Quantum PCPs meet derandomization

Alex Bredariol Grilo

CWI

QuSoft

joint work with Dorit Aharonov

Randomness helps...

- Communication complexity
- Query complexity
- Cryptography
- Non-local games

... in all cases?

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- Under believable assumptions, randomness does not increase computational power
  - If pseudo-random number generators exist, then probabilistic algorithms are as powerful as deterministic ones
- It should be true, but it is an open problem for decades!

A glimpse of its hardness

Polynomial identity testing problem

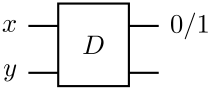
Input: Polynomial $p : \mathbb{F}_q^n \to \mathbb{F}_q$ of degree $d(n)$
Output: Decide if $\forall x_1, ..., x_n \in \mathbb{F}_q, p(x_1, ..., x_n) = 0$

- Simple randomized algorithm
  - Pick $x_1, ..., x_n$ uniformly at random from $\mathbb{F}_q^n$
  - If $p \neq 0$, $Pr[p(x_1, ..., x_n) = 0] \leq \frac{d}{q}$
- How to find such a "witness" deterministically?

MA vs. NP

MA vs. NP

Problem $L \in \text{NP}$



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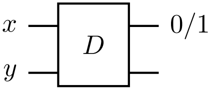

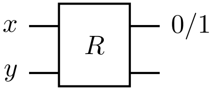

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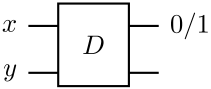

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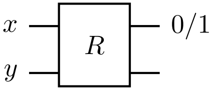

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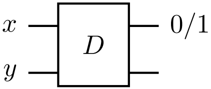

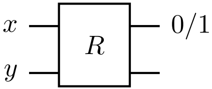

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Our result (informal)

Quantum PCP[1] conjecture is true iff MA = NP.

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Input: Local Hamiltonians H_1, ... H_m, each acting on k out of a n-qubit system; $H = \sum_i H_i$

yes-instance: $\langle\psi| H |\psi\rangle \leq \alpha m$ for some $|\psi\rangle$

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How hard is this problem?

Quantum proofs

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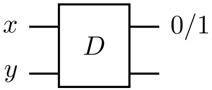

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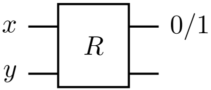

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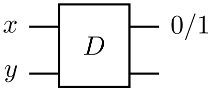

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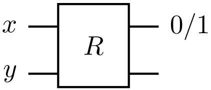

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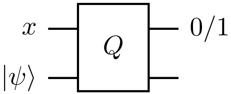

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- for some $\beta - \alpha \geq \frac{1}{poly(n)}$: QMA-complete (Kitaev'99)
- for $\beta - \alpha$ is a constant: open problem
  - Quantum PCP conjecture: it is also QMA-hard

Restrictions on the Hamiltonians

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  Model of first D-Wave machines

- Projector P_i onto the groundspace of H_i
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  - $|\phi_{i,j}\rangle = |T_{i,j}\rangle$, where $T_{i,j} \subseteq \{0,1\}^k$ and $\frac{1}{\sqrt{T_{i,j}}} \sum_{x \in T_{i,j}} |x\rangle$
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- Our work: if β is constant, it is in NP

Outline

Back to NP vs. MA

Theorem (BT '08)

Deciding if Unif. Stoq. LH is has groundenergy 0 or inverse polynomial is MA-complete.

Theorem (This work)

Deciding if Unif. Stoq. LH is has ground energy 0 or constant is NP-complete.

Back to NP vs. MA

Corollary

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Proof.

Problem in MA □

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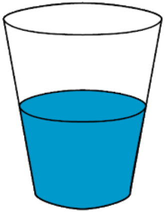

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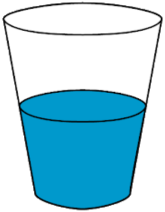

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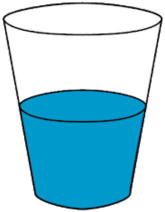

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Example

- 3-qubit system
  - $P_{1,2} = P_{2,3} = |\Psi^+\rangle\langle\Psi^+| + |\Phi^+\rangle\langle\Phi^+|$
  - $P_{1,3} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|$

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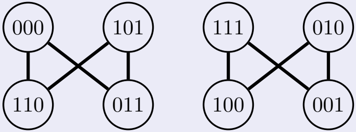

