Non-local binary games with noisy EPR states are decidable

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Workshop "Quantum protocols: Testing & Quantum PCPs"

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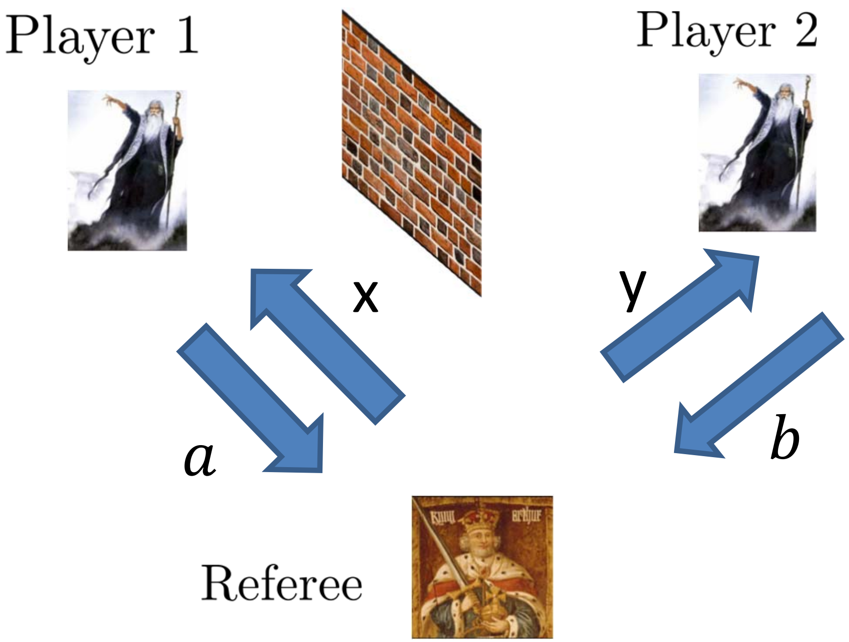
Classical non-local games



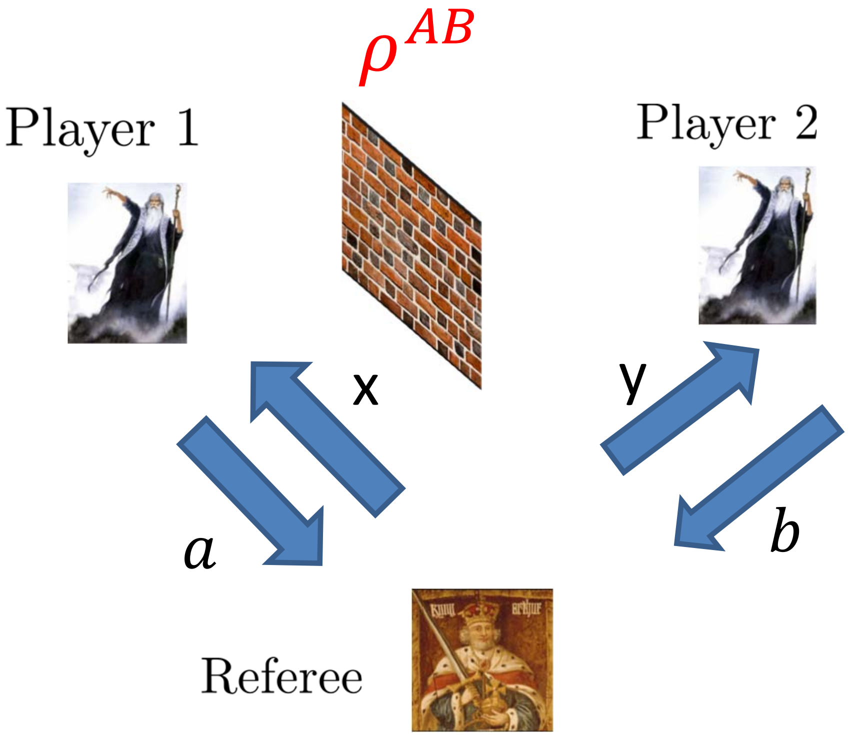
- $(x, y) \sim \mu$
- $V(x, y, a, b) \in \{0(lose), 1(win)\}$
- $\omega(G)$:=the highest probability to win the game

- It captures the complexity of many optimization problems.

- Computing $\omega(G)$ is NP-hard

PCP theorem.
It is NP-hard to approximate $\omega(G)$ to a constant precision.

