A polynomial time CVP algorithm for lattices related to zonotopes

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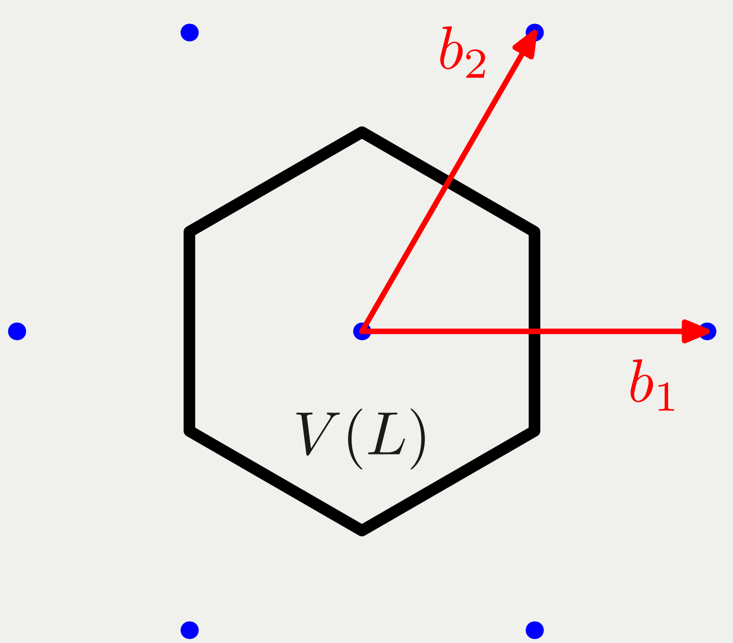
The closest vector problem

Input: Lattice $L = \left\{ \sum_{i=1}^{r} \alpha_i b_i : \alpha_i \in \mathbb{Z} \right\} \subseteq \mathbb{R}^n$ given by basis $b_1, \ldots, b_r$, and vector $x \in \mathbb{R}^n$ (wlog $x \in \operatorname{span} L$)

Output: lattice vector $u \in L$ with $|x - u| = \min_{v \in L} |x - v|$

Geometric interpretation:



Voronoi cell

$V(L) = \{x \in \mathbb{R}^n : |x| \leq |x - v| \text{ for } v \in L\}$

$V(L)$ tiles $\mathbb{R}^n$ by lattice translates $v + V(L)$

CVP: In which tile $u + V(L)$ does x lie?

Some words about algorithms and complexity

CVP has been studied intensively. Collection of important results:

CVP is NP-hard (van Emde Boas, 1981)

CVP is NP-hard to approximate within a factor $n^{c/\log\log n}$ for $c > 0$
(Dinur, Kindler, Raz, Safra, 2003)

Approximating CVP within a factor of $\sqrt{n}$ lies in NP $\cap$ co-NP.
(Aharonov, Regev, 2005)

$\tilde{O}(4^n)$-time, $\tilde{O}(2^n)$-space algorithm for exact CVP
(Micciancio, Voulgaris, 2013)

$2^{n+o(1)}$-time and space algorithm for exact CVP
(Aggarwal, Dadush, Stephens-Davidowitz, 2015)

If $V(L)$ compactly representable: reduced space complexity of MV algorithm
(Hunkenschröder, Reuland, Schymura, 2019)

