-q \frac{n-2}{n}},$$

with equality when $K = B_K$. In other words:

$$\left(\int_{G_{n,n-2}} |P_F K|^{-(n-1)} \sigma(dF) \right)^{-\frac{1}{n-1}} \geq \left(\int_{G_{n,n-2}} |P_F B_K|^{-(n-1)} \sigma(dF) \right)^{-\frac{1}{n-1}}.$$

Similarly, for all $k = 1, \ldots, n-1$:

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Lost affine-invariance; numerous other attempts could not improve.

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The Challenge

Rank one cases $k \in \{1, n-1\}$ proved by associating to K an auxiliary convex body $L_k(K) \subset \mathbb{R}^n$ ($K^\circ / \Pi^* K$) encapsulating $G_{n,k} \ni F \mapsto |P_F K|$, and establishing $|L_k(S_u K)| \geq |L_k(K)|$.

$L_k(K)$ is obtained by identifying $G_{n,k} \simeq \mathbb{RP}^{n-1}$, extending $G_{n,k} \ni F \mapsto |P_F K|$ to $\mathbb{R}^n$ homogeneously, and taking level set.

Higher rank cases $2 \leq k \leq n-2$: in which space does $L_k(K)$ reside?

Our solution: $L_{k,u}(K)$ resides in vector-bundle over $G_{u^\perp, k-1}$.

Instead of Lebesgue measure, $\mu_u(L_{k,u}(S_u K)) \geq \mu_u(L_{k,u}(K))$.

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The Projection Rolodex

Given $E \in G_{n,k-1}$, $x \in \mathbb{R}^n$, denote $|P_{E \wedge x} K| = |P_{E^\perp} x| \mathcal{L}^k(P_{\text{span}(E,x)} K)$.

Definition (E-projected polar-body)

$L_E(K) := \{x \in E^\perp \; ; \; |P_{E \wedge x} K| \leq 1\} \subset E^\perp$.

Lemma: $L_E(K)$ is always convex.
Note: $L_{\{0\}} K = \{x \in \mathbb{R}^n \; ; \; h_K(x) + h_K(-x) \leq 1\} = (K - K)^\circ$.

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Blaschke–Petkantschin-type formula

Let $u \in \mathbb{S}^{n-1}$ be fixed from now on.

Thm (Schneider–Weil monograph)

For any measurable function $f : G_{n,k} \to \mathbb{R}_+$:

$$c_{n,k} \int_{G_{n,k}} f(F)\sigma_{n,k}(dF) = \int_{G_{u^\perp,k-1}} \int_{\mathbb{S}(E^\perp)} f(\mathrm{span}(E,\theta_k))|\langle \theta_k, u\rangle|^{k-1} d\theta_k \; \sigma_{u^\perp,k-1}(dE).$$

Intuition: gives more weight to $F = \mathrm{span}(E,\theta_k)$'s that are far from $u^\perp$.

Corollary

Define $\mu_u := |\langle x_k, u\rangle|^{k-1} \mathcal{L}_{E^\perp}(dx_k)\sigma_{u^\perp,k-1}(dE)$. Then:

$$\mu_u(L_{k,u}(K)) = \frac{c_{n,k}}{n} \int_{G_{n,k}} \frac{1}{|P_F K|^n} \sigma_{n,k}(dF).$$

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Set $p(x_k) := |\langle x_k, u\rangle|^{k-1}$ for brevity.

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&= \int_{G_{u^\perp,k-1}} \int_{\mathbb{S}(E^\perp)} p(\theta_k) \int_0^{1/|P_{\mathrm{span}(E,x_k)}(K)|} r^{n-1} dr\, d\theta_k\, \sigma_{u^\perp,k-1}(dE) \\
&= \frac{1}{n} \int_{G_{u^\perp,k-1}} \int_{\mathbb{S}(E^\perp)} \frac{1}{|P_{\mathrm{span}(E,\theta_k)} K|^n} |\langle \theta_k, u\rangle|^{k-1} d\theta_k\, \sigma_{u^\perp,k-1}(dE) \\
&= \frac{c_{n,k}}{n} \int_{G_{n,k}} \frac{1}{|P_F K|^n} \sigma_{n,k}(dF). \quad \square
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Goal: $\Phi_k(K) \geq \Phi_k(S_u K) \quad \Leftrightarrow \quad \mu_u(L_{k,u}(K)) \leq \mu_u(L_{k,u}(S_u K))$

Now everything is aligned with u.

