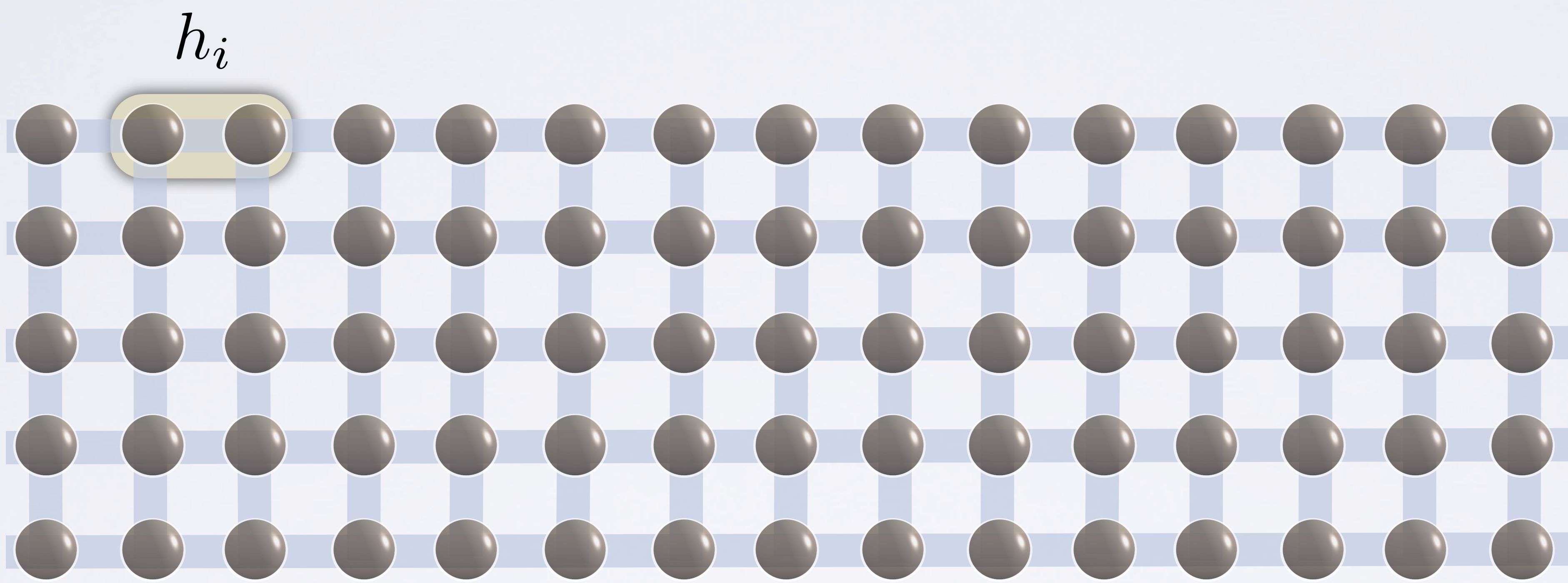
Correlations, area laws and stability
of open and thermal quantum many-body systems

Jens Eisert, FU Berlin

Complexity meets Condensed Matter, Simons Institute for the Theory of Computing, March 2014

Joint work with Michael Kastoryano, Martin Kliesch, Christian Gogolin, Arnau Riera

Hamiltonian complexity

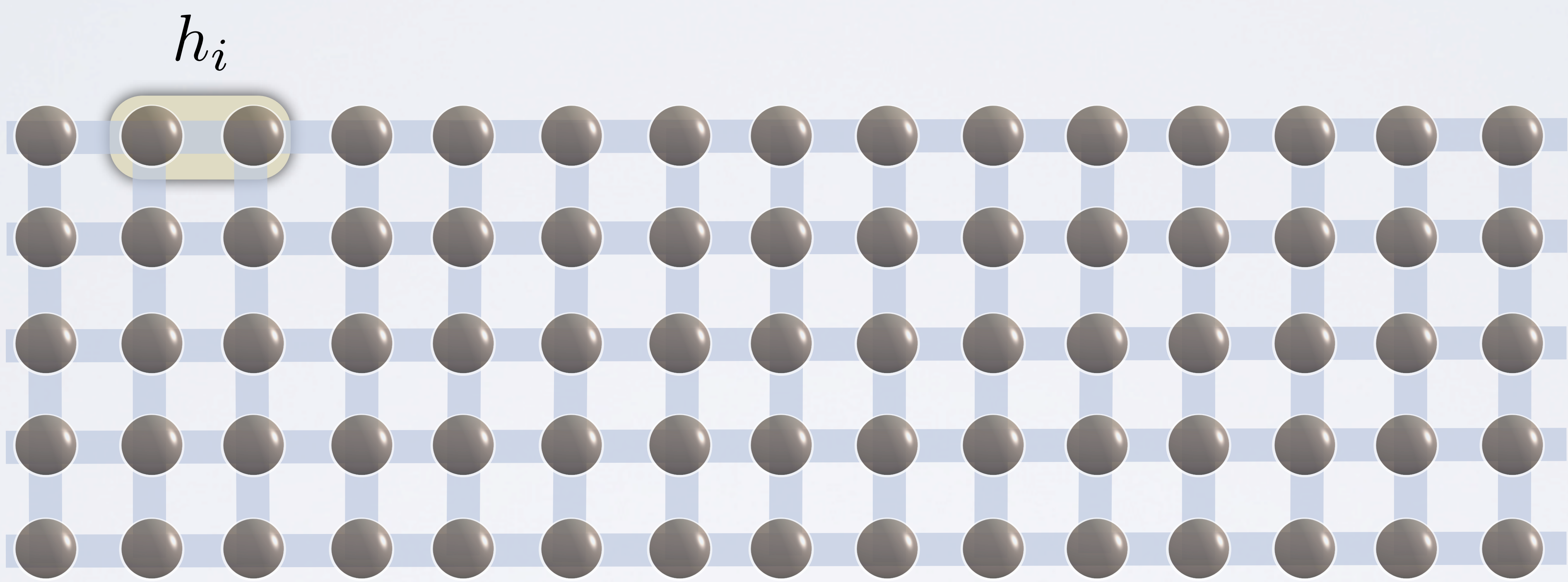


- Can ground states of “natural” quantum systems be described succinctly?
- Does the exponential complexity of general quantum systems persist at high temperature?
- Is the scientific method sufficiently powerful to understand general quantum systems?