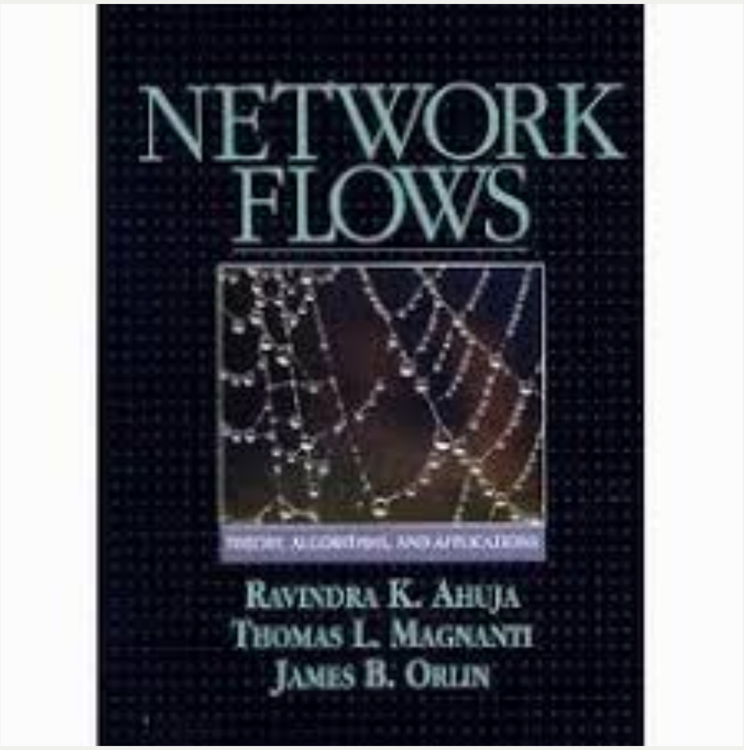
Special cases

Polynomial time algorithms for special classes of lattices:

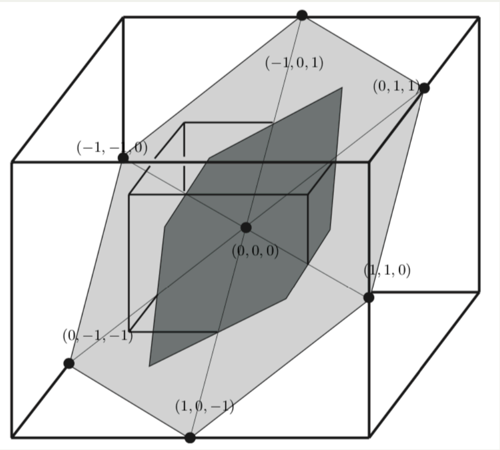
lattices of Voronoi's first kind
(McKilliam, Grant, Clarkson, 2014)

tensor products $A_n \otimes A_m$
(Ducas, van Woerden, 2018)

both based on network flows



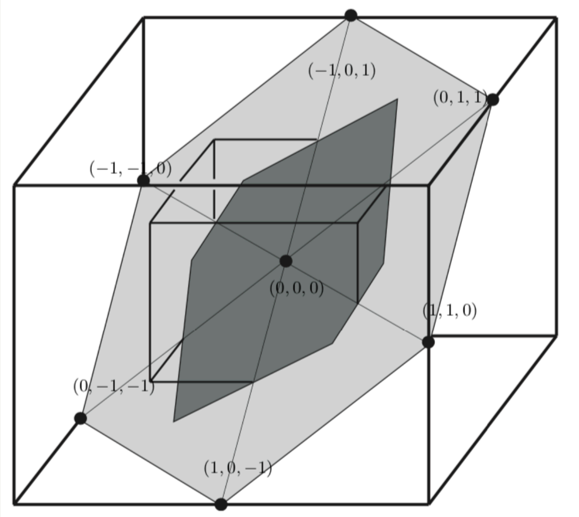
Goal: Unify and generalize these two cases.



Lattices and zonotopes - Setup

Consider *zonotopal lattices*: lattices L where $V(L)$ is a zonotope

$$\begin{aligned} \text{zonotope } &= \text{projection of regular cube } [-1,+1]^m \\ &= \text{Minkowski sum of line segments } \textstyle\sum_{i=1}^m [-s_i,+s_i] \end{aligned}$$



$\mathcal{L} \subseteq \mathbb{R}^m$ linear subspace

for $x \in \mathcal{L}$ define $\operatorname{supp} x = \{i : x_i \neq 0\}$

$x \in \mathcal{L} \setminus \{0\}$ is called *elementary* $\iff$ (i) x has minimal support in $\mathcal{L}$