Quantum non-local games

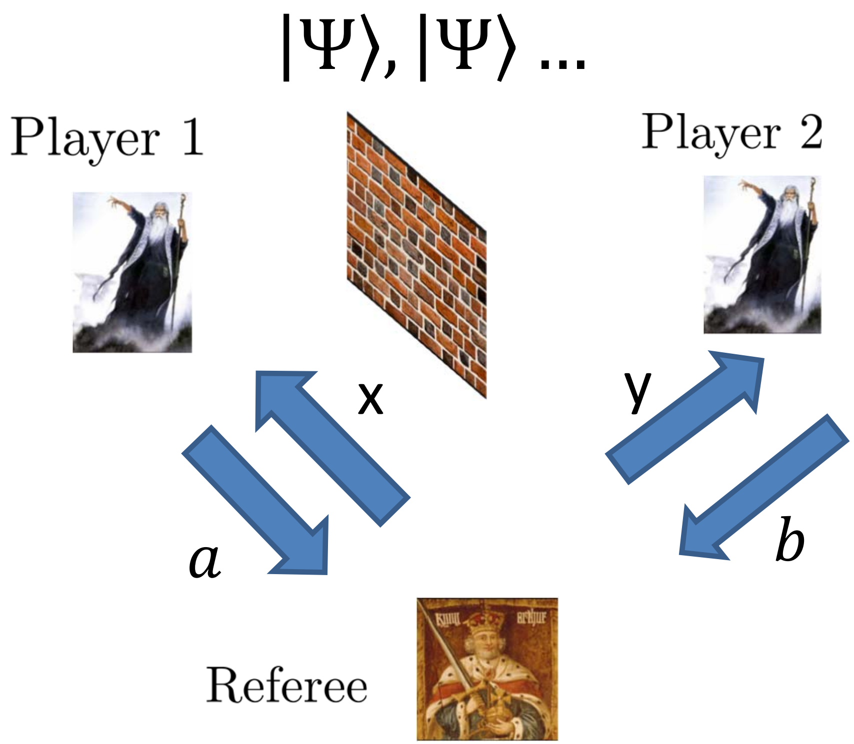


Q. What it the complexity of computing $\omega^*(G)$?

- [JNWVY] RE-complete to approximate $\omega^*(G)$ to a constant precision

$\omega^*(G)$:=the highest probability to win the game

Quantum non-local games



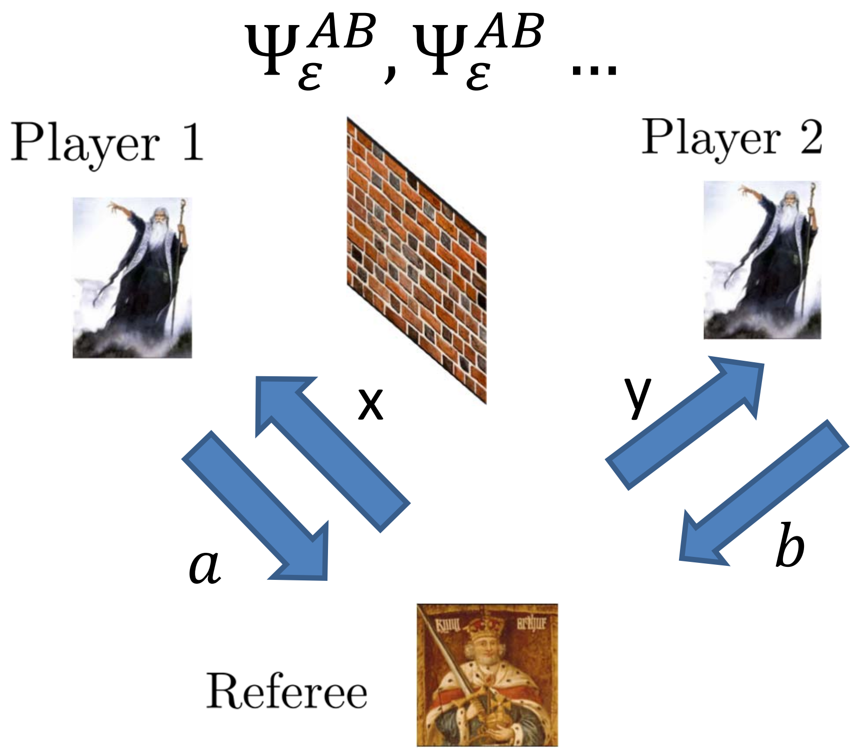
Q. What it the complexity of computing $\omega^*(G)$?

- [JNWVY] RE-complete to approximate $\omega^*(G)$ to a constant precision even if only sharing EPR states.

$\omega^*(G)$:=the highest probability to win the game

Q. Are strategies for non-local games are robust against noise?

Quantum non-local games



$$\Psi_{\varepsilon} := (1-\varepsilon)|\Psi\rangle\langle\Psi| + \varepsilon\frac{I}{4}$$

- $\omega(G, \Psi_{\varepsilon}, n) :=$ highest prob. if n copies of Ψ_{ε} are provided.
- $\omega^{*}(G, \Psi_{\varepsilon}) = \lim_{n\to\infty} \omega(G, \Psi_{\varepsilon}, n)$

- Ji. et al. undecidable to a constant precision if $\varepsilon = 0$
- The entanglement tends to infinity if ε is sufficiently small
- Are strategies robust against the noise from the shared states?

Binary games with noisy EPR states

Main result

For any **binary** game, if the players only share ε −noisy EPR states,

$$D = \exp\left(poly\left(|X|, |Y|, \exp\left(\frac{1}{\varepsilon}, \frac{1}{\delta}\right)\right)\right)$$

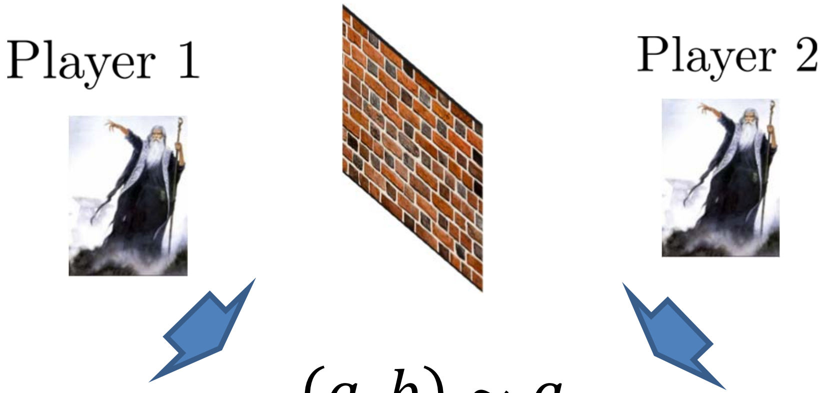
copies of the states are sufficient to achieve the optimal value to precision δ.

- Previous algorithms for $\omega^*(G)$ are via convex optimization, such as quantum XOR games, quantum unique games.
- We simulate high-entangled strategies by low-entangled strategies

High level of the proofs

Q. What distributions can the players sample?

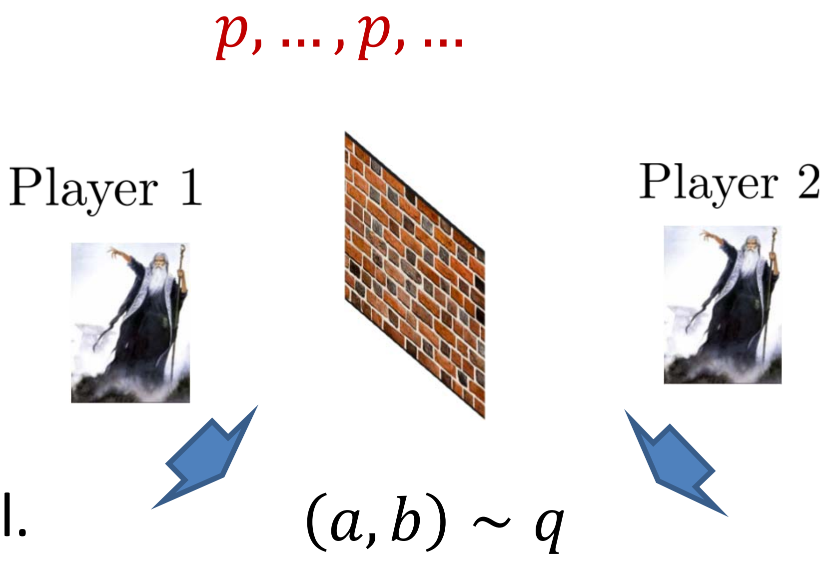
$\Psi_\varepsilon^{AB}, \Psi_\varepsilon^{AB} \ldots$



$(a, b) \sim q$

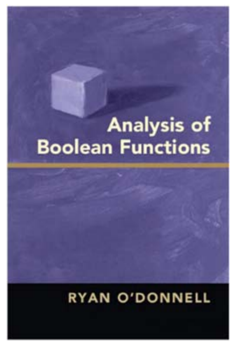
High level of the proofs

Q. What distributions can the players sample?



- Non-interactive simulation of Joint distributions
- It roots back to 70's by Gács et.al.