- **Local Hamiltonian** problem is QMA-complete
- Steps towards a **quantum PCP** theorem (Matt's and Dorit's talks)

Kitaev, Shen, Vyalyi, AMS MR1 907 291 (2002)
Osborne, arXiv:1106.5875
Aharonov, Aran Vidick, ACM SIACT 44, 47 (2013)
Hastings, QIC 13, 393 (2013)

Hamiltonian complexity



- Can ground states of "natural" quantum systems be described succinctly?
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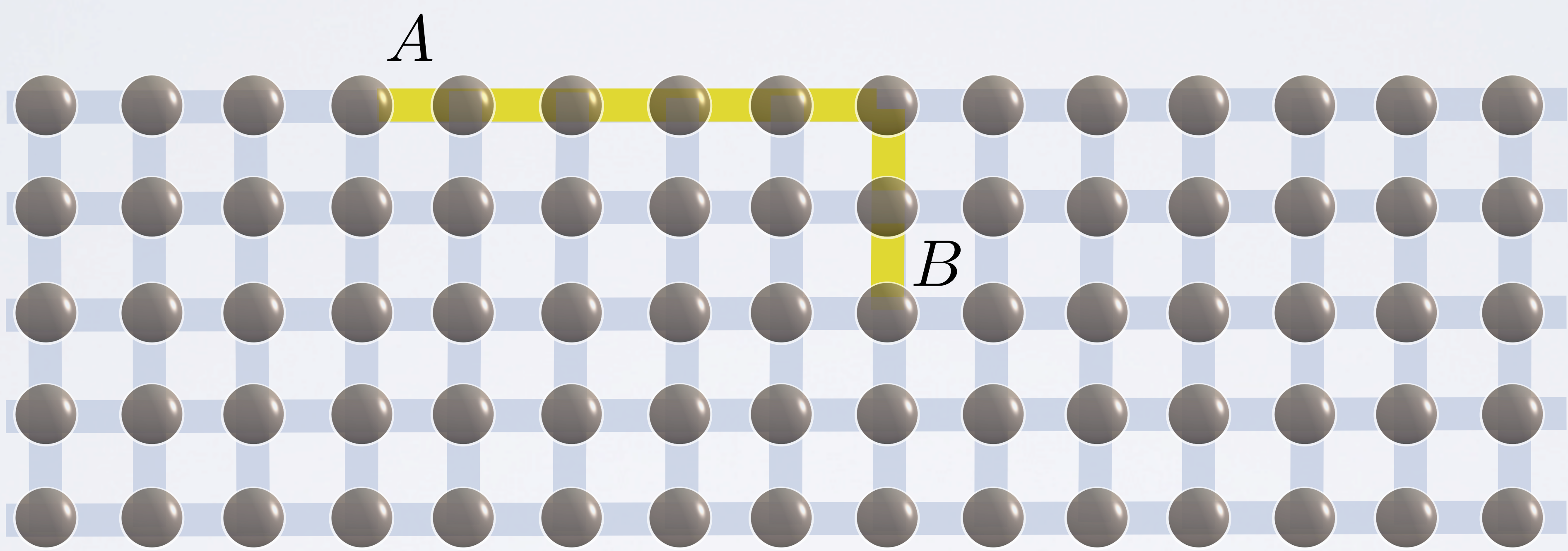
Ground states of local gapped models



- **Energy gap** $\Delta(H) = E_1 - E_0 > 0$
- Ground states of gapped models have **exponentially decaying correlations**

