The SIS Problem and Cryptographic Applications

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January 2020
Outline

1. The Short Integer Solution (SIS) Problem
2. Average Case Hardness
3. Efficiency and RingSIS
   - Small modulus
   - Ideal Lattices
4. Cryptographic Applications
   - 1: Compression and Hashing
   - 2: Regularity and Commitment Schemes
   - 3: Linearity and Digital Signatures
1 The Short Integer Solution (SIS) Problem

2 Average Case Hardness

3 Efficiency and RingSIS
   - Small modulus
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4 Cryptographic Applications
   - 1: Compression and Hashing
   - 2: Regularity and Commitment Schemes
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CVP and dual lattice

- Lattice $\Lambda$, target $t = v + e$
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- Dual lattice $\Lambda^* = \mathcal{L}(D)$.
The Short Integer Solution (SIS) Problem

CVP and dual lattice

- Lattice $\Lambda$, target $t = v + e$
- Dual lattice $\Lambda^* = L(D)$.
- Syndrome of $t$:
  
  $$s = \langle D, t \rangle \mod 1$$
  
  $$= \langle D, v \rangle + \langle D, e \rangle \mod 1$$
  
  $$= \langle D, e \rangle \mod 1.$$
**CVP and dual lattice**

- Lattice $\Lambda$, target $t = v + e$
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- $e$ belongs to coset
  
  $t + \Lambda = \{x : \langle D, x \rangle = s \mod 1\}$
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  \[
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  \]
- $e$ belongs to coset $t + \Lambda = \{ x : \langle D, x \rangle = s \mod 1 \}$

Problem (Syndrome Decoding)

*Find shortest $e$ such that* $\langle D, e \rangle = s \mod 1$
SIS/LWE as CVP

Candidate OWF
Key: a hard lattice $\mathcal{L}$
Input: $x$, $\|x\| \leq \beta$

Output: $f_L(x) = x \mod L$

$\beta < \lambda_1/2$: $f_L$ is injective

$\beta > \lambda_1/2$: $f_L$ is not injective

$\beta \geq \mu$: $f_L$ is surjective

$\beta \gg \mu$: $f_L(x)$ is almost uniform

Question: Are these functions cryptographically hard to invert?
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The Short Integer Solution (SIS) Problem

SIS/LWE as CVP

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Question
Are these functions cryptographically hard to invert?
Ajtai’s one-way function (SIS)

- Parameters: \( m, n, q \in \mathbb{Z} \)
- Key: \( A \in \mathbb{Z}_{q}^{n \times m} \)
- Input: \( x \in \{0, 1\}^{m} \)

Theorem (A’96)
For \( m > n \lg q \), if lattice problems (SIVP) are hard to approximate in the worst-case, then \( f_{A}(x) = Ax \mod q \) is a one-way function.

Applications: OWF [A’96], Hashing [GGH’97], Commit [KTX’08], IDs schemes [L’08], Signatures [LM’08, GPV’08, . . . , DDLL’13] . . .
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Cryptographic functions

Definition (Ajtai’s function)

\[ f_A(x) = Ax \mod q \quad \text{where } A \in \mathbb{Z}^{n \times m}_q \text{ and } x \in \{0, 1\}^m \]

\[ x \in \{0, 1\}^m \]

\[ A \in \mathbb{Z}^{n \times m}_q \]

\[ y = Ax \in \mathbb{Z}_q^n \]

Cryptanalysis (Inversion)

Given \( A \) and \( y \), find \( x \in \{0, 1\}^m \) such that \( Ax = y \)
Ajtai’s function and lattice problems

Cryptanalysis (Inversion)

Given $A$ and $y$, find small solution $x \in \{0, 1\}^m$ to inhomogeneous linear system $Ax = y \pmod{q}$

Inverting Ajtai’s function can be formulated as a lattice problem.
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Inverting Ajtai’s function can be formulated as a lattice problem.

