A Geometric Approach to Conic Stability of Polynomials

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Outline

Problem: Is there any relationship between the roots of two polynomials f, g and the roots of their average $(f + g)/2$?

- in general, no.
- the classical notion of *interlacing* and *common interlacing* polynomials. here
- The existence of common interlacing is equivalent to some real-rootedness condition.
- interlacing and real-rootedness are entirely univariate notions.
- can be viewed as restrictions of multivariate phenomena.
- Two important generalizations of real-rootedness to more than one variable: **real stability and hyperbolicity** (isomorpism).

1. A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called stable if every root $\mathbf{z} = (z_1, \ldots, z_n)$ satisfies $\mathsf{Im}(z_j) \leq 0$ for some j.
2. A polynomial f is real stable if it is stable and all of its coefficients are real.
3. A univariate polynomial is real stable if and only if it is real rooted.
4. A homogeneous $f \in \mathbb{R}[\mathbf{z}]$ is called hyperbolic w.r.t $\mathbf{e} \in \mathbb{R}^n$, if $f(\mathbf{e}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ the real function $t \to f(x + t\mathbf{e})$ has only real roots.

A polynomial $f \in \mathbb{R}[\mathbf{z}]$ is real stable

$\Updownarrow$

the (unique) homogenization polynomial w.r.t. the variable z_0 is hyperbolic w.r.t. every vector $\mathbf{e} \in \mathbb{R}^{n+1}$ such that $e_0 = 0$ and $e_j > 0$ for all $1 \leq j \leq n$ (Gårding89)

History

1. J. Borcea and P. Brändén,. Applications of stable polynomials to mixed determi- nants: Johnson's conjectures, unimodality, and symmetrized fischer products. Duke Mathematical Journal,
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6. Gurvits: Simple proof of a generalization of van der Waerden's Conjecture, Electron. J. Comb. 2008
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8. Marcus, Spielman, Srivastava: Existence of Ramanujan graphs, Ann. Math. 2015, FOCS 2013

Stable Polynomials

A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called **stable** provided whenever $\mathsf{Im}(\mathbf{z}) = (\mathsf{Im}(z_1), \ldots, \mathsf{Im}(z_n)) > 0$, $(\mathsf{Im}(z_j) > 0$ for all $j)$, $f(z_1, \ldots, z_n) \neq 0$.

Let $\mathcal{H}_{\mathbb{C}}^n$ denotes the set $\{\mathbf{z} \in \mathbb{C}^n : \mathsf{Im}(z_j) > 0, 1 \leq j \leq n\}$.

f is **stable** if it has no roots in $\mathcal{H}_{\mathbb{C}}^n$.

Note that $\mathsf{Im}(\mathcal{H}_{\mathbb{C}}^n) =: \mathbb{R}_{>0}^n$ is the positive orthant.

f is **stable** if and only if $\{\mathsf{Im}(\mathbf{z}) = (\mathsf{Im}(z_1), \ldots, \mathsf{Im}(z_n)) : f(\mathbf{z}) = 0\} \cap (\mathbb{R}_{>0})^n = \emptyset$ [Jörgens,Theobald, Wolff].

Question: Can this idea be generalized?

1. The cone
2. the imaginary projection of a polynomial?

Geometric Notion: **Imaginary projections of polynomials**

Definition

Given a polynomial $f \in \mathbb{C}[\mathbf{z}]$, define $\mathcal{I}(f) = \{\mathrm{Im}(\mathbf{z}) : \mathbf{z} \in \mathcal{V}(f)\}$.

We call $\mathcal{I}(f)$ the imaginary projection of f.

The underlying projection is

$$\mathrm{Im} : \mathbb{R}^{2n} \to \mathbb{R}^n, (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_n), \text{ for } z_j = x_j + iy_j \quad (1)$$