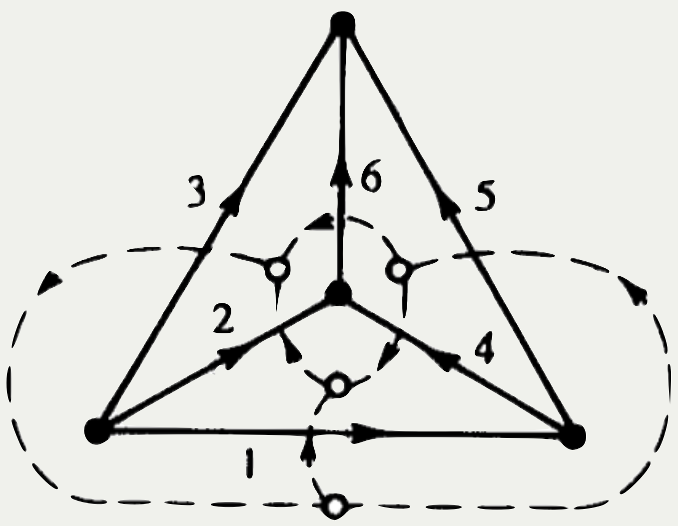
(ii) $x \in \{-1, 0, +1\}^m$

$\mathcal{L}$ is called *regular* $\iff$ for all $y \in \mathcal{L} \setminus \{0\}$ with minimal support there is $\alpha \in \mathbb{R}$ and $x \in \mathcal{L}$ elementary so that $y = \alpha x$

Lattices and zonotopes - Main examples

come from digraphs $D = (V, A)$

$M \in \{-1, 0, +1\}^{V \times A}$ incidence matrix



graphical case

$\mathcal{L}(D) = \{x \in \mathbb{R}^A : Mx = 0\}$ is regular

elementary vectors = circuits (unoriented)

cographical case

$\mathcal{L}(D)^\perp = \{y \in \mathbb{R}^A : x^\mathsf{T} y = 0 \text{ for all } x \in \mathcal{L}(D)\}$ is also regular

elementary vectors = minimal cuts (cocircuits)

Lattices and zonotopes - Voronoi cells

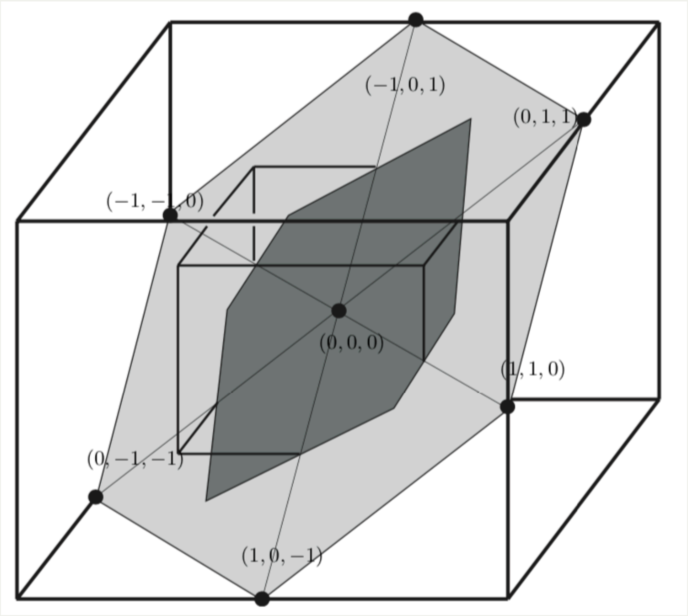
Regular subspaces define lattices $L = \mathcal{L} \cap \mathbb{Z}^m$

Positive vector $g \in \mathbb{R}^m_{>0}$ defined Euclidean structure on L $(x, y)_g = \sum_{i=1}^{m} g_i x_i y_i$

Facts about Voronoi cell of L:

$\{\text{Voronoi vectors}\} = \{\text{facet normals of } V(L)\} = \{\text{elementary vectors}\}$

$V(L) = \pi([-1/2, 1/2]^m)$ $\pi : \mathbb{R}^m \to \mathcal{L}$ orthogonal projection



L is zonotopal lattice; if $V(L)$ is zonotope, then L comes by this construction

Lattices and zonotopes - All examples

(a) Lattices of Voronoi's first kind

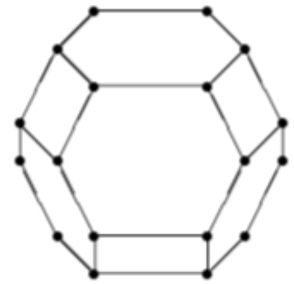
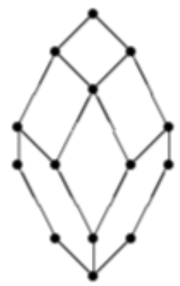
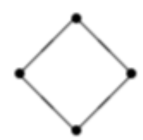
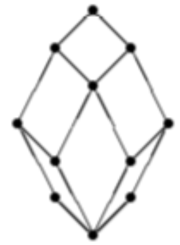
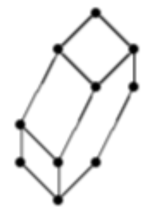
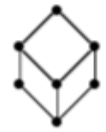
defined by obtuse superbasis $b_1, \ldots, b_n, b_{n+1}$
where $b_1, \ldots, b_n$ forms a lattice basis
where $b_i^\top b_j \leq 0$ if $i \neq j$ and $\sum_{i=1}^{n+1} b_i = 0$.