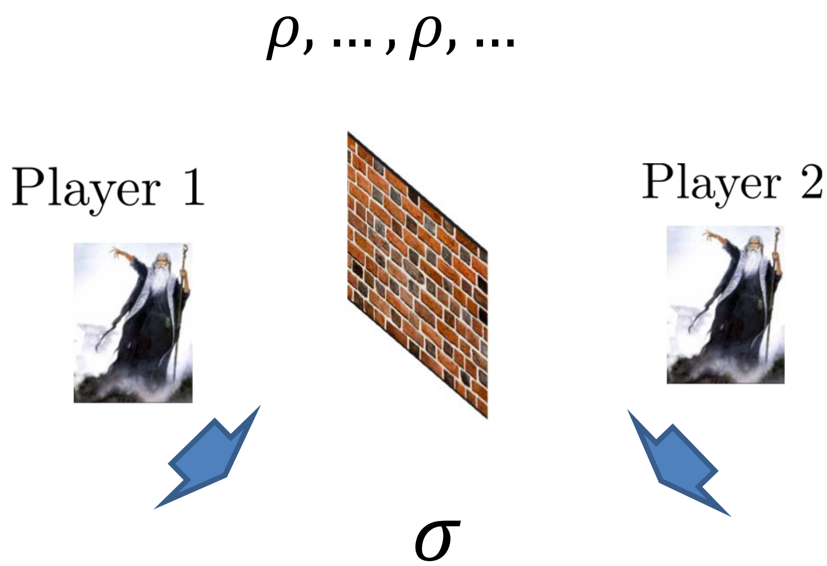
- The decidability was resolved recently by Ghazi et al. in a series of work using Boolean analysis
  - Yes. $\exists\, n$ and strategy to jointly sample q' s.t. $||q - q'||_1 \leq \varepsilon$
  - No. $\forall\, n$ and strategy jointly sample q', $||q - q'||_1 > 2\,\varepsilon$



High level of the proofs

Q. What quantum states can players create?

- Local state transformation
- It was first studied by Beigi



High level of the proofs

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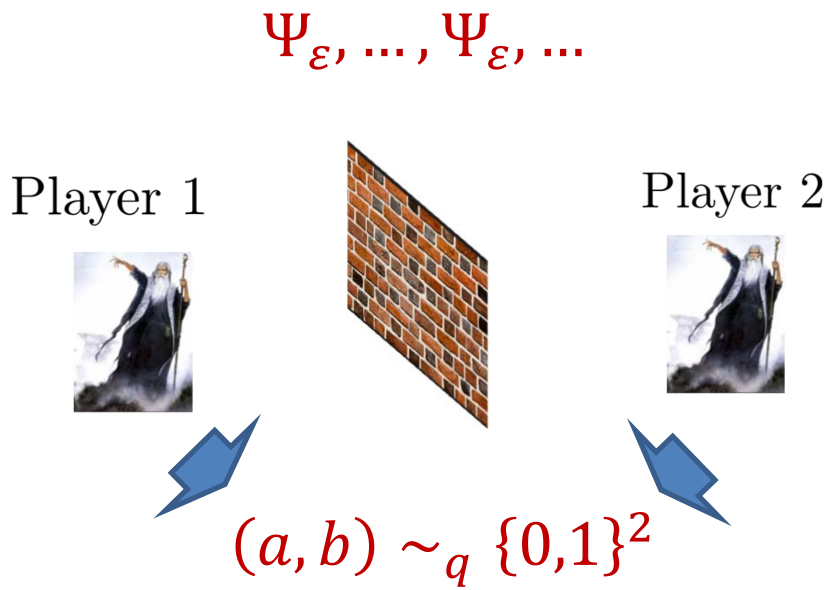

- Local state transformation
- It was first studied by Beigi
- We proved that it is decidable if sharing noisy EPR states and the target state is a binary distribution
- It is via generalizing Boolean analysis to Hermitian operators and random operators.

High level of the proofs

Player 1: $(P, I-P) \in H_2^{\otimes n}$
Player 2: $(Q, I-Q) \in H_2^{\otimes n}$

→

Player 1: $(P', I-P') \in H_2^{\otimes D}$
Player 2: $(Q', I-Q') \in H_2^{\otimes D}$

Requirements.

$\exists D \forall n$

1. $0 \leq P', Q' \leq I$
2. $\Pr[Player\ 1\ outputs\ 1] = \frac{Tr\ P'}{2^D} \approx \frac{Tr\ P}{2^n}$
3. $\Pr[Player\ 2\ outputs\ 1] = \frac{Tr\ Q'}{2^D} \approx \frac{Tr\ Q}{2^n}$
4. $\Pr[both\ output\ 1] = Tr(P \otimes Q)\Psi_\varepsilon^{\otimes n} \approx Tr(P' \otimes Q')\Psi_\varepsilon^{\otimes D}$
5. P' only depends on P and Q' only depends on Q (applicable for non-local games)

Fourier expansion

Pauli basis

$\sigma_0 = I, \sigma_1 = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$P \in \mathbb{M}_2$: $P = \hat{P}(0)\,\sigma_0 + \hat{P}(1)\,\sigma_1 + \hat{P}(2)\,\sigma_2 + \hat{P}(3)\,\sigma_3$

$P \in \mathbb{H}_2$: $\hat{P}(i)$'s are reals

$P \in \mathbb{M}_2^{\otimes n}: P = \sum_{s \in \{0,1,2,3\}^n} \hat{P}(s) \sigma_s$.