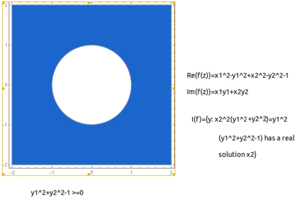

Figure: Imaginary Projections of $f(z_1, z_2) = z_1^2 + z_2^2 + 1$

Pictures

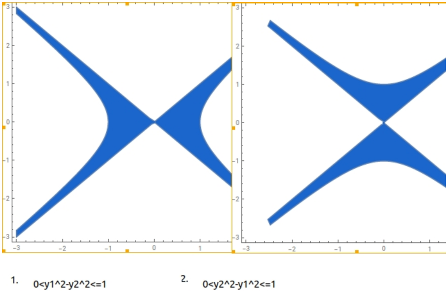

Figure: Imaginary Projections of $f(z_1, z_2) = z_1^2 - z_2^2 - 1$ and $f(z_1, z_2) = -z_1^2 + z_2^2 - 1$

Properties of the Imaginary projection

- $\mathcal{I}(f)$ is a semialgebraic set as it is the projection of a real algebraic variety.
- It is not always closed.
- For $n \geq 2$, it is always unbounded.
- If f is irreducible, then $\mathcal{I}(f)$ is connected since the map (1) is continuous.
- Components of the complement are convex and finite in number [Jörgens, Theobald, Wolff]

Motivation:

- $\mathcal{V}(f) \to \mathbb{R}^n, \mathbf{z} \mapsto (|z_1|, \ldots, |z_n|)$, (known as semialgebraic amoeba)

Definition

- The amoeba
$$A(f) = \{(\log |z_1|, \ldots, \log |z_n|) : \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\},$$
- the coamoeba
$\mathrm{co}A(f) = \{(\arg(z_1), \ldots, \arg(z_n)) : \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\},$
- $\mathcal{V}(f) \to \mathbb{R}^n, \mathbf{z} \mapsto \mathsf{Im}(\mathbf{z})$ or $\mathbf{z} \mapsto \mathsf{Re}(\mathbf{z})$

Conic Stable polynomials

Definition

Let $K \subseteq \mathbb{R}^n$ be a proper cone. A multivariate polynomial $f \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots, z_n]$ is called K-**stable** if $\mathcal{I}(f) \cap \operatorname{Int} K = \emptyset$, where $\operatorname{Int} K$ is the interior of K.

f is **stable** if and only if $\mathcal{I}(f) \cap (\mathbb{R}_{>0})^n = \emptyset$, K is the non-negative orthant.

Examples: PSD stable and Determinantal polynomials

If $f \in \mathbb{R}[Z]$ on the symmetric matrix variables $Z = (z_{ij})_{n \times n}$ is S_n^+-stable, then f is called positive semidefinite-stable (for short, psd-stable).

- Psd-stability of $f \in \mathbb{C}(Z)$ can be viewed as stability w.r.t the Siegel upper half-space

$$\mathcal{H}_g = \{A \in \mathbb{C}^{g \times g} \text{ symmetric } : \mathsf{Im}(A) = (\mathsf{Im}(a_{ij}))_{g \times g} \text{ is positive definite}\}$$

- The determinantal polynomial $f(\mathbf{z}) = \det(A_0 + \sum_{j=1}^{n} A_j z_j)$ is real stable or the zero polynomial where A_j's are positive semidefinite $d \times d$-matrices and A_0 is a Hermitian $d \times d$-matrix [Borcea, Brändén].

Relationship

Question: The class of stable polynomials $\underbrace{\subseteq}_{?}$ the class of psd stable polynomials

Not all **stable** polynomials are **psd-stable**

- The determinantal polynomial

$$f(z_1, z_2, z_3) = (z_1 + z_3)^2 - z_2^2 = (z_1 + z_3 - z_2)(z_1 + z_3 + z_2)$$

  is not stable, because $(1, 2, 1) \in \mathcal{I}(f) \cap \mathbb{R}^3_{>0}$.

- In the matrix variables $Z = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix}$, the polynomial $f(Z) = f(z_1, z_2, z_3)$ is psd-stable.

Not all **determinantal** polynomials are **psd-stable**

Example

A non psd-stable determinantal polynomial is the determinant of the spectrahedral representation of the open Lorentz cone $g(\mathbf{z}) = \det \begin{pmatrix} z_1 + z_3 & z_2 \\ z_2 & z_1 - z_3 \end{pmatrix} = z_1^2 - z_2^2 - z_3^2$.

Imaginary Projections and Hyperbolic polynomials

Definition

Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous. Then f is called **hyperbolic** w.r.t $\mathbf{e} \in \mathbb{R}^n$, if $f(\mathbf{e}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ the real function $t \mapsto f(\mathbf{x} + t\mathbf{e})$ has only real roots.