- Easy problem: find (arbitrary) integer solution $t$ to system of linear equations $At = y \pmod{q}$
- All solutions to $Ax = y$ are of the form $t + \Lambda^\perp$ where

$$\Lambda^\perp(A) = \{ x \in \mathbb{Z}^m : Ax = 0 \pmod{q} \}$$
Ajtai’s function and lattice problems

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- Cryptanalysis problem: find a small vector in $t + \Lambda^\perp(A)$
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  \]

- Cryptanalysis problem: find a small vector in \( t + \Lambda^\perp(A) \)
- Equivalently: find a lattice vector \( v \in \Lambda^\perp(A) \) close to \( t \)
Ajtai’s function and lattice problems

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- Cryptanalys...
Ajtai’s function: collision resistance

- The kernel set $\Lambda^\perp(A)$ is a lattice

$$\Lambda^\perp(A) = \{ z \in \mathbb{Z}^m : Az = 0 \pmod{q} \}$$

- Collisions $Ax = Ay \pmod{q}$ can be represented by a single vector $z = x - y \in \{-1, 0, 1\}$ such that

$$z = x - y$$
Ajtai’s function: collision resistance

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- Collisions are lattice vectors $z \in \Lambda^\perp(A)$ with small norm

  $$\|z\|_\infty = \max_i |z_i| = 1.$$
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- Collisions are lattice vectors $z \in \Lambda^\perp(A)$ with small norm $\|z\|_\infty = \max_i |z_i| = 1$.

- ...there is a much deeper and interesting relation between breaking $f_A$ and lattice problems.
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Provable security (from average case hardness)

Example 1: (Rabin) modular squaring

- $f_N(x) = x^2 \mod N$, where $N = p \cdot q$
- Inverting $f_N$ is at least as hard as factoring $N$
Provable security (from average case hardness)

Example 1: (Rabin) modular squaring

- \( f_N(x) = x^2 \mod N \), where \( N = p \cdot q \)
- Inverting \( f_N \) is at least as hard as factoring \( N \)

**Theorem**

\( f_N \) is cryptographically hard to invert, provided most \( N = p \cdot q \) are hard to factor

![Diagram showing the relationship between \( N \) and \( f_N \)'](image-url)
Provable security (from average case hardness)

Example 2: Ajtai’s function

- \( f_A(x) = Ax \mod q \)
- Finding collisions in \( f_A \) is as hard as \( \ell_\infty\text{-SVP} \) in \( \Lambda(A) \)

\[\text{All } \Lambda(A)'s \quad \rightarrow \quad \text{hard } \Lambda(A)'s \]

\[\text{hard } \Lambda(A)'s \quad \rightarrow \quad \text{All } f_A's \]

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- \( f_A(x) = Ax \mod q \)
- Finding collisions in \( f_A \) is as hard as \( \ell_\infty \)-SVP in \( \Lambda(A) \)

**Theorem**

\( f_A \) is collision resistant, provided \( \ell_\infty \)-SVP is hard for most lattices \( \Lambda(A) \)
Average-case Complexity

Average-case complexity depends on input distribution

Example (Factoring problem)

Given a number $N$, output $a, b > 1$ such that $N = ab$
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Given a number $N$, output $a, b > 1$ such that $N = ab$

Factoring can be easy on average

if $N$ is uniformly random, then $N = 2 \cdot \frac{N}{2}$ with probability 50%!
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Example (Factoring problem)
Given a number $N$, output $a, b > 1$ such that $N = ab$

Factoring can be easy on average
if $N$ is uniformly random, then $N = 2 \cdot \frac{N}{2}$ with probability 50%!

- Factoring $N = pq$ is believed to be hard when $p, q$ are randomly chosen primes
- How do we know $\Lambda^\perp(A)$ is a hard distribution for SVP?
Provable security (from worst case hardness)

- Any fixed lattice $\mathcal{L}$ is mapped to a random $A$
- Finding collisions in $f_A$ allows to find (relatively) short vectors in $\mathcal{L}$.