$\sigma_s \coloneqq \sigma_{s_1} \otimes \cdots \otimes \sigma_{s_n}$

Parseval identity $\sum_s \left|\hat{P}(s)\right|^2 = |||P|||_2^2$

Normalized p-norm $|||P|||_{\mathrm{p}} \coloneqq \left(\frac{1}{2^n} ||P||_p^p\right)^{1/p}$

Constructions

1. Smooth the operators by applying noise operators(depolarizing channels) T_ρ

$$P = \sum_{s \in \{0,1,2,3\}^n} \widehat{P}(s)\sigma_s$$

Constructions

1. Smooth the operators by applying noise operators(depolarizing channels) T_ρ

Exponentially small

$$P_1 = T_\rho P = \sum_{s \in \{0,1,2,3\}^n} \hat{P}(s) \rho^{|s|} \sigma_s = P_1^{\leq d} + \cancel{P_1^{>d}}$$

2. Number of high influential coordinates are bounded

$$Inf_i\,(P) := |||P - P_{-i} \otimes I/2|||_2^2 = \sum_{i:s_i \neq 0} \left|\hat{P}(s)\right|^2$$

$$\sum_i Inf_i\,(P) = \sum_{i,s:s_i \neq 0} \left|\hat{P}(s)\right|^2 \leq \deg(P)\,|||P|||_2^2 \leq \deg(P)$$

Constructions

2. Number of high influential coordinates are bounded

$$P_1 = \sum_{s \in \{0,1,2,3\}^n} \widehat{P_1}(s) \overbrace{\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3}}^{\text{H}} \otimes \cdots\cdots \otimes \sigma_{s_{n-1}} \otimes \sigma_{s_n}$$

3. Substitute all basis elements in low influential coordinates by independent Gaussian variables

$$\left(\boldsymbol{g}_0^{(h+1)} = 1, \boldsymbol{g}_1^{(h+1)}, \boldsymbol{g}_2^{(h+1)}, \boldsymbol{g}_3^{(h+1)}, \dots, \boldsymbol{g}_3^{(n)}\right)$$

$$P_1 = \sum_{s \in \{0,1,2,3\}^n} \widehat{P_1}(s) \sigma_{s_1} \otimes \cdots \otimes \underset{\Uparrow \atop \boldsymbol{g}_{s_{h+1}}^{(h+1)}}{\sigma_{s_{h+1}}} \otimes \underset{\Uparrow \atop \boldsymbol{g}_{s_{h+2}}^{(h+2)}}{\sigma_{s_{h+2}}} \otimes \cdots\cdots \otimes \underset{\Uparrow \atop \boldsymbol{g}_{s_n}^{(n)}}{\sigma_{s_n}}$$

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$$\boldsymbol{P_2} = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s) \sigma_{s_1} \otimes \cdots \otimes \sigma_{s_h} \cdot \left(\boldsymbol{g}_{s_{h+1}}^{(h+1)} \cdot \boldsymbol{g}_{s_{h+2}}^{(h+2)} \cdots \boldsymbol{g}_{s_n}^{(n)}\right)$$

Constructions

3. Substitute all basis elements in low influential coordinates by independent Gaussian variables

$$\boldsymbol{P_2} = \sum_{s \in \{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1} \otimes \cdots \otimes \sigma_{s_h} \cdot \left(\boldsymbol{g}_{s_{h+1}}^{(h+1)} \cdot \boldsymbol{g}_{s_{h+2}}^{(h+2)} \cdots \boldsymbol{g}_{s_n}^{(n)}\right)$$

$$= \sum_{s \in \{0,1,2,3\}^n} f_{s_h}(\vec{\boldsymbol{g}})\sigma_{s_1} \otimes \sigma_{s_2} \otimes \cdots \otimes \sigma_{s_h}$$

f_{s_h} ’s are low-deg polynomials.

4. Dimension reduction for low-deg polynomials on Gaussian space to reduce the # of Gaussian variables.

$$\boldsymbol{P_3} = \sum_{s \in \{0,1,2,3\}^{n_0}} f'_{s_h}\left(\boldsymbol{g}_{s_{h+1}} \dots, \boldsymbol{g}_{s_{n_0}}\right)\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \otimes \cdots \otimes \sigma_{s_h}$$

Constructions

4. Dimension reduction for low-deg polynomials on Gaussian space to reduce the # of Gaussian variables.

$$\boldsymbol{P_3} = \sum_{s\in\{0,1,2,3\}^{n_0}} {f'}_{s_h}\left(\boldsymbol{g}_{s_{h+1}} \dots, \boldsymbol{g}_{s_{n_0}}\right) \sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \otimes \cdots \otimes \sigma_{s_h}$$

$$= \sum_{\substack{s\in\{0,1,2,3\}^h \\ t\in Z_{\geq 0}^m}} \widehat{P_3}(s,t)\, H_{t_1}(\boldsymbol{g}_{h+1}) H_{t_2}(\boldsymbol{g}_{h+2}) \dots H_{t_m}(\boldsymbol{g}_{h+m}) \sigma_{s_1} \otimes \cdots \otimes \sigma_{s_h}$$

$H_i(x)$: Hermite polynomials

5. With some extra work, substitute Gaussian vars. by matrix bases

$$P_4 = \sum_{s\in\{0,1,2,3\}^D} \hat{P}_4(s) \sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \otimes \cdots \otimes \sigma_{s_D}$$

Constructions

Noise is required.

Smooth

$\sigma_i \to \rho^{\delta_{i,0}} \sigma_i$

$$P = \sum_{s\in\{0,1,2,3\}^n} \hat{P}(s)\sigma_s$$

Bounded # of high inf. Coord.

$$P_1 = \sum_{s\in\{0,1,2,3\}^n} \hat{P}(s)\rho^{|s|}\sigma_s = P_1^{\leq d} + P_1^{>d}$$

Exponentially small

$$P_1 = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \otimes \sigma_{s_4} \otimes \cdots \otimes \sigma_{s_{n-1}} \otimes \sigma_{s_n}$$

Substitution

$\sigma_i \to \boldsymbol{g}_i^{(j)}$

$$\boldsymbol{P_2} = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \cdot \boldsymbol{g}_{s_4}^{(4)} \cdots \boldsymbol{g}_{s_{n-1}}^{(n-1)} \cdot \boldsymbol{g}_{s_n}^{(n)}$$

Dimension reduction

$$\boldsymbol{P_3} = \sum_{s\in\{0,1,2,3\}^{n_0}} f_{s_h}\left(\boldsymbol{g}_{s_{h+1}} \ldots, \boldsymbol{g}_{s_{n_0}}\right)\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \otimes \cdots \otimes \sigma_{s_h}$$

Substitution

$\boldsymbol{g}_i \to \sigma_i$

$$P_4 = \sum_{s\in\{0,1,2,3\}^D} \hat{P}_4(s)\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \otimes \cdots \otimes \sigma_{s_D}$$