Definition

If f is hyperbolic w.r.t $\mathbf{e} \in \mathbb{R}^n$, we call $C(f, \mathbf{e}) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x} + t\mathbf{e}) = 0 \Rightarrow t < 0\}$ the hyperbolicity cone of f with respect to $\mathbf{e}$.

- $C(f, \mathbf{e})$ is open and convex (Gårding, 1959).
- f is hyperbolic to every point $\mathbf{e}^{'}$ in its hyperbolicity cone and $C(f, \mathbf{e}) = C(f, \mathbf{e}^{'})$.

Theorem:Jörgens-Theobald

Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous. Then the hyperbolicity cones of f coincide with the complement components of $\mathcal{I}(f)$.

Connection:Hyperbolic Polynomials

A hyperbolic polynomial f w.r.t $\mathbf{e}$ is $\mathrm{cl}(C(f,\mathbf{e}))$-stable.
The FAE:

1. A hyperbolic polynomial $f \in \mathbb{R}[\mathbf{z}]$ is K-stable
2. f is hyperbolic w.r.t every point in int K
3. Int $K \subseteq C(f,\mathbf{e})$ for some hyperbolicity direction $\mathbf{e}$ of f.

- The initial form of f, denoted by $\mathrm{in}(f)$, is defined as $\mathrm{in}(f)(\mathbf{z}) = f_h(0,\mathbf{z})$, where f_h is the homogenization of f w.r.t. the variable z_0.

Theorem:[Dey, Gardoll, Thoebald]

If a degree d polynomial $f = \det(A_0 + \sum_{j=1}^{n} z_j A_j)$ where $A_j, j = 0,\ldots,n$ are Hermitian matrices, and there exists an $\mathbf{e} \in \mathbb{R}^n$ with $\sum_{j=1}^{n} A_j e_j > 0$, then

1. $\mathrm{in}(f)$ is hyperbolic and
2. every hyperbolicity cone of $\mathrm{in}(f)$ is contained in $\mathcal{I}(f)^c$.

Idea of the proof

- Since f is of degree d, $\mathrm{in}(f) = \det(\sum_{j=1}^n A_j z_j)$.
- The initial form $\mathrm{in}(f)$ has exactly the two hyperbolicity cones $C_1 = \{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n A_j x_j \succ 0\}$ and $C_2 = \{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n A_j x_j \prec 0\}$ [Mario19].
- Show that $C_1 \subseteq \mathcal{I}(f)^{\mathsf{c}}$. Suppose $\mathbf{e} \in C_1$.
- For every $\mathbf{x} \in \mathbb{R}^n$, we have

$$f(\mathbf{x} + t\mathbf{e}) \quad = \quad \det(A_0 + \sum_{j=1}^n A_j x_j + t \sum_{j=1}^n A_j e_j).$$

- Since $\sum_{j=1}^n A_j e_j \succ 0$, we obtain

$$f(\mathbf{x} + t\mathbf{e}) \quad = \quad \det(\sum_{j=1}^n A_j e_j) \det\left((\sum_{j=1}^n A_j e_j)^{-1/2} (A_0 + \sum_{j=1}^n A_j x_j)(\sum_{j=1}^n A_j e_j)^{-1/2} + tI \right).$$

- There cannot be a non-real vector $\mathbf{a} + i\mathbf{e}$ s.t $f(\mathbf{a} + i\mathbf{e}) = 0$.
- $\mathbf{e} \in \mathcal{I}(f)^{\mathsf{c}}$.

Quadratic Polynomials

Known Classification

Every real quadric in $\mathbb{R}^n$ is affinely equivalent to a quadric given by one of the three (normal form) types,

$$\begin{array}{lll} \text{(I)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 & \left(1 \le p \le r,\ r \ge 1,\ p \ge \frac{r}{2}\right), \\ \text{(II)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 + 1 & \left(0 \le p \le r,\ r \ge 1\right), \\ \text{(III)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 + z_{r+1} & \left(1 \le p \le r,\ r \ge 1,\ p \ge \frac{r}{2}\right). \end{array}$$

- Let $f \in \mathbb{R}[\mathbf{z}]$ be a quadratic polynomial of the form

$$f = \mathbf{z}^T A \mathbf{z} + \mathbf{b}^T \mathbf{z} + c \qquad (2)$$

  with $A \in \mathsf{sym}_n$, $\mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.
- It is well known that a non-degenerate quadratic form $f \in \mathbb{R}[\mathbf{z}]$ is hyperbolic if and only if A has signature $(n-1, 1)$ [Gårding59]
- There are two unbounded components in the complement $\mathcal{I}(f)^{\mathsf{c}}$ [Jörgens, Theobald].