Theorem (Ajtai,...,Micciancio&Regev)

If $f_A$ is collision resistant, provided SIVP is hard to approximate (within $\gamma = \frac{n}{2}$) for some $\mathcal{L}$.
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All lattices $L$ hard $f_A$’s
Provable security (from worst case hardness)

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Theorem (Ajtai,...,Micciancio&Regev)

$f_A$ is collision resistant, provided SIVP is hard to approximate (within $\gamma = n$) for some $\mathcal{L}$
Blurring a lattice

Consider a lattice $\Lambda$, and add noise to each lattice point until the entire space is covered. Increase the noise until the space is uniformly covered. How much noise is needed?

$$\|r\| \leq (\log n) \cdot \sqrt{n} \cdot \lambda_n / 2$$

Each point in $a \in \mathbb{R}^n$ can be written $a = v + r$ where $v \in \Lambda$ and $\|r\| \approx \sqrt{n} \cdot \lambda_n$. $a \in \mathbb{R}^n / \Lambda$ is uniformly distributed.
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Daniele Micciancio (UCSD)

The SIS Problem and Cryptographic Applications
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\[ 
\begin{array}{c}
\textbf{v} \rightarrow \textbf{a} \\
\end{array} 
\]
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Consider a lattice $\Lambda$, and add noise to each lattice point until the entire space is covered. Increase the noise until the space is uniformly covered.

How much noise is needed? [MR]

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How much noise is needed? [MR]

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- Each point in $a \in \mathbb{R}^n$ can be written $a = v + r$ where $v \in \mathcal{L}$ and $\|r\| \approx \sqrt{n} \lambda_n$.
- $a \in \mathbb{R}^n/\Lambda$ is uniformly distributed.
- Think of $\mathbb{R}^n \approx \frac{1}{q} \Lambda$ [GPV'07]
Security of Ajtai’s function (sketch)

- Generate random points \( a_i = v_i + r_i \), where
  - \( v_i \) is a random lattice point
  - \( r_i \) is a random error vector of length \( \|r_i\| \approx \sqrt{n} \lambda_n \)
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- $A = [a_1, \ldots, a_m]$ is distributed almost uniformly at random in $\mathbb{R}^{n \times m}$, $q = n^{O(1)}$, $m = O(n \log q) = O(n \log n)$, so
Security of Ajtai’s function (sketch)

- Generate random points $\mathbf{a}_i = \mathbf{v}_i + \mathbf{r}_i$, where
  - $\mathbf{v}_i$ is a random lattice point
  - $\mathbf{r}_i$ is a random error vector of length $||\mathbf{r}_i|| \approx \sqrt{n} \lambda_n$
- $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$ is distributed almost uniformly at random in $\mathbb{R}^{n \times m}$, $q = n^{O(1)}$, $m = O(n \log q) = O(n \log n)$, so
  - if we can break Ajtai’s function $f_\mathbf{A}$, then
  - we can find a vector $\mathbf{z} \in \{-1, 0, 1\}^m$ such that
    \[
    \sum \mathbf{a}_i z_i = 0
    \]
Security of Ajtai’s function (sketch)

- Generate random points \( \mathbf{a}_i = \mathbf{v}_i + \mathbf{r}_i \), where
  - \( \mathbf{v}_i \) is a random lattice point
  - \( \mathbf{r}_i \) is a random error vector of length \( \|\mathbf{r}_i\| \approx \sqrt{n} \lambda_n \)

- \( \mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_m] \) is distributed almost uniformly at random in \( \mathbb{R}^{n \times m} \),
  - \( q = n^{O(1)} \), \( m = O(n \log q) = O(n \log n) \), so
  - if we can break Ajtai’s function \( f_{\mathbf{A}} \), then
  - we can find a vector \( \mathbf{z} \in \{-1, 0, 1\}^m \) such that

\[
\sum (\mathbf{v}_i + \mathbf{r}_i)z_i = \sum \mathbf{a}_iz_i = 0
\]

- Rearranging the terms yields a lattice vector

\[
\sum \mathbf{v}_iz_i = -\sum \mathbf{r}_iz_i
\]

of length at most \( \|\sum \mathbf{r}_iz_i\| \approx \sqrt{m} \cdot \max \|\mathbf{r}_i\| \approx n \cdot \lambda_n \)
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Theorem (A’96)

For large enough $m, n, q$, the function $f_A$ is collision resistant
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For large enough $m, n, q$, the function $f_A$ is collision resistant

- Original proof required $q = n^{O(1)}$ to be a large polynomial
- Improved to $q \approx n^{2.5}$ in [MR’04]
- Further improved in [GPV’08] to $q \approx n$, making seemingly optimal use of known techniques
- Question: How can we prove hardness for smaller values of $q$?
Ajtai’s connection

Theorem (A’96)

*For large enough* $m, n, q$, the function $f_A$ is collision resistant

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- Question: How can we prove hardness for smaller values of $q$?