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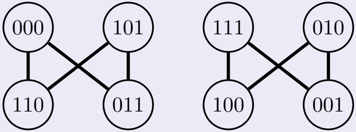

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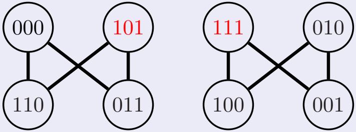

Stoquastic Hamiltonians in MA (BT '08)

- MA-verification:
  1. Given a initial string x_0
  2. Perform a random walk for poly(n) steps.
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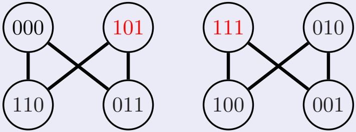

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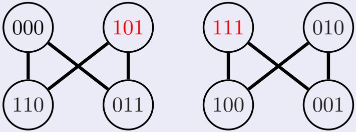

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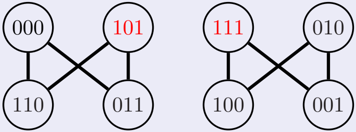

$\mathbf{00}0 \xrightarrow{P_{1,2}}$

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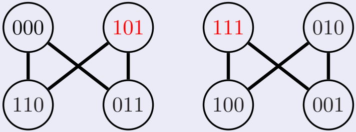

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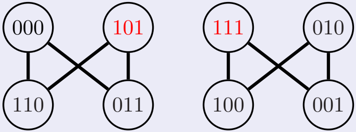

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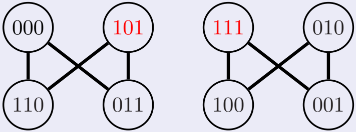

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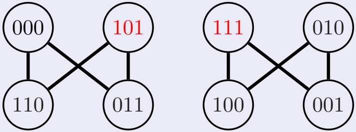

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Stoquastic Hamiltonians in MA (BT '08)

Theorem

If H has groundenergy 0 and x_0 is in some groundstate of H, then the verifier never reaches a bad string.

If H has groundenergy $1/poly(n)$, then the random-walk rejects with constant probability for any x_0.

Very frustrated case

Theorem

If H is εm frustrated for some constant ε, then from every initial string there is a constant-size path that leads to a bad string.

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Check if any of the constant-size paths reaches a bad string.

- For yes-instances, this is never the case (BT' 08).
- For no-instances, this is always the case (previous theorem).

□

Structure of the proof

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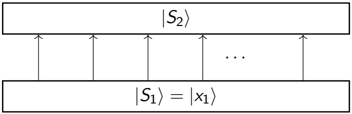
1. There is a **constant-depth** "circuit" of non-overlapping projectors that achieves state with a bad string
    1. Construct circuit layer by layer: either there is a bad string, or we can add a new layer that brings us closer to a bad string
2. From the constant-depth circuit, we can use a **lightcone**-argument to retrieve a constant-size path.

States with a bad string

$\|S_1\rangle = \|x_1\rangle$	1 string

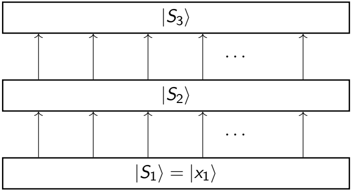
States with a bad string



$(1+\frac{\varepsilon}{4})^{\frac{2\varepsilon m}{2kd}}$ strings

1 string

States with a bad string



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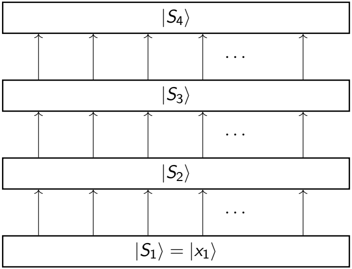

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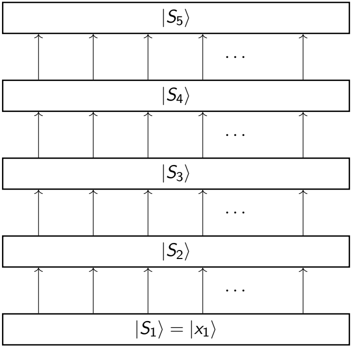