Homogeneous and non-homogeneous

Homogeneous $f = \mathbf{z}^T A\mathbf{z}$	Non-homogeneous $f = \mathbf{z}^T A\mathbf{z} + \mathbf{b}^T\mathbf{z} + c$
f is of type (I) with $r = 1$ -A has Lorentzian signature $(n-1, 1)$	f is of type (II) with $p = 1$ (sub-case I) and f is of type (II) with $p = n - 1$ (sub-case II)
$\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^n \ : \ \mathbf{y}^T A\mathbf{y} < 0\}$	$\mathcal{I}(f) = \begin{cases} \{\mathbf{y} \in \mathbb{R}^n \ : \ y_1^2 - \sum_{j=2}^{r} y_j^2 \leq 1\}, p = 1 \,, \\ \{\mathbf{y} \in \mathbb{R}^n \ : \ \sum_{j=1}^{n-1} y_j^2 > y_n^2\} \cup \{\mathbf{0}\}, p = n \end{cases}$
Hyperbolicity cone is Lorentz cone here	$p = 1$,no suitable connected components $p = n - 1$, Int $S \subset C(\text{in}(f))$ for every full dimensional cone S.

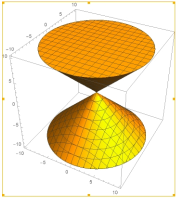

Figure: Lorentz cone:$(y_1, y_2, y_3) = y_3^2 - y_1^2 - y_2^2 > 0$

Back to there

Spectrahedral Representation:Quadratic Polynomials

Hyperbolicity cones are spectrahedral

Theorem

Let $n \geq 3$ and $f = \mathbf{z}^T A\mathbf{z} + \mathbf{b}^T\mathbf{z} + c \in \mathbb{R}[\mathbf{z}]$ be quadratic of the form of type (II) with $p = n - 1$. Then there exists a linear form $\ell(\mathbf{z})$ in $\mathbf{z}$ such that $-\ell(\mathbf{z})^{n-2}\,\mathrm{in}(f)$ has a determinantal representation. In particular, the closure of each unbounded component of $\mathcal{I}(f)^{\mathsf{c}}$ is a spectrahedral cone.

Computational Algorithm

- $-A$ has Lorentzian signature.
- Find normal form of $\mathrm{in}(f) = \mathbf{z}^T A\mathbf{z}$, i.e., $\mathrm{in}(f)(\mathbf{z}) = \mathrm{in}(g)(T\mathbf{z})$ where
  - $g = \sum_{j=1}^{n-1} z_j^2 - z_n^2 + 1$
  - $A = LDL^T$, $D = \mathrm{Diag}(d_1, \ldots, d_{n-1}, d_n)$ such that $d_1, \ldots, d_{n-1} > 0$ and $d_n < 0$ and $T = \mathrm{Diag}(\sqrt{d_1}, \ldots, \sqrt{d_{n-1}}, \sqrt{|d_n|}).L^T$.
- Let $g \in \mathbb{C}[\mathbf{z}]$ and $S \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then, $\mathcal{I}(g(S\mathbf{z})) = S^{-1}\mathcal{I}(g(\mathbf{z}))$.

Computational Algorithm:continuation

- $\mathcal{I}(g)^{\mathsf{c}}$ has the two unbounded conic components
- These are the open Lorentz cone and its negative.
- Their closures are exactly the closures of the hyperbolicity cones of the initial form $\mathrm{in}(g)$ of g.
- Open Lorentz cone has the spectrahedral representation

$$L(\mathbf{z}) := \left(\begin{array}{ccc|c} & & & z_1 \\ & z_n I & & \vdots \\ & & & z_{n-1} \\ \hline z_1 & \cdots & z_{n-1} & z_n \end{array}\right) \succ 0,$$

- Note that $z_n^{n-2}\,\mathrm{in}(g) = -\det(L(\mathbf{z}))$
- $(T\mathbf{z})_n$ provides $\ell(\mathbf{z})$.
- $-\det F(\mathbf{z}) = ((T\mathbf{z})_n)^{n-2}\,\mathrm{in}(f)$

Key Idea: Spectrahedral Representations

The cone K and the conic components of $\mathcal{I}(f)^{\mathsf{c}}$ are spectrahedral, conic stability turns into a problem of spectrahedral containment.