This defines a graph D with vertices $b_1, \ldots, b_{n+1}$
and weighted edges (b_i, b_j) if $i < j$ with weight $g_{ij} = -b_i^\top b_j$.

Then: $L(D)^\perp \simeq L$

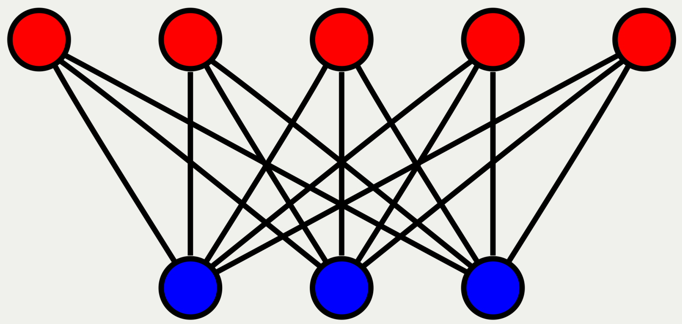
In particular: $L(C_{n+1})^\perp = A_n$ and $L(K_{n+1})^\perp = A_n^*$

3-dimensional lattices

d	Delone Graph	Polytope	Form	Name
6	 K_4		$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$	**TRUNCATED OCTAHEDRON**
5	 $K_4 - 1$		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	**HEXA-RHOMBIC DODECAHEDRON**
4	 C_4		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	**RHOMBIC DODECAHEDRON**
4	 $K_3 + 1$		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	**HEXAGONAL PRISM**
3	 $1 + 1 + 1$		$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	**CUBE**

(b) tensor products

$$A_m \otimes A_n = L(K_{m+1,n+1})$$



(c) Seymour (1980)

Classification: Every zonotopal lattice is 1-,2-,3-sum of cographical, graphical lattices or R_{10} (exceptional 5-dim. lattice)

Minimum mean cycle canceling

Karzanov, McCormick (1997):

Can solve the following problem in polynomial time

$M \in \{-1, 0, +1\}^{n \times m}$ totally unimodular matrix $\quad L = \{v \in \mathbb{Z}^m : Mv = 0\}$

$w_i : \mathbb{R} \to \mathbb{R}$ convex functions, $i = 1, \ldots, m$

minimize $\sum_{i=1}^{m} w_i(v_i)$ subject to $v \in L$

separable convex objective function

Observation: That is a perfect fit for the CVP of zonotopal lattices.

For (L, g) zonotopal lattice and $x \in \mathbb{R}^m$ set

$w_i(v_i) = g_i(v_i - x_i)^2$ convex quadratic

Then: $\sum w_i(v_i) = (x - v, x - v)_g = |x - v|_g^2$

Idea of algorithm

For $v \in L$ and for elementary vector $u \in L$

define cost of u at v by

$$c(v,u) = \sum_{i:u_i=+1} c_i^+(v_i) - \sum_{i:u_i=-1} c_i^-(v_i)$$

where
$$c_i^+(v_i) = g_i(v_i - x_i + 1)^2 - g_i(v_i - x_i)^2$$
$$c_i^-(v_i) = g_i(v_i - x_i)^2 - g_i(v_i - x_i - 1)^2$$

If cost $c(u,v)$ is negative then $v+u$ is closer to x than v:

$$(v+u-x, v+u-x)_g = (v-x, v-x)_g + c(v,u)$$

$$\lambda(v) = \max\left\{0, -\min_{u \text{ elementary}} \frac{c(v,u)}{|\text{supp } u|}\right\}$$

minimizer u defines "minimum mean cycle at v"

1. $\lambda(v) = 0 \iff v$ is closest vector to x
2. Can determine $\lambda(v)$ and u elementary attaining minimum by LP
3. Pivot: $v \leftarrow v + \varepsilon u$ for suitable step size ε
4. λ-value descreases geometrically

$\Longrightarrow$ polynomial time algorithm for CVP