$(1+\frac{\varepsilon}{4})^{\frac{5\varepsilon m}{2kd}}$ strings

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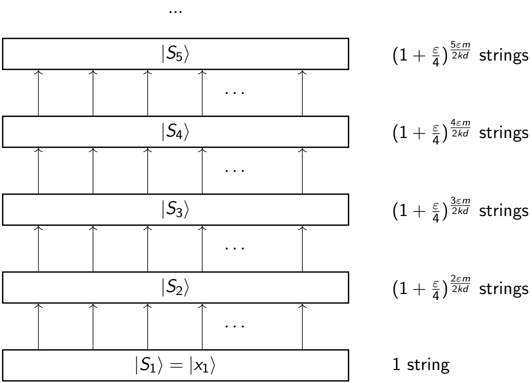

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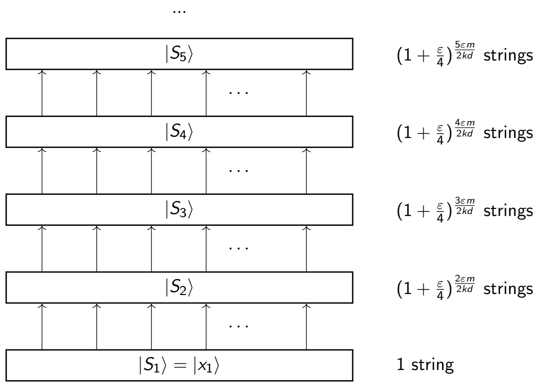

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Pick $L = \frac{\varepsilon m}{2kd}$, the frustration is at least $\frac{\varepsilon}{2}$, there is a constant T such that $|S_T\rangle = |+\rangle^{\otimes n}$

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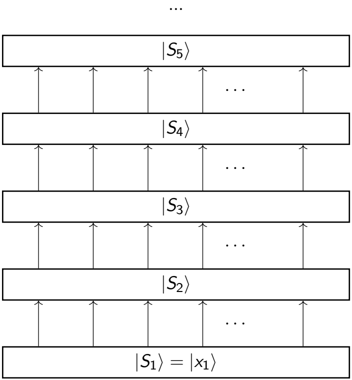

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Finding a bad string

Pick $L = \frac{\varepsilon m}{2kd}$, the frustration is at least $\frac{\varepsilon}{2}$, there is a constant T such that $|S_T\rangle = |+\rangle^{\otimes n} \Rightarrow$ there is a bad string in $|S_T\rangle$.

One term expansion

Lemma

Assume

- $|S\rangle$ *be a subset state*
- P *be a k-local stoquastic projector*

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Intuition of the proof

If P does not contain a bad string, the frustration must come from missed strings and $P |S\rangle$ will "add" such strings.

Non-overlapping expansion

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Proof.

Apply one-term expansion lemma l times. □

From shallow non-overlapping transitions to short paths

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If a bad string is reached after a constant number of non-overlapping projections, then there is a constant-size path to a bad string.

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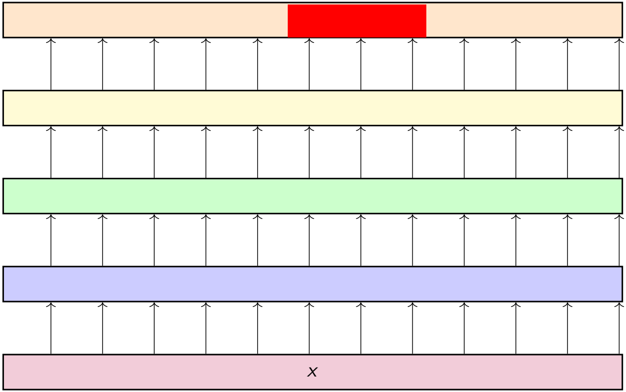

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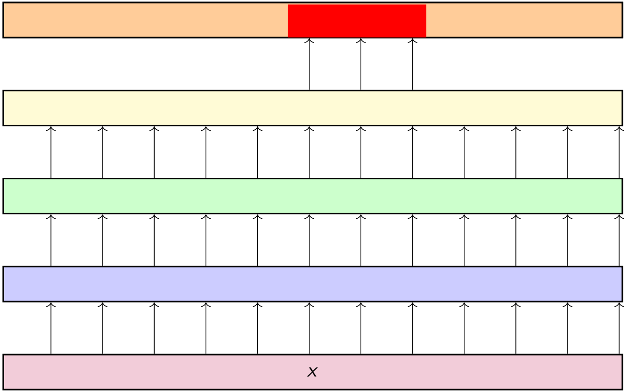