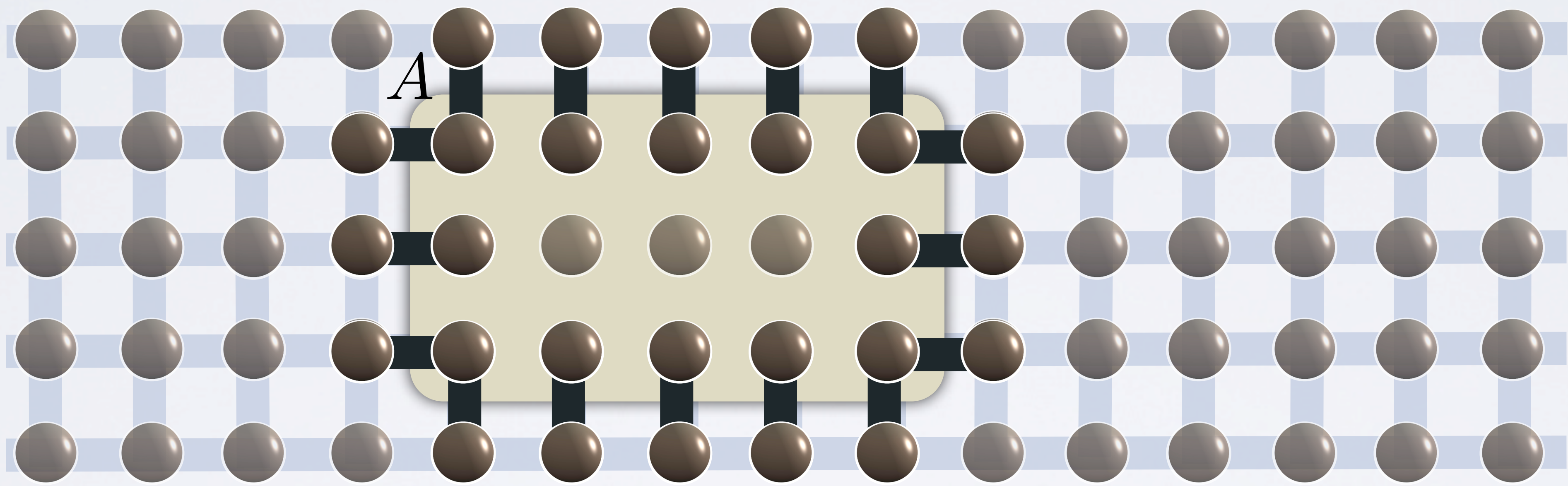
- Proof based on Lieb-Robinson bounds

  Hastings, Koma, Commun Math Phys 265, 781 (2006)
  Nachtergaele, Sims, Commun Math Phys 265, 119 (2006)

- Combinatorical proof (detectability lemma)

  Aharonov, Arad, Landau, Vazirani, arXiv:1011.3445

Area laws

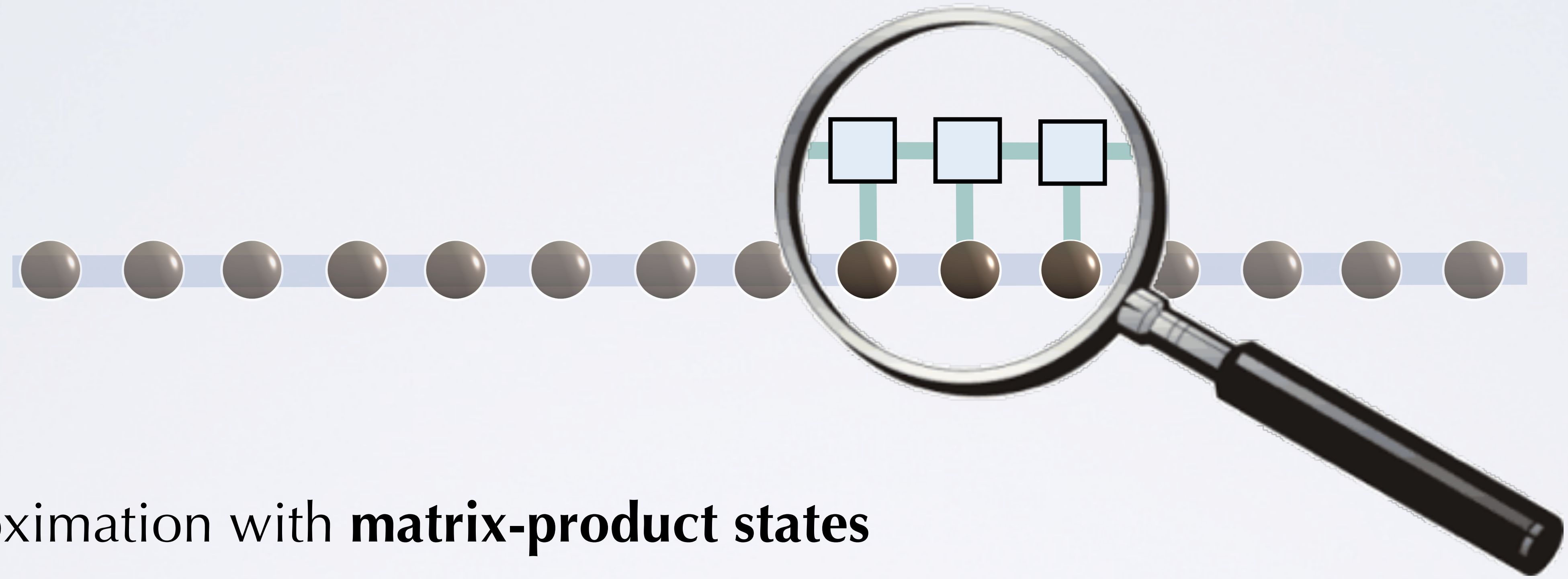


- **Area laws for the entanglement entropy** $S(\rho_A) = O(|\partial A|)$

- Proven for gapped quasi-free **bosonic and fermionic systems** in any dimension, **1D gapped local models** and ones with exponentially **decaying correlations**

Eisert, Cramer, Plenio, Rev Mod Phys 82, 277 (2010)
Hastings, Koma, Commun Math Phys 265, 781 (2006)
Aharonov, Arad, Landau, Vazirani, arXiv:1011.3445
Brandao, Horodecki, arXiv:1206.2947
Plenio, Eisert, Dreissig, Cramer, Phys Rev Lett 94, 060503 (2005)

