Introduction to the Low-Degree Polynomial Method

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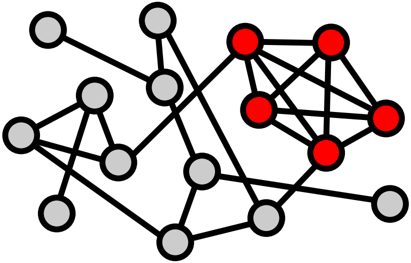
Part I: Why Low-Degree Polynomials?

Problems in High-Dimensional Statistics

Example: finding a large clique in a random graph



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What makes problems easy vs hard?

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Today: self-contained motivation (without SoS)

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- Approximate message passing: $v \leftarrow Y h(v)$ — $O(1)$ rounds
- Local algorithms on sparse graphs — radius $O(1)$
- Or any of the above applied to $\tilde{Y} = g(Y)$ — $\deg g = O(1)$

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"Low-degree conjecture" (informal): low-degree polynomials are as powerful as all poly-time algorithms for "natural" high-dimensional problems [Hopkins '18]

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Settings:

- Detection

  [Hopkins, Steurer '17]

  [Hopkins, Kothari, Potechin, Raghavendra, Schramm, Steurer '17]

  [Hopkins '18] (PhD thesis)

  [Kunisky, W., Bandeira '19] (survey)

- Recovery

  [Schramm, W. '20]

- Optimization

  [Gamarnik, Jagannath, W. '20]

Part II: Detection

Detection (e.g. [Hopkins, Steurer '17])

Goal: hypothesis test with error probability $o(1)$ between:

- Null model $Y \sim \mathbb{Q}_n$ e.g. $G(n, 1/2)$
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$$= \begin{cases} \omega(1) & \text{"degree-}D\text{ polynomial succeed"} \\ O(1) & \text{"degree-}D\text{ polynomials fail"} \end{cases}$$

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Extended low-degree conjecture [Hopkins '18]:

$$\text{degree-}D \text{ polynomials} \Leftrightarrow n^{\tilde{\Theta}(D)}\text{-time algorithms}$$

$$D = n^{\delta} \quad \Leftrightarrow \quad \exp(n^{\delta \pm o(1)}) \text{ time}$$

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Proof: $\hat{L}_S = \mathbb{E}_{Y \sim \mathbb{Q}}[L(Y)Y^S] = \mathbb{E}_{Y \sim \mathbb{P}}[Y^S]$ $\qquad \hat{f}^*_S = \mathbb{E}_{Y \sim \mathbb{P}}[Y^S]\,\mathbb{1}_{|S| \leq D}$

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- These predictions are "correct" for: planted clique, sparse PCA, community detection, tensor PCA, spiked Wigner/Wishart, ... [BHKKMP16,HS17,HKPRSS17,Hop18,BK**W**19,K**W**B19,DK**W**B19]

Part III: Recovery

Recovery [Schramm, W. '20]

Example (planted submatrix): observe $n \times n$ matrix $Y = X + Z$

- Signal: $X = \lambda v v^\top \quad \lambda > 0 \quad v_i \sim \mathrm{Bernoulli}(\rho)$
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Equivalent to low-degree maximum correlation:

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Fact: $\mathrm{MMSE}_{\le D} = \mathbb{E}[v_1^2] - \mathrm{Corr}_{\le D}^2$

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$$f = \sum_{|S| \leq D} \hat{f}_S Y^S$$

<u>Numerator</u>: $\mathbb{E}[f(Y) \cdot v_1] = \sum_{|S| \leq D} \hat{f}_S \, \mathbb{E}[Y^S \cdot v_1] =: \langle \hat{f}, c \rangle$

<u>Denominator</u>: $\mathbb{E}[f(Y)^2] = \sum_{S,T} \hat{f}_S \hat{f}_T \, \mathbb{E}[Y^S \cdot Y^T] = \hat{f}^\top M \hat{f}$

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- if $\lambda \gg \min\{1, \frac{1}{\rho\sqrt{n}}\}$ then $\mathrm{MMSE}_{\leq O(\log n)} = o(\rho)$
  low-degree polynomials succeed in the "easy" regime

Part IV: Optimization

Optimization [Gamarnik, Jagannath, W. '20]

Example (spherical spin glass): for $Y \in \mathbb{R}^{n \times n \times n \times n}$ i.i.d. $\mathcal{N}(0,1)$,

$$\max_{\|v\|=1} \frac{1}{\sqrt{n}} \langle Y, v^{\otimes 4} \rangle$$

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Forthcoming: improve $1 + \frac{1}{\sqrt{2}} \to 1 + \epsilon$ (optimal)

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- Overlap gap property (OGP)
  [Gamarnik, Sudan '13]
  [Chen, Gamarnik, Panchenko, Rahman '17]
  [Gamarnik, Jagannath '19]

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With non-trivial probability (over path), f's output is "smooth"

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Proof: first moment method [Gamarnik, Sudan '13]

Optimization [Gamarnik, Jagannath, W. '20]

Ensemble OGP: with high probability, $\forall i, j$ on the interpolation path

$$Y^{(0)} \quad Y^{(1)} \quad Y^{(2)} \quad \dots \quad Y^{(m-1)} \quad Y^{(m)}$$

there is no occurrence of

- S independent set in $Y^{(i)}$
- T independent set in $Y^{(j)}$
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Contradicts OGP

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  - E.g. random regular graphs
- (Recovery) bound $\mathrm{MMSE}_{\leq D}$ when not "signal + noise"
  - E.g. sparse regression, phase retrieval

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- Implications for other algorithms?
  - E.g. convex programming, MCMC

References

- Detection (survey article)
  Notes on Computational Hardness of Hypothesis Testing: Predictions using the Low-Degree Likelihood Ratio
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- Optimization
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(extra scratch paper)