Theorem (MP’13)

*If one can break* $f_A$ *for some* $\sqrt{n} < q < n$, *then one can also break it for larger* $q' = q^c, c > 1$. 
Reducing $q$ in SIS (proof sketch, toy version)

- For simplicity, assume $f_A$ takes binary inputs $\mathbf{x} \in \{0,1\}^m$. 
Reducing \( q \) in SIS (proof sketch, toy version)

- For simplicity, assume \( f_A \) takes binary inputs \( x \in \{0, 1\}^m \).
- Say we can solve SIS for some \( n, m, q \).
Reducing $q$ in SIS (proof sketch, toy version)

- For simplicity, assume $f_A$ takes binary inputs $x \in \{0, 1\}^m$.
- Say we can solve SIS for some $n, m, q$. \( A'(\mathbb{Z}_{q}^{n \times m}) \)
- We solve SIS with parameters $n, m^2, q^2$ as follows:

\[
A (\mathbb{Z}_{q^2}^{n \times m^2})
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\[
A_1 \quad A_2 \quad \cdots \quad A_m
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- For simplicity, assume $f_A$ takes binary inputs $x \in \{0, 1\}^m$.
- Say we can solve SIS for some $n, m, q$.  \[ A'(\mathbb{Z}_q^{n \times m}) \]
- We solve SIS with parameters $n, m^2, q^2$ as follows:

\[
\begin{align*}
A & (\mathbb{Z}_{q^2}^{n \times m^2}) \\
A_1 & \\
A_2 & \\
\cdots & \\
A_m & \\
A'_1 + qA''_1 & \\
A'_2 + qA''_2 & \\
\cdots & \\
A'_m + qA''_m & 
\end{align*}
\]

- $A'_i, A''_i \in \mathbb{Z}_q^{n \times m}$ for all $i$
Reducing $q$ in SIS (toy version, cont.)

\[
A \in \mathbb{Z}_{q^2}^{n \times m^2}
\]

\[
\begin{align*}
A_1' + qA_1'' & \\
A_2' + qA_2'' & \\
& \cdots \\
A_m' + qA_m'' & 
\end{align*}
\]
Reducing $q$ in SIS (toy version, cont.)

\[
\begin{align*}
A & \in (\mathbb{Z}_q^{n \times m^2}) \\
A_1' + qA_1'' & \quad A_2' + qA_2'' & \cdots & \quad A_m' + qA_m''
\end{align*}
\]

- Find SIS($n,m,q$) collisions $A_i'z_i \equiv_q 0$, $z_i \in \{0, \pm 1\}$
Reducing $q$ in SIS (toy version, cont.)

Find SIS($n,m,q$) collisions $A'_i z_i \equiv_q 0$, $z_i \in \{0, \pm 1\}$

Compute $b_i = \frac{1}{q} (A'_i + qA''_i) z_i$
Reducing $q$ in SIS (toy version, cont.)

\[
\begin{array}{|c|c|c|c|}
\hline
A & (\mathbb{Z}_q^{n \times m^2}) \\
\hline
A_1' + qA_1'' & A_2' + qA_2'' & \cdots & A_m' + qA_m'' \\
\hline
\end{array}
\]

- Find SIS($n,m,q$) collisions $A_i'z_i \equiv_q 0$, $z_i \in \{0, \pm 1\}$
- Compute $b_i = \frac{1}{q}(A_i' + qA_i'')z_i = \frac{1}{q}(A_i'z_i) + \frac{q}{q}(A_i''z_i) \in \mathbb{Z}_q^n$
Reducing $q$ in SIS (toy version, cont.)

Find SIS($n,m,q$) collisions $A'_i z_i \equiv_q 0$, $z_i \in \{0, \pm 1\}$

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Reducing $q$ in SIS (toy version, cont.)