Matrix-product states and efficient descriptions

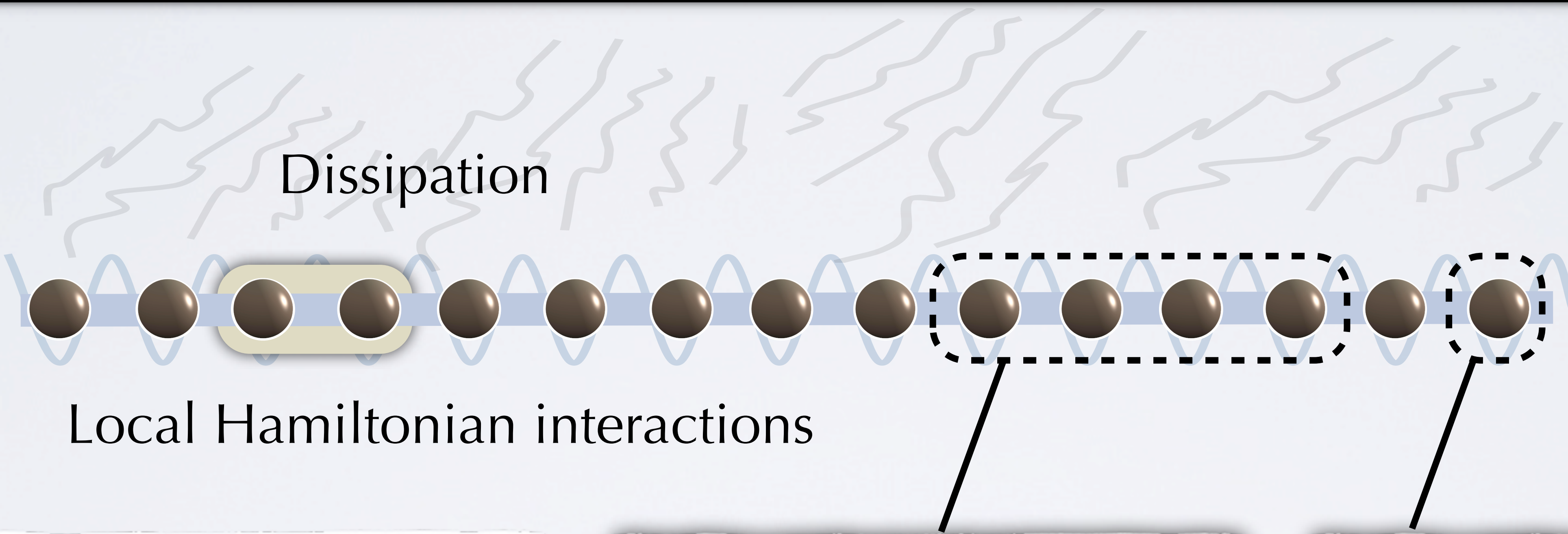


- Approximation with **matrix-product states**
- Polynomial-time algorithm for ground states of 1D gapped local Hamiltonians

Landau, Vazirani, Vidick, arXiv:1307.5143

- **This talk:** Correlations in **thermal** and **open** quantum many-body systems