Decompose $E^\perp = \mathrm{span}(u) \oplus (E^\perp \cap u^\perp)$ and write:

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\end{aligned}$$

where $L_{E,u,s}(K) := \{y \in E^\perp \cap u^\perp \; ; \; |P_{E\wedge(y+su)}K| \leq 1\}$ is section of $L_E(K)$ parallel to $u^\perp$ at height s.

It remains to show the key inequality $|L_{E,u,s}(K)| \leq |L_{E,u,s}(S_u K)|$.

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$$= \int_{G_{u^\perp,k-1}} \int_{\mathbb{R}} |s|^{k-1} |L_{E,u,s}(K)|\, ds\, \sigma_{u^\perp,k-1}(dE),$$

where $L_{E,u,s}(K) := \{y \in E^\perp \cap u^\perp \; ; \; |P_{E \wedge (y+su)} K| \leq 1\}$ is section of $L_E(K)$ parallel to $u^\perp$ at height s.

It remains to show the key inequality $|L_{E,u,s}(K)| \leq |L_{E,u,s}(S_u K)|$.

Convexity of Shadow Systems

Denote $A^{(y)}$ = one-dim section of A in direction of u over $y \in u^\perp$.
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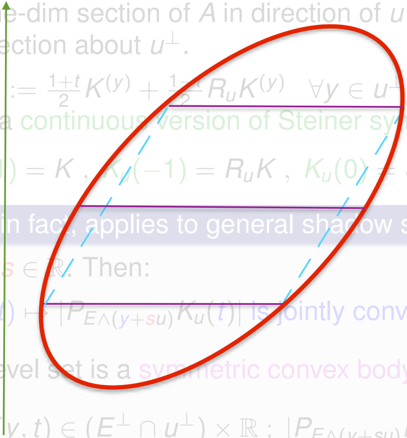

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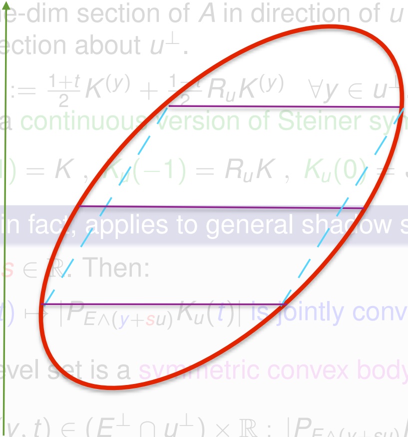

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Proof: enough for $x \in E^\perp$. Denote $K^{(w)} = (K - w) \cap E^\perp$ for $w \in E$.

$$|P_{E\cap x} K| = \int_E |P_x K^{(w)}| dw = \int_E (h_{K^{(w)}}(x) + h_{K^{(w)}}(-x)) dw.$$

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L^p-moment Quermassintegrals

The L^p-moment k-th quermassintegral of K is defined as:

$$\mathcal{Q}_{k,p}(K) := \frac{|B_2^n|}{|B_2^k|} \left(\int_{G_{n,k}} |P_F K|^p \sigma(dF) \right)^{\frac{1}{p}} .$$

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$p = 1$ classical, $p = -n$ affine, $p = -1$ harmonic (Hadwiger, Lutwak).

Corollary (by applying Jensen)

Fix $k = 1, \ldots, n-1$ and $p > -n$.
Then $\mathcal{Q}_{k,p}(K) \geq \mathcal{Q}_{k,p}(B_K)$ with equality iff K is Euclidean ball.

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