Why? and How? $\operatorname{int} K \subseteq C(\operatorname{in}(f))$

Usual stability: K non-negative orthant, is the positive semidefiniteness region of the linear matrix pencil

$$M^{\geq 0}(\mathbf{x}) \;=\; \sum_{j=1}^{n} M_j^{\geq 0} x_j$$

with $M_j^{\geq 0} = E_{jj}$, where E_{ij} is the matrix with a one in position (i,j) and zeros elsewhere.

PSD-stability: K is the cone of psd matrices. The matrix pencil is

$$M^{\mathrm{psd}}(X) \;=\; \sum_{i,j=1}^{n} M_{ij}^{\mathrm{psd}} x_{ij}$$

with symmetric matrix variables $X = (x_{ij})$ and $M_{ij}^{\mathrm{psd}} = \frac{1}{2}(E_{ij} + E_{ji}) = \frac{1}{2}(e_i e_j^T + e_j e_i^T)$

Positive maps

Set-Up

- Let $U(\mathbf{x}) = \sum_{j=1}^{n} U_j x_j$ and $V(\mathbf{x}) = \sum_{j=1}^{n} V_j x_j$
- The spectrahedra $S_U := \{x \in \mathbb{R}^n \ : \ U(\mathbf{x}) \succeq 0\}$, and $S_V := \{x \in \mathbb{R}^n \ : \ V(\mathbf{x}) \succeq 0\}$ are cones.
- Let $\mathcal{U} = \operatorname{span}(U_1, \ldots, U_n) \subseteq \mathsf{Herm}_k$ (or sym_k) and $\mathcal{V} = \operatorname{span}(V_1, \ldots, V_n) \subseteq \mathsf{Herm}_k$ (or sym_l).
- If $U_1, \ldots, U_n$ are linearly independent, then the linear mapping $\Phi_{UV} : \mathcal{U} \to \mathcal{V}$, $\Phi_{UV}(U_i) := V_i$, $1 \leq i \leq n$, is well defined.
- A linear map $\Phi \ : \ \mathcal{U} \to \mathcal{V}$ is called *positive* if $\Phi(U) \succeq 0$ for any $U \in \mathcal{U}$ with $U \succeq 0$ for given two linear subspaces $\mathcal{U} \subseteq \mathsf{Herm}_k$ and $\mathcal{V} \subseteq \mathsf{Herm}_l$ (or $\mathcal{U} \subseteq \mathcal{S}_k$ and $\mathcal{V} \subseteq \mathcal{S}_l$).
- The *d-multiplicity map* Φ_d on the set of all Hermitian $d \times d$ block matrices with symmetric $n \times n$-matrix entries is defined by

$$(A_{ij})_{i,j=1}^{d} \ \mapsto \ \left(\Phi\left(A_{ij}\right)\right)_{i,j=1}^{d}.$$

- The map Φ is called *d-positive* if the d-multiplicity map Φ_d (viewed as a map on a Hermitian matrix space) is a positive map.
- Φ is called *completely positive* if Φ_d is a positive map for all $d \geq 1$.

Spectrahedral Containment

Let $U_1, \ldots, U_n \subset \mathsf{Herm}_k$ (or, $U_1, \ldots, U_n \subset \mathsf{sym}_k$, respectively) be linearly independent and $S_U \neq \emptyset$. Then for the properties

1. the semidefinite feasibility problem

$$C = (C_{ij})_{i,j=1}^{k} \succeq 0 \text{ and } V_p = \sum_{i,j=1}^{k} (U_p)_{ij} C_{ij} \text{ for } p = 1, \ldots, n \tag{3}$$

   has a solution with Hermitian (respectively symmetric) matrix C,
2. Φ_{UV} is completely positive,
3. Φ_{UV} is positive,
4. $S_U \subseteq S_V$ (*containment problem for spectrahedra*),

the implications and equivalences (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4) hold, and if $\mathcal{U}$ contains a positive definite matrix, (1) $\Longleftrightarrow$ (2).