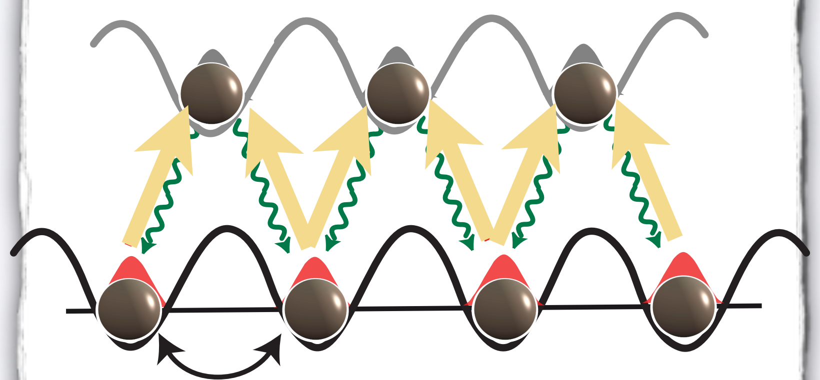
Open quantum many-body systems



Many-body physics

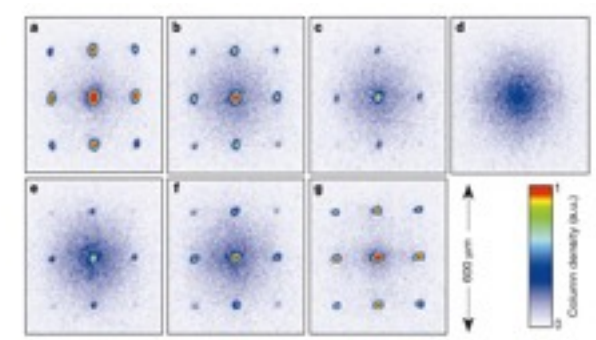
Liouvillian dynamics

Quantum optics



Engineered dissipation

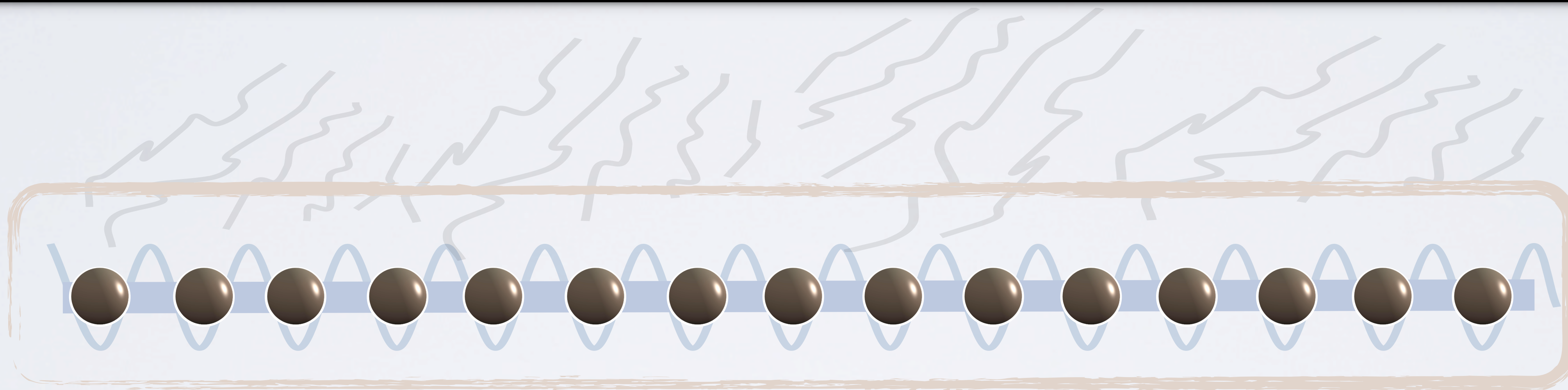
Cold atoms



Optical lattices

Diehl, Micheli, Kantian, Kraus, Buechler, Zoller, Nature Physics 4, 878 (2008)
Kraus, Diehl, Micheli, Kantian, Buechler, Zoller, Phys Rev A 78, 042307 (2008)
Verstraete, Wolf, Cirac, Nature Physics 5, 633 (2009)
Eisert, Prosen, arXiv:1012.5013
Bravyi, Chesi, Loss, Terhal, New J Phys 12, 025013 (2010)
Kastoryano, Wolf, Eisert, Phys Rev Lett 110, 110501 (2013)

Open quantum many-body systems



- Dissipative quantum phase transitions, noise-driven criticality, topological order
- Dissipative quantum computing
- Dissipative passive quantum memories?

Diehl, Micheli, Kantian, Kraus, Buechler, Zoller, Nature Physics 4, 878 (2008)
Kraus, Diehl, Micheli, Kantian, Buechler, Zoller, Phys Rev A 78, 042307 (2008)
Verstraete, Wolf, Cirac, Nature Physics 5, 633 (2009)
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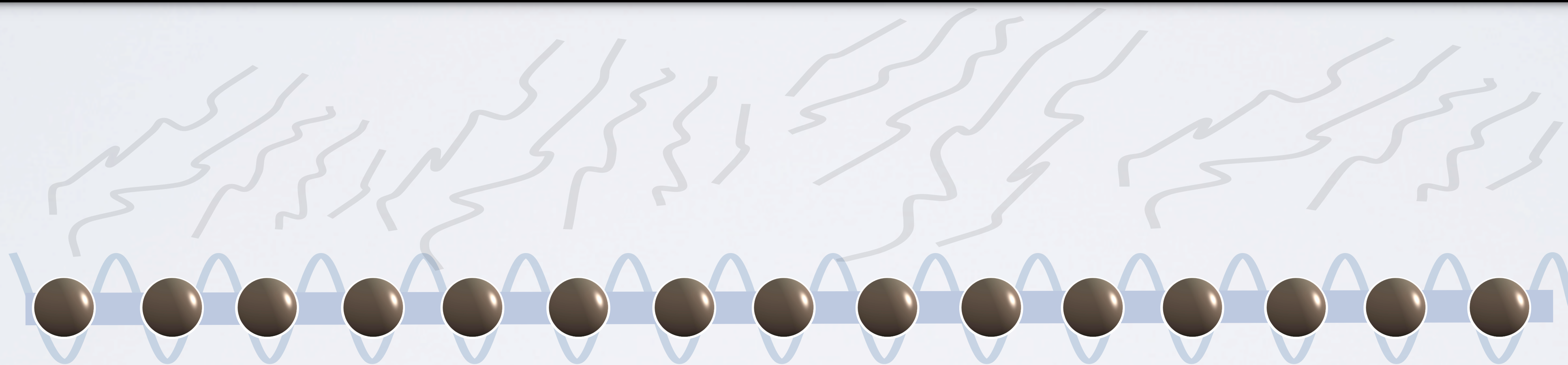
Questions of the rest of talk

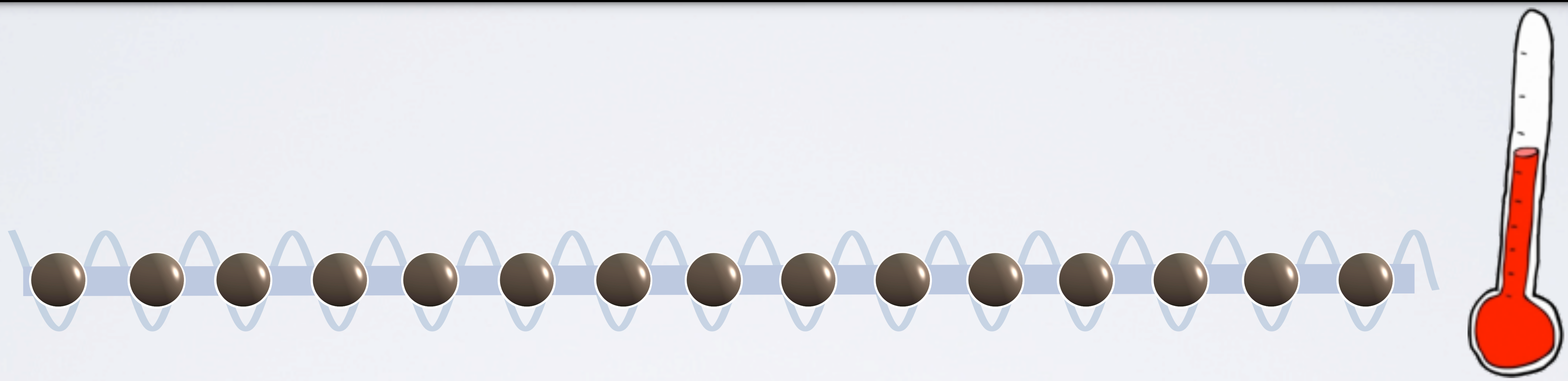


- "Liouvillian complexity", resembling Hamiltonian complexity

- **How is closing of Liouvillian gaps related to clustering of correlations?**
- **Area laws in dissipative systems? Stability? Topological dissipative memories?**

Kastoryano, Eisert, J Math Phys 54, 102201 (2013)
Cubitt, Lucia, Michalakis, Perez-Garcia, arXiv:1303.4744