Find SIS($n,m,q$) collisions $A'_i z_i \equiv_q 0, z_i \in \{0, \pm 1\}$

Compute $b_i = \frac{1}{q} (A'_i + qA''_i) z_i = \frac{1}{q} (A'_i z_i) + \frac{q}{q} (A''_i z_i) \in \mathbb{Z}_q^n$

Solve SIS($n,m,q$) instance $B = [b_1, \ldots, b_m]$ to find collision $w$
Reducing $q$ in SIS (toy version, cont.)

<table>
<thead>
<tr>
<th>$b_1$</th>
<th>$b_2$</th>
<th>...</th>
<th>$b_m$</th>
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</table>

Find SIS($n,m,q$) collisions $A'_i z_i \equiv_q 0$, $z_i \in \{0, \pm 1\}$

Compute $b_i = \frac{1}{q} (A'_i + q A''_i) z_i = \frac{1}{q} (A'_i z_i) + \frac{q}{q} (A''_i z_i) \in \mathbb{Z}_q^n$

Solve SIS($n,m,q$) instance $B = [b_1, \ldots, b_m]$ to find collision $w$

Output collision $A(w \otimes z_*) \equiv q^2 0$

$$(w \otimes z_*) = (w_1 \cdot z_1, \ldots, w_m \cdot z_m) \in \{-1, 0, +1\}^{m^2}$$
Reducing $q$ in SIS (toy version, cont.)

Find SIS($n, m, q$) collisions $A'_i z_i \equiv_q 0$, $z_i \in \{0, \pm 1\}$

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$$(w \otimes z_*) = (w_1 \cdot z_1, \ldots, w_m \cdot z_m) \in \{-1, 0, +1\}^{m^2}$$

Actual proof used discrete gaussian sampling (DGS $\leq$ DGS)
Efficiency of Ajtai’s function

- $q = n^{O(1)}$, $m = O(n \log n) > n \log_2 q$
- E.g., $n = 64$, $q = 2^8$, $m = 1024$
- $f_A$ maps 1024 bits to 512.
Efficiency of Ajtai’s function

- \( q = n^{O(1)} \), \( m = O(n \log n) > n \log_2 q \)
- E.g., \( n = 64, q = 2^8, m = 1024 \)
- \( f_A \) maps 1024 bits to 512.
- Key size: \( nm \log q = O(n^2 \log^2 n) = 2^{19} = 64KB \)
- Runtime: \( nm = O(n^2 \log n) = 2^{16} \) arithmetic operations
Efficiency of Ajtai’s function

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- $f_A$ maps 1024 bits to 512.
- Key size: $nm \log q = O(n^2 \log^2 n) = 2^{19} = 64$KB
- Runtime: $nm = O(n^2 \log n) = 2^{16}$ arithmetic operations
- Usable, but inefficient
  - Source of inefficiency: quadratic dependency in $n$

Problem

Can we do better than $O(n^2)$ complexity?
Efficient lattice based hashing

Idea

Use structured matrix

\[
A = [A^{(1)} \mid \ldots \mid A^{(m/n)}]
\]

where \(A^{(i)} \in \mathbb{Z}_q^{n \times n}\) is circulant

\[
A^{(i)} = \begin{bmatrix}
    a_1^{(i)} & a_n^{(i)} & \cdots & a_2^{(i)} \\
    a_2^{(i)} & a_1^{(i)} & \cdots & a_3^{(i)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_n^{(i)} & a_{n-1}^{(i)} & \cdots & a_1^{(i)}
\end{bmatrix}
\]
Efficient lattice based hashing