Constructions

Step 1	Smooth	Low degree
Step 2	Regularization	Bounded # of high influential coordinates
Step 3	Substitution: Pauli→ Gaussian	Bounded # of matrix bases & unbounded # of gaussian
Step 4	Dimension reduction	Bounded # of matrix bases & bounded # of gaussian
Step 5	Substitution: Gaussian→Pauli	Bounded # of matrix bases

Constructions

Step 1	Smooth (unital & linear)	Low degree
Step 2	Regularization	Bounded # of high influential coordinates
Step 3	Substitution Pauli→ Gaussian	Bounded # of matrix bases & unbounded # of gaussian
Step 4	Dimension reduction (unital & linear)	Bounded # of matrix bases & bounded # of gaussian
Step 5	Substitution Gaussian→Pauli	Bounded # of matrix bases

Constructions

Step 1	Smooth (unital & linear)	Low degree
Step 2	Regularization	Bounded # of high influential coordinates
Step 3	Substitution & **rounding**: Pauli→ Gaussian	Bounded # of matrix bases & unbounded # of gaussian
Step 4	Dimension reduction (unital & linear)	Bounded # of matrix bases & bounded # of gaussian
Step 5	Substitution & **rounding**: Gaussian→Pauli	Bounded # of matrix bases

Constructions

Step 1	Smooth (unital & linear)	Low degree
Step 2	Regularization	Bounded # of high influential coordinates
Step 3	Substitution & rounding: Pauli→ Gaussian (nonlinear)	Bounded # of matrix bases & unbounded # of gaussian
Step 4	Dimension reduction (unital & linear)	Bounded # of matrix bases & bounded # of gaussian
Step 5	Substitution & rounding: Gaussian→Pauli (nonlinear)	Bounded # of matrix bases

Constructions

Step 1	Smooth (unital & linear)	Low degree
Step 2	Regularization	Bounded # of high influential coordinates
Step 3	Substitution & rounding: Pauli→ Gaussian (nonlinear) Quantum Invariance principle & Quantum hypercontractive ineq.	Bounded # of matrix bases & unbounded # of gaussian
Step 4	Dimension reduction (unital & linear)	Bounded # of matrix bases & bounded # of gaussian
Step 5	Substitution & rounding: Gaussian→Pauli (nonlinear) Quantum Invariance principle & Quantum hypercontractive ineq.	Bounded # of matrix bases

Constructions

3. Substitute all bases in low influential coordinates by independent **Gaussian variables** ($\boldsymbol{g}_{\mathbf{0}}^{(j)} := 1, \boldsymbol{g}_{\mathbf{1}}^{(j)}, \boldsymbol{g}_{\mathbf{2}}^{(j)}, \boldsymbol{g}_{\mathbf{3}}^{(j)}$).

$$P_1 = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1} \otimes \cdots \otimes \sigma_{s_{h+1}} \otimes \sigma_{s_{h+2}} \otimes \cdots\cdots \otimes \sigma_{s_n}$$

$$\sigma_{s_{h+1}} \Leftarrow \boldsymbol{g}_{s_{h+1}}^{(h+1)}, \quad \sigma_{s_{h+2}} \Leftarrow \boldsymbol{g}_{s_{h+2}}^{(h+2)}, \quad \sigma_{s_n} \Leftarrow \boldsymbol{g}_{s_n}^{(n)}$$

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Requirement.

1. $0 \le P', Q' \le I$
2. $\Pr[Player\ 1\ outputs\ 1] = \frac{Tr\ P'}{2^D} \approx \frac{Tr\ P}{2^n}$
3. $\Pr[Player\ 2\ outputs\ 1] = \frac{Tr\ Q'}{2^D} \approx \frac{Tr\ Q}{2^n}$
4. $\Pr[both\ output\ 1] = \mathrm{Tr}(P' \otimes Q')\Psi_\varepsilon^{\otimes n} \approx Tr(P \otimes Q)\Psi_\varepsilon^{\otimes D}$
5. P' only depends on P and Q' only depends on Q (applicable for non-local games)

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Requirement.

1. $0 \leq \boldsymbol{P}_2, \boldsymbol{Q}_2 \leq I$
2. $\Pr[Player\ 1\ outputs\ 1] = E\left[\frac{Tr\ \boldsymbol{P}_2}{2^D}\right] \approx \frac{Tr\ P}{2^n}$
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4. $\Pr[both\ output\ 1] = E\left[\mathrm{Tr}(\boldsymbol{P}_2 \otimes \boldsymbol{Q}_2)\Psi_\varepsilon^{\otimes h}\right] \approx Tr(P \otimes Q)\Psi_\varepsilon^{\otimes n}$
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2. $\Pr[Player\ 1\ outputs\ 1] = E\left[\frac{Tr\ \boldsymbol{P}_2}{2^D}\right] \approx \frac{Tr\ P}{2^n}$ √

3. $\Pr[Player\ 2\ outputs\ 1] = E\left[\frac{Tr\ \boldsymbol{Q}_2}{2^D}\right] \approx \frac{Tr\ Q}{2^n}$ √

4. $\Pr[both\ output\ 1] = E\left[\mathrm{Tr}(\boldsymbol{P}_2\otimes\boldsymbol{Q}_2)\Psi_\varepsilon^{\otimes h}\right] \approx \mathrm{Tr}(P\otimes Q)\Psi_\varepsilon^{\otimes n}$ √

5. $\boldsymbol{P}_{\mathbf{2}}$ only depends on P and $\boldsymbol{Q}_{\mathbf{2}}$ only depends on Q (applicable for non-local games) √

Constructions

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Requirement.