Questions of the rest of talk



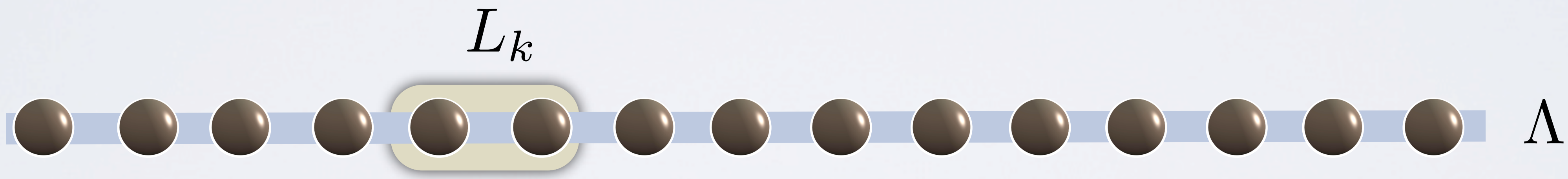
- Thermal states at high temperatures

- **Is temperature intensive/local?**
- **Correlations in thermal many-body states?**
- **Computational complexity of computing expectation values?**

Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816

Correlations in open many-body systems

Local Liouvillian setting

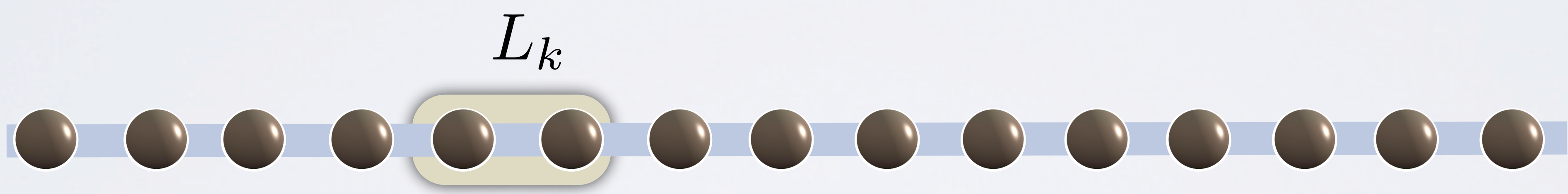


- **Liouvillian setting**, reflecting Markovian dynamics

$$\frac{d}{dt}\rho = \mathcal{L}(\rho) = i[H,\rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

- Effective system dynamics
- Lindblad operators are **geometrically local** on **some graph** Λ
- **Bounded** interactions, i.e., $\|L_k\| < K$ for all k

Stationary states and mixing properties



- Role of ground state taken over by **stationary state** σ, satisfying

$$\mathcal{L}(\sigma) = 0$$

here often taken to be full rank (primitive), with detailed balance

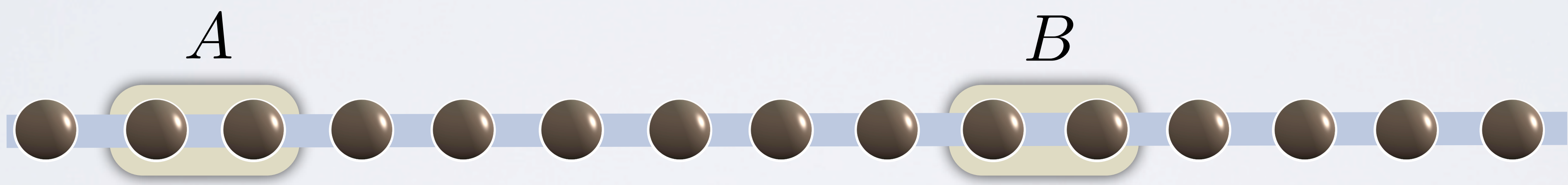
- **Mixing properties:** For any primitive local Liouvillian,

$$\|e^{t\mathcal{L}}(\rho_0) - \sigma\|_1^2 \leq \|\sigma^{-1}\| e^{-2\lambda t}$$

for any initial state ρ_0

- λ is the **Liouvillian gap**, resembling the Hamiltonian gap in closed systems

Correlation measures



- **Covariance:** For arbitrary regions $A, B \subset \Lambda$

$$C_\rho(A,B) = \sup_{\|f\|=\|g\|=1} |\mathrm{tr}((f \otimes g)(\rho_{A,B} - \rho_A \otimes \rho_B))|$$

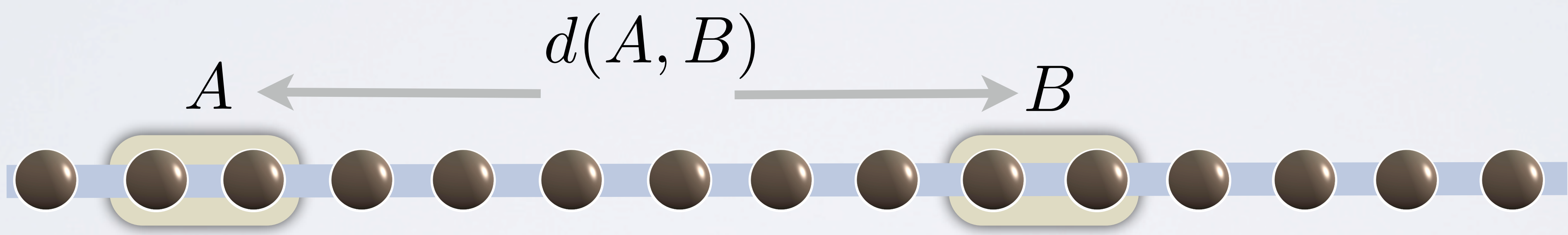
"largest connected correlation function"

- Related to other **standard correlation measures**

  - *trace distance* $T_\rho(A,B) := \|\rho_{A,B} - \rho_A \otimes \rho_B\|_1$
  - *mutual information* $I_\rho(A,B) := S(\rho_{A,B} \| \rho_A \otimes \rho_B)$

Kastoryano, Eisert, J Math Phys 54, 102201 (2013)

Clustering of correlations and gaps?