Idea

Use structured matrix

\[ A = [A^{(1)} | \ldots | A^{(m/n)}] \]

where \( A^{(i)} \in \mathbb{Z}_q^{n \times n} \) is circulant

- Proposed by [M02], where it is proved that \( f_A \) is one-way under plausible complexity assumptions
- Similar idea first used by NTRU public key cryptosystem (1998), but with no proof of security
- Wishful thinking: finding short vectors in \( \Lambda_\perp_q(A) \) is hard, and therefore \( f_A \) is collision resistant
Can you find a collision? (mod 10)

| 1 4 3 8 | 6 4 9 0 | 2 6 4 5 | 3 2 7 1 |
| 8 1 4 3 | 0 6 4 9 | 5 2 6 4 | 1 3 2 7 |
| 3 8 1 4 | 9 0 6 4 | 4 5 2 6 | 7 1 3 2 |
| 4 3 8 1 | 4 9 0 6 | 6 4 5 2 | 2 7 1 3 |
Can you find a collision? (mod 10)

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\[
x^n - 1 = (x - 1) \cdot (x^n - 1 + \cdots + 1)
\]
Can you find a collision? (mod 10)

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\[
x_n - 1 = (x - 1) \cdot (x^{n-1} + \cdots + 1)
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\[ x^n - 1 = (x - 1) \cdot (x^{n-1} + \cdots + 1) \]
**Can you find a collision? (mod 10)**

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\[ + 1 \times \begin{bmatrix} 6 \\ 9 \\ 6 \end{bmatrix} - 1 \times \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix} + 0 \times \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} + 1 \times \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \]

\[ x^n - 1 = (x - 1) \cdot (x^{n-1} + \cdots + 1) \]
Remarks about proofs of security

- This function is essentially the compression function of hash function LASH, modeled after NTRU
- You can still “prove” security based on average case assumption: Breaking the above hash function is as hard as finding short vectors in a random lattice $\Lambda([A^{(1)}|\ldots|A^{(m/n)}])$
- ...but we know the function is broken: The underlying random lattice distribution is weak!
- Conclusion: Assuming that a problem is hard on average-case is a really tricky business!
Can you find a collision now? (mod 10)

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Theorem (trivial)

*Finding collisions on the average is at least as hard as finding short vectors in the corresponding random lattices*
Can you find a collision now? (mod 10)

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<td>-4</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>8</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>0</td>
<td>6</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

Theorem (trivial)

Finding collisions on the average is at least as hard as finding short vectors in the corresponding random lattices

Theorem (LM’07, PR’07)

Provably collision resistant, assuming the worst case hardness of approximating SVP and SIVP over anti-cyclic lattices.

- $x^n + 1$ is irreducible (for $n = 2^k$)
Efficiency of anti-cyclic hashing

- Key size: \((m/n) \cdot n \log q = m \cdot \log q = \tilde{O}(n)\) bits
- Anti-cyclic matrix-vector multiplication can be computed in quasi-linear time \(\tilde{O}(n)\) using FFT
- The resulting hash function can also be computed in \(\tilde{O}(n)\) time
- For appropriate choice of parameters, this can be very practical (SWIFFT [LMPR])
- The hash function is linear: \(A(x+y) = Ax + Ay\)
- This can be a feature rather than a weakness
Isomorphism: $\mathbb{A}^{cyc} \leftrightarrow \mathbb{Z}[X]/(X^n - 1)$

Cyclic SIS:

$$f_{a_1, \ldots, a_k}(u_1, \ldots, u_k) = \sum_i a_i(X) \cdot u_i(X) \pmod{X^n - 1}$$

where $a_i, u_i \in R = \mathbb{Z}[X]/(X^n - 1)$.

More generally, use $R = \mathbb{Z}[X]/p(X)$ for some monic polynomial $p(X) \in \mathbb{Z}[X]$.

If $p(X)$ is irreducible, then finding collisions to $f_a$ for random $a$ is as hard as solving lattice problems in the worst case in ideal lattices.

Can set $R$ to the ring of integers of $K = \mathbb{Q}[X]/p(X)$. 
1. The Short Integer Solution (SIS) Problem

2. Average Case Hardness

3. Efficiency and RingSIS
   - Small modulus
   - Ideal Lattices

4. Cryptographic Applications
   - 1: Compression and Hashing
   - 2: Regularity and Commitment Schemes
   - 3: Linearity and Digital Signatures
SIS: Properties and Applications

Properties:
1. Compression
2. Regularity
3. Homomorphism

Applications:
1. Collision Resistant Hashing
2. Commitment Schemes
3. Digital Signatures
SIS Property: Compression

**SIS Function**

\[ A \in \mathbb{Z}_q^{n \times m}, \quad x \in \{0, 1\}^m, \quad f_A(x) = Ax \mod q \in \mathbb{Z}_q^n \]

Main security parameter: \( n \). (Security largely independent of \( m \).)
SIS Property: Compression

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- \( f_A \): \( m \) bits \( \rightarrow \) \( n \log q \) bits.