1. $E[\text{dist}(\boldsymbol{P}_2, \text{ POVM elements})] \approx 0$. Same for $\boldsymbol{Q}_2$
2. $\Pr[Player\ 1\ outputs\ 1] = E\left[\frac{Tr\ \boldsymbol{P}_2}{2^D}\right] \approx \frac{Tr\ P}{2^n}$ √
3. $\Pr[Player\ 2\ outputs\ 1] = E\left[\frac{Tr\ \boldsymbol{Q}_2}{2^D}\right] \approx \frac{Tr\ Q}{2^n}$ √
4. $\Pr[both\ output\ 1] = E\left[\text{Tr}(\boldsymbol{P}_2 \otimes \boldsymbol{Q}_2)\Psi_\varepsilon^{\otimes h}\right] \approx \text{Tr}(P \otimes Q)\Psi_\varepsilon^{\otimes n}$ √
5. $\boldsymbol{P}_{\mathbf{2}}$ only depends on P and $\boldsymbol{Q}_{\mathbf{2}}$ only depends on Q (applicable for non-local games) √

Quantum invariance principle

2. Choose a bounded set H of coordinates with high influence

$$P_1 = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s) \overbrace{\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3}}^{H} \otimes \cdots\cdots \otimes \sigma_{s_{n-1}} \otimes \sigma_{s_n}$$

3. Substitute all bases in low influential coordinates by Gaussian variables

$$\boldsymbol{P_2} = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1} \otimes \cdots \otimes \sigma_{s_h} \cdot \left(\boldsymbol{g}_{s_{h+1}}^{(h+1)} \cdot \boldsymbol{g}_{s_{h+2}}^{(h+2)} \cdots \boldsymbol{g}_{s_n}^{(n)}\right)$$

Quantum Invariance Principle:

$$E\left[|||\boldsymbol{P}_2 - POVM\ elements|||_2^2\right] \leq \varepsilon$$

Quantum invariance principle

$$P_1 = \sum_{s \in \{0,1,2,3\}^n} \widehat{P_1}(s) \sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_3 \otimes \cdots \otimes \sigma_{s_{n-1}} \otimes \sigma_{s_n} \Rightarrow \boldsymbol{P_2} = \sum_{s \in \{0,1,2,3\}^n} \widehat{P_1}(s) \sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \otimes \cdots \otimes \boldsymbol{g}_{s_{n-1}}^{(n-1)} \sigma_0 \otimes \boldsymbol{g}_{s_n}^{(n)} \sigma_0$$

$$R(x) := \begin{cases} 1 & \text{if } x \geq 1 \\ x & \text{if } 0 \leq x < 1\,. \\ 0 & \text{otherwise} \end{cases}$$

$$R(P_1) = P_1 \;\; \Rightarrow E[\|\|R(\boldsymbol{P}_2) - \boldsymbol{P}_2\|\|_2^2] \approx 0$$

$$\zeta(x) := |x - R(x)|^2 = \begin{cases} x^2 & \text{if } x \leq 0 \\ 0 & \text{if } 0 \leq x \leq 1\,. \\ (1-x)^2 & \text{if } x \geq 1. \end{cases}$$

$$Tr\zeta(P_1) = 0 \Rightarrow E[Tr\, \zeta(\boldsymbol{P}_2)] \approx 0$$

Quantum invariance principle

$$P_1 = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1}\otimes\sigma_{s_2}\otimes\sigma_{3}\otimes\cdots\otimes\sigma_{s_{n-1}}\otimes\sigma_{s_n} \Rightarrow \boldsymbol{P_2} = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1}\otimes\sigma_{s_2}\otimes\sigma_{s_3}\otimes\cdots\otimes\boldsymbol{g}_{s_{n-1}}^{(n-1)}\sigma_0\otimes\boldsymbol{g}_{s_n}^{(n)}\sigma_0$$

Goal: $Tr\zeta(P_1) = 0 \Rightarrow E[Tr\,\zeta(\boldsymbol{P}_2)] \approx 0$

$$\boldsymbol{P}^{(i)} = \sum_{s} \widehat{P_1}(s)\sigma_{s_1}\otimes\cdots\otimes\sigma_{s_h}\otimes\boldsymbol{g}_{s_{h+1}}^{(h+1)}\sigma_0\otimes\cdots\otimes\boldsymbol{g}_{s_{h+i}}^{(h+i)}\sigma_0\otimes\sigma_{s_{h+i+1}}\otimes\cdots$$

$$\boldsymbol{Q} := \sum_{s_{h+i+1}=0} \widehat{P_1}(s)\sigma_{s_1}\otimes\cdots\otimes\sigma_{s_h}\otimes\boldsymbol{g}_{s_{h+1}}^{(h+1)}\sigma_0\otimes\cdots\otimes\boldsymbol{g}_{s_{h+i+1}}^{(h+i)}\sigma_0\otimes\sigma_{s_{h+i+1}}\otimes\cdots$$

$$\boldsymbol{R}^{(i)} := \sum_{s_{h+i+1}\neq 0} \widehat{P_1}(s)\sigma_{s_1}\otimes\cdots\otimes\sigma_{s_h}\otimes\boldsymbol{g}_{s_{h+1}}^{(h+1)}\sigma_0\otimes\cdots\otimes\boldsymbol{g}_{s_{h+i}}^{(h+i)}\sigma_0\otimes\sigma_{s_{h+i+1}}\otimes\cdots$$

$$\boldsymbol{P}^{(i)} = \boldsymbol{Q} + \boldsymbol{R}^{(i)}, \boldsymbol{P}^{(i+1)} = \boldsymbol{Q} + \boldsymbol{R}^{(i+1)}$$

$$P_1 = \boldsymbol{P}^{(0)}, \boldsymbol{P}_2 = \boldsymbol{P}^{(n-h)}$$

Quantum invariance principle

$$P_1 = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_3 \otimes \cdots \otimes \sigma_{s_{n-1}} \otimes \sigma_{s_n} \Rightarrow \boldsymbol{P_2} = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \otimes \cdots \otimes \boldsymbol{g}_{s_{n-1}}^{(n-1)} \sigma_0 \otimes \boldsymbol{g}_{s_n}^{(n)} \sigma_0$$

$$Tr\zeta(P_1) = 0 \Rightarrow E[Tr\,\zeta(\boldsymbol{P}_2)] \approx 0$$

$$\boldsymbol{P}^{(i)} = \boldsymbol{Q} + \boldsymbol{R}^{(i)}, \boldsymbol{P}^{(i+1)} = \boldsymbol{Q} + \boldsymbol{R}^{(i+1)}$$

$E\left[|||\boldsymbol{R}^{(i)}|||_2^2\right] = \sum_{i:s_i\neq 0}\left|\widehat{P_1}(s)\right|^2 = Inf_i(P_1)$ is small!

$$E\left[Tr\,\zeta\left(\boldsymbol{P}^{(i)}\right)\right] \approx E\left[Tr\,\zeta\left(\boldsymbol{P}^{(i+1)}\right)\right]$$

Taylor expanding $Tr\,\zeta(\cdot)$ at $\boldsymbol{Q}$

$\zeta(\cdot) \in C^1$ not very smooth !

$$\zeta(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ 0 & \text{if } 0 \le x \le 1\,. \\ (1-x)^2 & \text{if } x \ge 1. \end{cases}$$