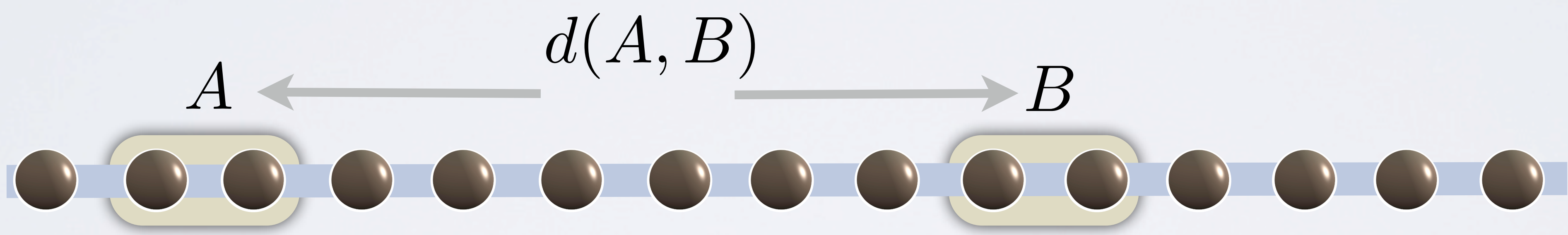
- Gapped Hamiltonians, away from **phase transitions**, show **clustering of correlations**

$$C_\rho(A, B) \leq C e^{-d(A,B)/\xi}$$

- **How about gapped Liouvillians?**

Kastoryano, Eisert, J Math Phys 54, 102201 (2013)

Clustering of correlations and gaps

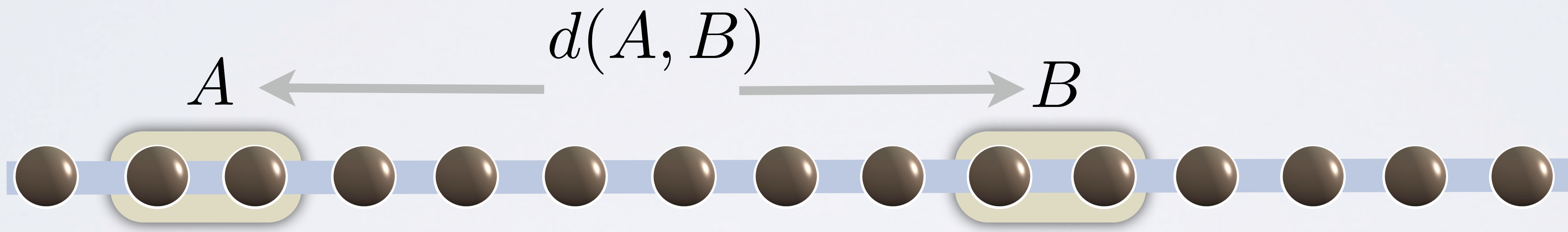


- **Clustering of correlations:** $A, B \subset \Lambda$ non-overlapping subsets, consider local, bounded Liouvillian with stationary state σ, gap λ, and Lieb-Robinson velocity v. Then there ex constant $c > 0$, such that

$$C_\sigma(A,B) \leq c d(A,B)^{\mathcal{D}-1} e^{-\frac{\lambda d(A,B)}{v+2\lambda}}$$

Kastoryano, Eisert, J Math Phys 54, 102201 (2013)

Flavour of (simple) proof



Hölder's inequality and mixing time tools:
Variational characterisation of gap, ... $\leq \|f\| \, \|g\| e^{-2t\lambda}$

Set $f_t = e^{t\mathcal{L}^*}(f)$, then

$$|\mathrm{Cov}_\sigma(f, g)| \leq |\mathrm{Cov}_\sigma(f_t, g_t)| + |\mathrm{Cov}_\sigma(f_t, g_t) - \mathrm{Cov}_\sigma(f, g)|$$

Choose suitable t

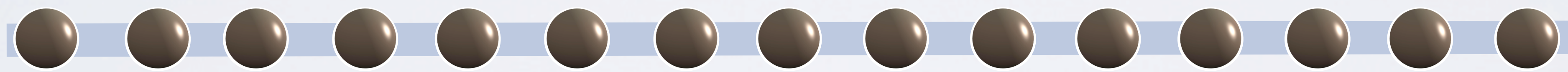
- **Dissipative Lieb-Robinson bound:** For observables f, g supported on $A, B \subset \Lambda$, respectively,

$$\|(fg)_t - f_t g_t\| \leq C d(A, B)^{\mathcal{D}-1} \|f\| \, \|g\| e^{vt - d(A,B)/2}$$

for all $t \geq 0$, where v is the Lieb-Robinson velocity and $C > 0$ constant

Kliesch, Barthel, Phys Rev Lett 108, 230504 (2012)
Nachtergaele, Vershynina, Zagrebnov, AMS Cont Math 552, 161 (2011)
Kliesch, Gogolin, Eisert, arXiv:1306.0716
Kastoryano, Eisert, J Math Phys 54, 102201 (2013)

Slightly stronger mixing tools



- Stronger concept of mixing, based on **Log-Sobolev constant**
- Log-Sobolev-constant α bounded from above by Liouvillian gap λ
- Variational characterisation of α, related to hypercontractivity