\( \{0, 1\}^m \) \( \xrightarrow{f_A} \) \( \mathbb{Z}_q^n \)

\( m \) bits \( \rightarrow \) \( n \log q \) bits
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\[ \mathbf{A} \in \mathbb{Z}_q^{n \times m}, \quad \mathbf{x} \in \{0, 1\}^m, \quad f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \mod q \in \mathbb{Z}_q^n \]

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- \( f_{\mathbf{A}} \): \( m \) bits \( \rightarrow \) \( n \log q \) bits.
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The SIS Problem and Cryptographic Applications

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- E.g., \( m = 2n \log q \):
  \[ f_A : \{0,1\}^m \rightarrow \{0,1\}^{m/2} \]
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Ajtai’s theorem requires \( (m > n \log q) \).
Collision Resistant Hashing

Keyed function family \( f_A : X \rightarrow Y \) with \(|X| > |Y|\)
(E.g., \( X = Y^2 \) and \( f_A : Y^2 \rightarrow Y \).)
Collision Resistant Hashing

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Definition (Collision Resistance)
Finding $x_1 \neq x_2 \in X$ such that $f_A(x_1) = f_A(x_2)$ is hard.
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Finding $x_1 \neq x_2 \in X$ such that $f_A(x_1) = f_A(x_2)$ is hard.

Classic application: Merkle Trees
- Leaves are user data
- Each internal node is the hash of its children
- Root $r$ commits to all $y_1, \ldots, y_n$
- Each $y_i$ can be shown to be consistent with $r$ by revealing $\log(n)$ values
SIS Application: Collision Resistant Hashing

Definition (Collision Resistance)

\[ f_A: X \rightarrow Y. \] No adversary, given a random \( A \), can efficiently find \( x \neq x' \in X \) such that \( f_A(x) = f_A(x') \)
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\[ f_A : X \rightarrow Y. \text{ No adversary, given a random } A, \text{ can efficiently find } x \neq x' \in X \text{ such that } f_A(x) = f_A(x') \]

Theorem

If \( f_A : \{0, \pm 1\}^m \rightarrow \mathbb{Z}_q^n \) is one-way, then \( f_A : \{0, 1\}^m \rightarrow \mathbb{Z}_q^n \) is collision resistant.
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**Definition (Collision Resistance)**

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- Add \( y \) to random column \( a'_i = a_i + y \).
- Find collision for \( A' : A'x = A'x' \)
- If \( x'_i = 1 \) and \( x_i = 0 \), then \( A(x - x') = y \)
SIS Property: Regularity

\( f : X \rightarrow Y \) is regular if all \( y \in Y \) have same \( |f^{-1}(y)| \).
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Pairwise independence:

- Fix \( x_1 \neq x_2 \in \{0, 1\}^m \),
- Random \( A \)
- \( f_A(x_1) \) and \( f_A(x_2) \) are independent.
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Lemma (Leftover Hash Lemma)

Pairwise Independence + Compression \( \implies \) Regular
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*Pairwise Indepencence + Compression \( \Rightarrow \) Regular*

\[ f_A : (U(\{0,1\}^n)) \approx U(\mathbb{Z}_q^n) \text{ maps uniform to uniform.} \]
Perfectly Hiding Commitments

Analogy:
Lock message in a box
Give box, keep key
Later: give key to open box

Implementation
Randomized function $C(m; r)$

Commit($m$): give $c = C(m; r)$ for random $r \leftarrow \$$. 

Open: reveal $m, r$ such that $C(m; r) = c$.

Security properties:
Hiding: $c = C(m; \$) \text{ is independent of } m$ 
Binding: hard to find $m \neq m' \text{ and } r, r'$ such that $C(m; r) = C(m'; r')$. 