Quantum invariance principle

$$P_1 = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_3 \otimes \cdots \otimes \sigma_{s_{n-1}} \otimes \sigma_{s_n} \Rightarrow \boldsymbol{P_2} = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \otimes \cdots \otimes \boldsymbol{g}_{s_{n-1}}^{(n-1)}\sigma_0 \otimes \boldsymbol{g}_{s_n}^{(n)}\sigma_0$$

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$$E\left[Tr\,\zeta\left(\boldsymbol{P}^{(i)}\right)\right] \approx E\left[Tr\,\zeta\left(\boldsymbol{P}^{(i+1)}\right)\right]$$

- $\zeta_\lambda \approx \zeta$
- $\zeta_\lambda \in C^2$

$$E\left[Tr\,\zeta_\lambda\left(\boldsymbol{P}^{(i)}\right)\right] \approx E\left[Tr\,\zeta_\lambda\left(\boldsymbol{P}^{(i+1)}\right)\right]$$

$$\zeta_\lambda(x) := \begin{cases} x^2 + \frac{1}{3}\lambda^2 & \text{if } x \le -\lambda \\ \frac{(\lambda - x)^3}{6\lambda} & \text{if } -\lambda \le x \le \lambda \\ 0 & \text{if } \lambda \le x \le 1-\lambda \\ \frac{(x-1+\lambda)^3}{6\lambda} & \text{if } 1-\lambda \le x \le 1+\lambda \\ (1-x)^2 + \frac{1}{3}\lambda^2 & \text{if } x \ge 1+\lambda. \end{cases}.$$

Quantum invariance principle

$$P_1 = \sum_{s \in \{0,1,2,3\}^n} \widehat{P_1}(s) \sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_3 \otimes \cdots \otimes \sigma_{s_{n-1}} \otimes \sigma_{s_n} \Rightarrow \boldsymbol{P_2} = \sum_{s \in \{0,1,2,3\}^n} \widehat{P_1}(s) \sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \otimes \cdots \otimes \boldsymbol{g}_{s_{n-1}}^{(n-1)} \sigma_0 \otimes \boldsymbol{g}_{s_n}^{(n)} \sigma_0$$

$$Tr\zeta(P_1) = 0 \Rightarrow E[Tr\, \zeta_\lambda(\boldsymbol{P}_2)] \approx 0$$

$$\boldsymbol{P}^{(i)} = \boldsymbol{Q} + \boldsymbol{R}^{(i)}, \boldsymbol{P}^{(i+1)} = \boldsymbol{Q} + \boldsymbol{R}^{(i+1)}$$

Fréchet derivative:

$$Df(A)(B) = \frac{d}{dt} f(A + tB)\Big|_{t=0}$$

$$\mathrm{E}\left[Tr\, \zeta_\lambda\left(\boldsymbol{P}^{(i)}\right)\right] - \mathrm{E}[Tr\, \zeta_\lambda(\boldsymbol{Q})] =$$

$$\mathrm{E}\left[Tr\, f_1(\boldsymbol{Q})\boldsymbol{R}^{(i)}\right] + \mathrm{E}\left[Tr\, \left(f_2(\boldsymbol{Q})\boldsymbol{R}^{(i)} f_3(\boldsymbol{Q})\boldsymbol{R}^{(i)} + f_4(\boldsymbol{Q})\left(\boldsymbol{R}^{(i)}\right)^2\right)\right] + 3^{rd} \text{ term}$$

$$\mathrm{E}\left[Tr\, \zeta_\lambda\left(\boldsymbol{P}^{(i+1)}\right)\right] - \mathrm{E}[Tr\, \zeta_\lambda(\boldsymbol{Q})] =$$

$$\mathrm{E}\left[Tr\, f_1(\boldsymbol{Q})\boldsymbol{R}^{(i+1)}\right] + \mathrm{E}\left[Tr\left(f_2(\boldsymbol{Q})\boldsymbol{R}^{(i+1)} f_3(\boldsymbol{Q})\boldsymbol{R}^{(i+1)} + f_4(\boldsymbol{Q})\left(\boldsymbol{R}^{(i+1)}\right)^2\right)\right] + 3^{rd} \text{ term}$$

Quantum invariance principle

$$P_1 = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_3 \otimes \cdots \otimes \sigma_{s_{n-1}} \otimes \sigma_{s_n} \Rightarrow \boldsymbol{P_2} = \sum_{s\in\{0,1,2,3\}^n} \widehat{P_1}(s)\sigma_{s_1} \otimes \sigma_{s_2} \otimes \sigma_{s_3} \otimes \cdots \otimes \boldsymbol{g}_{s_{n-1}}^{(n-1)}\sigma_0 \otimes \boldsymbol{g}_{s_n}^{(n)}\sigma_0$$

$$\mathrm{E}\left[Tr\,\zeta_\lambda\left(\boldsymbol{P}^{(i)}\right)\right] - \mathrm{E}[Tr\,\zeta_\lambda(\boldsymbol{Q})] =$$
$$\mathrm{E}\left[Tr\,f_1(\boldsymbol{Q})\boldsymbol{R}^{(i)}\right] + \mathrm{E}\left[Tr\,\left(f_2(\boldsymbol{Q})\boldsymbol{R}^{(i)}f_3(\boldsymbol{Q})\boldsymbol{R}^{(i)} + f_4(\boldsymbol{Q})\left(\boldsymbol{R}^{(i)}\right)^2\right)\right] + 3^{\mathrm{rd}}\text{ term}$$

$$\mathrm{E}\left[Tr\,\zeta_\lambda\left(\boldsymbol{P}^{(i+1)}\right)\right] - \mathrm{E}[Tr\,\zeta_\lambda(\boldsymbol{Q})] =$$
$$\mathrm{E}\left[Tr\,f_1(\boldsymbol{Q})\boldsymbol{R}^{(i+1)}\right] + \mathrm{E}\left[Tr\left(f_2(\boldsymbol{Q})\boldsymbol{R}^{(i+1)}f_3(\boldsymbol{Q})\boldsymbol{R}^{(i+1)} + f_4(\boldsymbol{Q})\left(\boldsymbol{R}^{(i+1)}\right)^2\right)\right] + 3^{\mathrm{rd}}\text{ term}$$

- 1st order and 2nd order match.
- 3rd order terms $\leq O\left(E\left[|||\boldsymbol{R}^{(i)}|||_3^3\right]\right)$
- $E[Tr\,\zeta(\boldsymbol{P}_2)] \leq O\left(\sum_i E\left[|||\boldsymbol{R}^{(i)}|||_3^3\right]\right)$
- $E\left[|||\boldsymbol{R}^{(i)}|||_2^2\right] = Inf_i(P)$ is small