Determinantal polynomials

Main Result

Let $f = \det(A_0 + \sum_{j=1}^n A_j z_j)$ with Hermitian matrices $A_0, \ldots, A_n$ be a degree d determinantal polynomial such that

- $\mathrm{in}(f)$ is irreducible and
- there exists $\mathbf{e} \in \mathbb{R}^n$ with $\sum_{j=1}^n A_j e_j \succ 0$.

Let $M(\mathbf{x}) = \sum_{j=1}^n M_j x_j$ with symmetric $l \times l$-matrices be a pencil of the cone K. If there exists a Hermitian block matrix $C = (C_{ij})_{i,j=1}^l$ with blocks C_{ij} of size $d \times d$ and

$$C = (C_{ij})_{i,j=1}^l \succeq 0, \quad \forall p = 1, \ldots, n \ : \ \sigma A_p = \sum_{i,j=1}^l (M_p)_{ij} C_{ij} \tag{4}$$

for some $\sigma \in \{-1, 1\}$, then f is K-stable.

Idea:

$$A^h(\mathbf{x}) = (I \cdots I)(M(\mathbf{x}) * C) \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix}$$

Deciding whether such a block matrix C exists is a semidefinite feasibility problem.

Borcea-Brändén stability criterion

Revisit: the stability criterion for a determinantal polynomial.

- View Choi matrix C as a block diagonal matrix $C = (C_{ij})_{i=1}^{l}$ with diagonal blocks C_{ii} of size $d \times d$ and vanishing non-diagonal blocks C_{ij} $(i \neq j)$.
- such that

$$A_p = C_{pp} \quad \text{for } p = 1, \ldots, n,$$

- stability criterion in main Theorem is satisfied if and only if the matrices $A_1, \ldots, A_n$ are positive semidefinite

> The determinantal polynomial $f(\mathbf{z}) = \det(A_0 + \sum_{j=1}^{n} A_j z_j)$ is real stable or the zero polynomial if and only if the matrices $A_1, \ldots, A_n$ are positive semidefinite.

Example

- Let $g(z_1, z_2, z_3) := 31z_1^2 + 32z_1z_3 + 8z_3^2 - 8z_1z_2 - 16z_2^2$.
- A determinantal representation of g is given by $\det \begin{pmatrix} 4z_1 + 2z_3 & z_1 + 4z_2 \\ z_1 + 4z_2 & 8z_1 + 4z_3 \end{pmatrix}$, and
- at $\mathbf{z} = (0, 0, 1)^T$, the matrix polynomial is positive definite.
- Let $M(\mathbf{x})$ denote the linear matrix pencil of the psd cone sym_2^+.
- Then the psd-stability of g follows from the above Theorem
- by the Choi matrix

$$C = \begin{pmatrix} 4 & 1 & 0 & 2 \\ 1 & 8 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 4 \end{pmatrix} \succeq 0.$$

Open problems

- Characterization (includes certification)
- Closure property:operations which preserve conic stability)
- Connection with log-concave (Lorentzian) polynomials
- generalize *Hyperbolic programming*?

Thank You for your attention!

Definition

Let f be a degree n polynomial with real roots $\{\alpha_i\}$, and let g be degree n or $n-1$ with real roots $\{\beta_i\}$ (ignoring β_n in the degree $n-1$ case). We say that g interlaces f if their roots alternate, i.e.,

$$\beta_n \leq \alpha_n \leq \beta_{n-1} \leq \ldots \beta_1 \leq \alpha_1,$$

and the largest root belongs to f.

If there is a single g which interlaces a family of polynomials $f_1, \ldots, f_m$, we say that they have a common interlacing. Back to there

Theorem

Let $f_1, \ldots, f_m$ be degree n polynomials. All of their convex combinations $\sum_{i=1}^{m} \mu_i f_i$ have real roots if and only if they have a common interlacing.

- For example, $f << g$, if the univariate polynomials $f(x + t\mathbf{e}), g(x + t\mathbf{e})$ are in proper position for all $\mathbf{x} \in \mathbb{R}^n, \mathbf{e} \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$.