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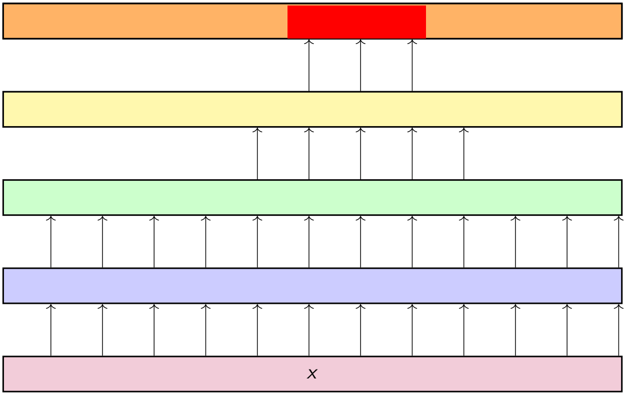

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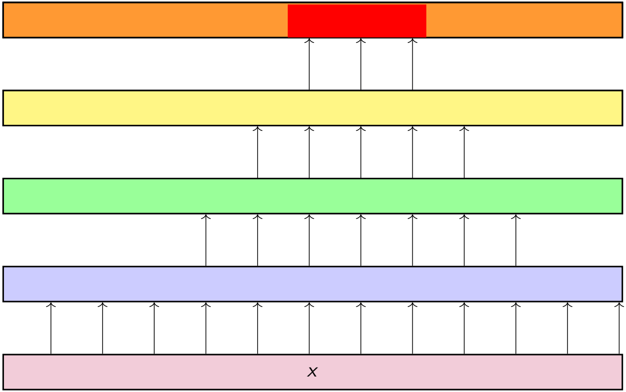

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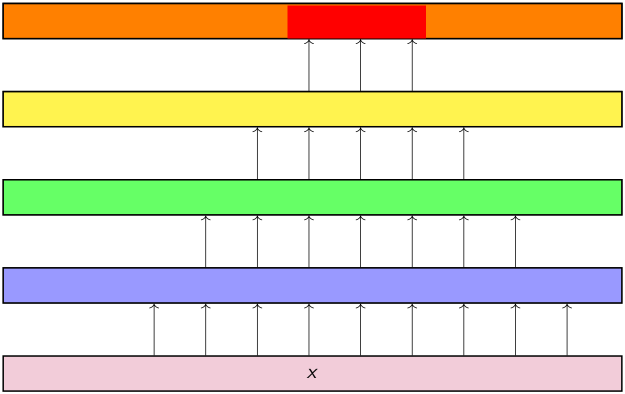

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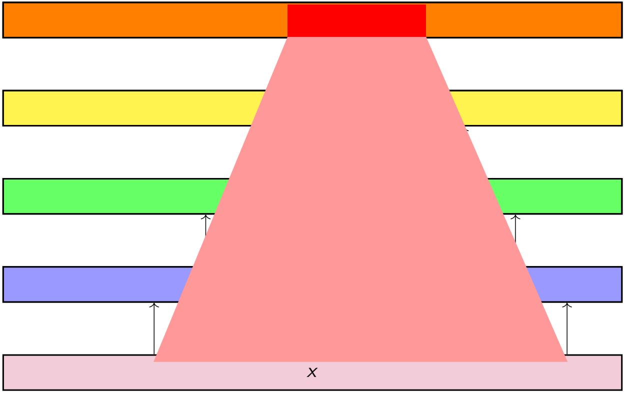

Related results

- Extension to tiny frustration *vs*. large frustration
  - In the tiny frustration case, there is a string far from all bad strings
- "Classical" definition of the problem
  - SetCSP: extension of CSPs for *sets of strings*
  - Gap amplification for SetCSP $\Leftrightarrow$ MA = NP
  - Details in arxiv:2003.13065

Open problems

- Prove/disprove Stoquastic PCP conjecture
  - Smaller promise gap in NP
  - Completeness parameter far from 0
- Non-uniform case
  - There are highly frustrated Hamiltonians with no bad strings
  - Frustration comes from incompatibility of amplitudes
    $\sqrt{1-\varepsilon}\,|0\rangle + \sqrt{\varepsilon}\,|1\rangle$ vs. $\sqrt{\varepsilon}\,|0\rangle + \sqrt{1-\varepsilon}\,|1\rangle$
  - Add more tests
    BT has a consistency test, but not clear that it is “local”
- Advances in adiabatic evolution of stoquastic Hamiltonians

Thank you for your attention!