Quantum hypercontractive inequality

Quantum hypercontractive inequality

$$|||R|||_2 \le |||R|||_3 \le 3^{\frac{\deg(R)}{2}} |||R|||_2$$

$$E[Tr\,\zeta(\boldsymbol{P}_2)] \le O\left(\sum_i E\left[|||\boldsymbol{R}_2^{(i)}|||_3^3\right]\right)$$

$$\le 3^{O(d)} \sum_i E\left[|||\boldsymbol{R}_2^{(i)}|||_2^2\right]^{3/2}$$

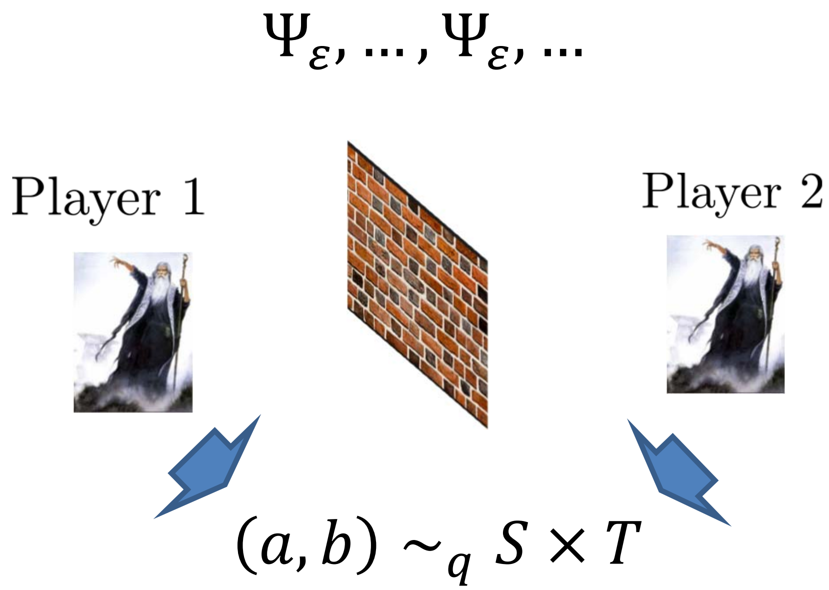
$$\le 3^{O(d)} \sum_i Inf_i(P)^{\frac{3}{2}}$$

$$\le 3^{O(d)} \cdot \sum_j Inf_j(P) \cdot \left(\max_i \sqrt{Inf_i(P)}\right)$$

$$\le 3^{O(d)} d \cdot \epsilon$$

Future directions & open problems

- Target distributions are non-binary?

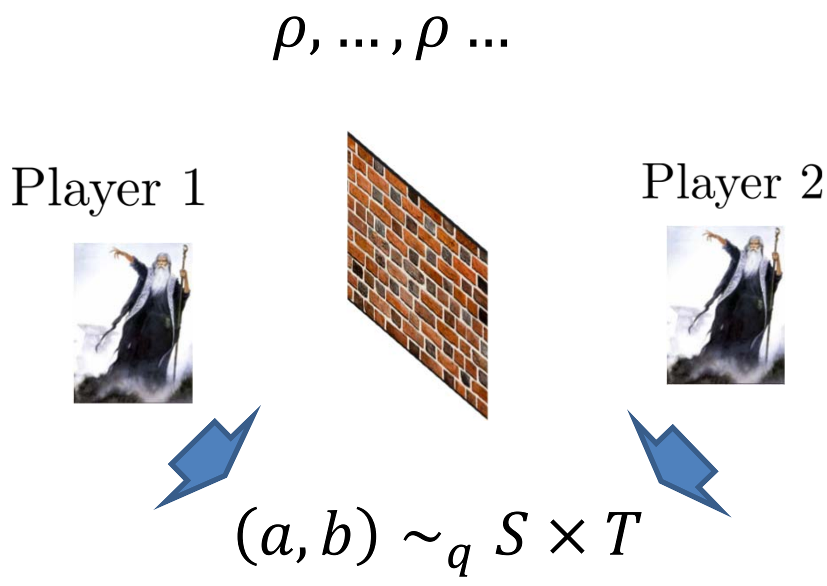


Future directions & open problems

- Target distributions are non-binary?

- Decidability of general mono-state games

Conjecture: decidable if shared states are noisy.



Future directions & open problems

- Target distributions are non-binary?
- Decidability of general mono-state games

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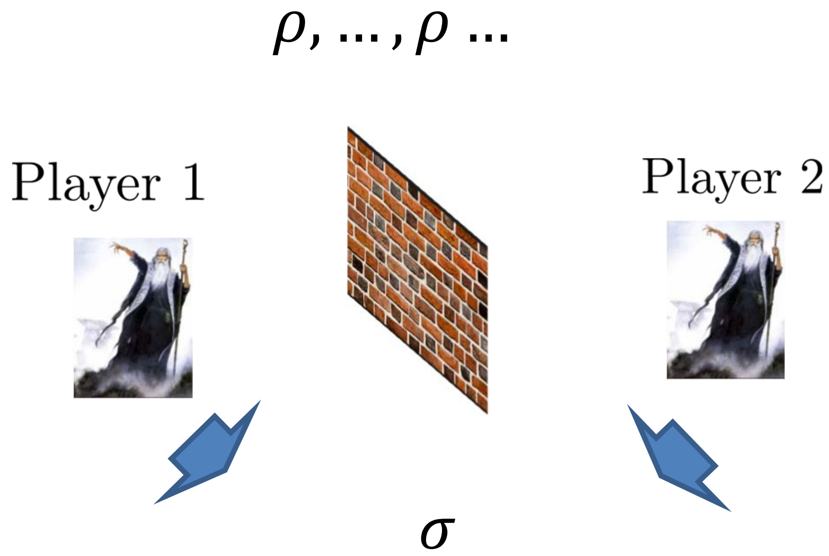

- Decidability of local state transformations and non-local games with quantum answers and quantum predicates?

Future directions & open problems

- High dimensional quantum hypercontractive inequality?
- Multivariate quantum invariance principle?
- Computability of tensored quantities: quantum channel capacities, regularized entanglement measures, quantum information complexity etc.
- Other applications of Boolean analysis on quantum information theory and quantum complexity theory

Thank you