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SIS Application: Commitment

- Choose $A_1, A_2$ at random

Commitment: $C(m, r) = f[A_1, A_2](m, r) = A_1 m + A_2 r$.

Hiding Property: $C(m)$ hides the message because $A_2 r = f[A_2](r) \approx U(Z_n)$

Binding Property: Finding $(m, r) \neq (m', r')$ such that $C(m, r) = C(m', r')$ breaks the collision resistance of $f[A_1, A_2]$. 
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  - Domain of \( f_A \) is not closed under +

- \( f_A \) is also key-homomorphic:
  \[ f_{A_1}(x) + f_{A_2}(x) = f_{A_1 + A_2}(x) \]
(One-Time) Digital Signatures

- Digital Signature Scheme:
  - Key Generation Algorithm: \((pk, sk) \leftarrow KeyGen\)
  - Signing Algorithm: \(Sign(sk, m) = \sigma\)
  - Verification Algorithm: \(Verify(pk, m, \sigma)\)
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- (One-Time) Security:
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General Signatures: Adversary is allowed an arbitrary number of signature queries
SIS Application: One-Time Signatures

- Extend $f_A$ to matrices $X = [x_1, \ldots, x_l]$:

$$f_A(X) = [f_A(x_1), \ldots, f_A(x_l)] = AX \pmod{q}$$
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  \]

- Key Generation:
  - Public Parameter: SIS function key $A$
  - Secret Key: $sk = (X, x)$ two (small) inputs to $f_A$
  - Public Key: $pk = (Y = f_A(X), y = f_A(x))$ image of $sk$ under $f_A$
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- **Message:** short vector $m \in \{0, 1\}^l$

- **Sign$(sk, m) = Xm + x$, linear combination of secret key**
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- Message: short vector $m \in \{0, 1\}^l$

- $Sign(sk, m) = Xm + x$, linear combination of secret key

- $Verify(pk, m, \sigma)$ uses homomorphhic properties to check that

  $$f_A(\sigma) = f_A(Xm + x) = f_A(X)m + f_A(x) = Ym + y$$
One-time signatures from anti-cyclic lattices

Fix hash function key $A = [A^{(1)} | \ldots | A^{(m/n)}]$

**Definition (Secret signing key)**

$x = [x^{(1)}, \ldots, x^{(m/n)}]$  
$y = [y^{(1)}, \ldots, y^{(m/n)}]$  

- Signing $m \in \{0, 1\}^n$:
  
  $\sigma_i = x^{(i)}M + y^{(i)}$  
  $\sigma = (\sigma_1, \ldots, \sigma_{m/n})$

- Verification:

  Check if $h_A(\sigma) = X M + Y$

**Definition (Public verif. key)**

$X = h_A(x) = \sum A^{(i)} x^{(i)}$  
$Y = h_A(y) = \sum A^{(i)} y^{(i)}$

$$
M = \begin{bmatrix}
m_1 & -m_n & \cdots & -m_2 \\
m_2 & m_1 & \cdots & -m_3 \\
\vdots & \vdots & \ddots & \vdots \\
m_n & m_{n-1} & \cdots & m_1
\end{bmatrix}
$$
Efficiency and security

- Key generation, signing and verifying all require just 1 or 2 hash function computations in $\tilde{O}(n)$ time
- Secret key, public key and signature size are also $\tilde{O}(n)$ bits

**Theorem (Lyubashevsky&Micciancio)**

The one-time signature scheme is secure based on the worst-case hardness of approximating SVP/SIVP on anti-cyclic lattices within a factor $\gamma = n^2$

- Forgery $(\mathbf{M}, \sigma)$: $h_{A}(\sigma) = X\mathbf{M} + Y$
- Use $x, y$ to sign $\mathbf{M}$: $h_{A}(\sigma') = X\mathbf{M} + Y$
- If $\sigma \neq \sigma'$, then $h_{A}(\sigma) = X\mathbf{M} + Y = h_{A}(\sigma')$ is a collision!
That’s all folks!

Later today:

- LWE: injective version of SIS, many more applications
- RingLWE: efficient version of LWE