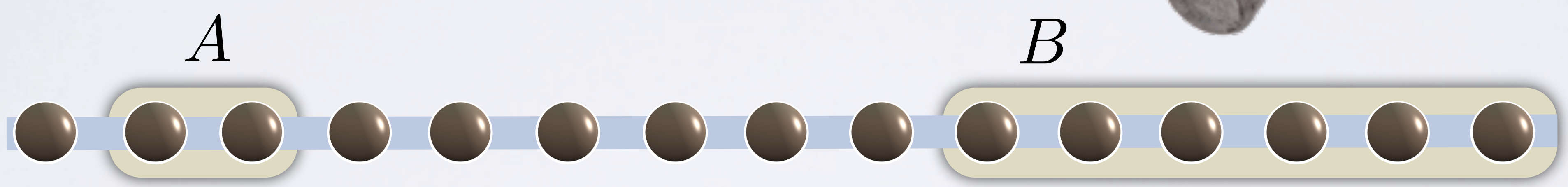
- **Mixing properties:** For any primitive local Liouvillian,

$$\|e^{t\mathcal{L}}(\rho_0) - \sigma\|_1^2 \leq 2\log(\|\sigma^{-1}\|)e^{-2\alpha t}$$

for any initial state ρ_0

Kastoryano, Temme, arXiv:1207.3261
Temme, Kastoryano, Ruskai, Wolf, Verstraete, J Stat Mech (2010)
Kastoryano, Eisert, J Math Phys 54, 102201 (2013)

Stability results

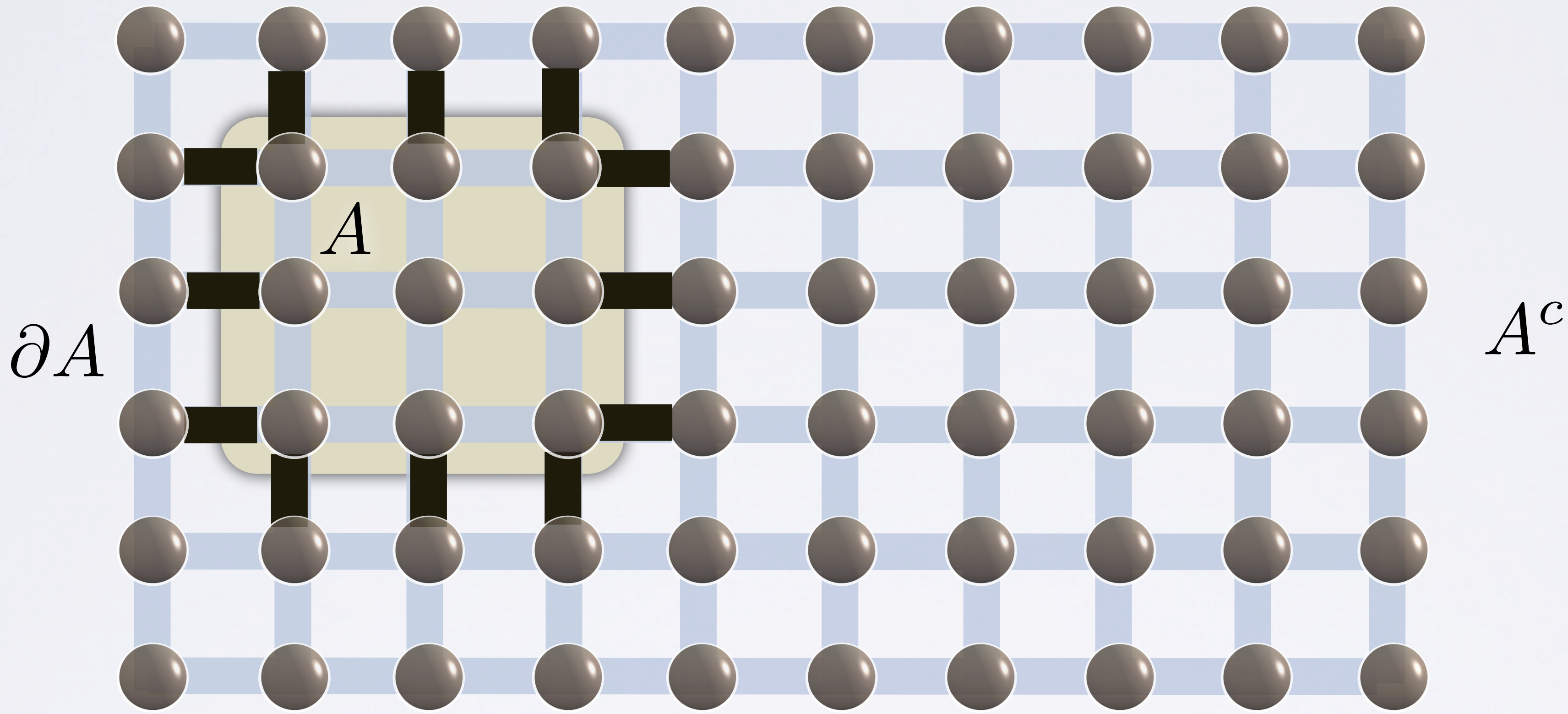


- **Local perturbations perturb locally:** Let $\mathcal{L}$ be a local Liouvillian with Log-Sobolev-constant α and stationary state ρ, let $\mathcal{Q}_B$ be a perturbation on B only, with stationary state σ of $\mathcal{L} + \mathcal{Q}_B$, (...), then

$$\|\rho_A - \sigma_A\|_1 \leq Ce^{-\alpha d(A,B)/(v+\alpha)}$$

Kastoryano, Eisert, J Math Phys 54, 102201 (2013)
Cubitt, Lucia, Michalakis, Perez-Garcia, arXiv:1303.4744

Area laws



- **Area law for mutual information:** (...)

$$I_\rho(A, A^c) \leq \left(\gamma_1 + \gamma_2 \log\log \|\rho^{-1}\|\right) |\partial A| + \epsilon$$

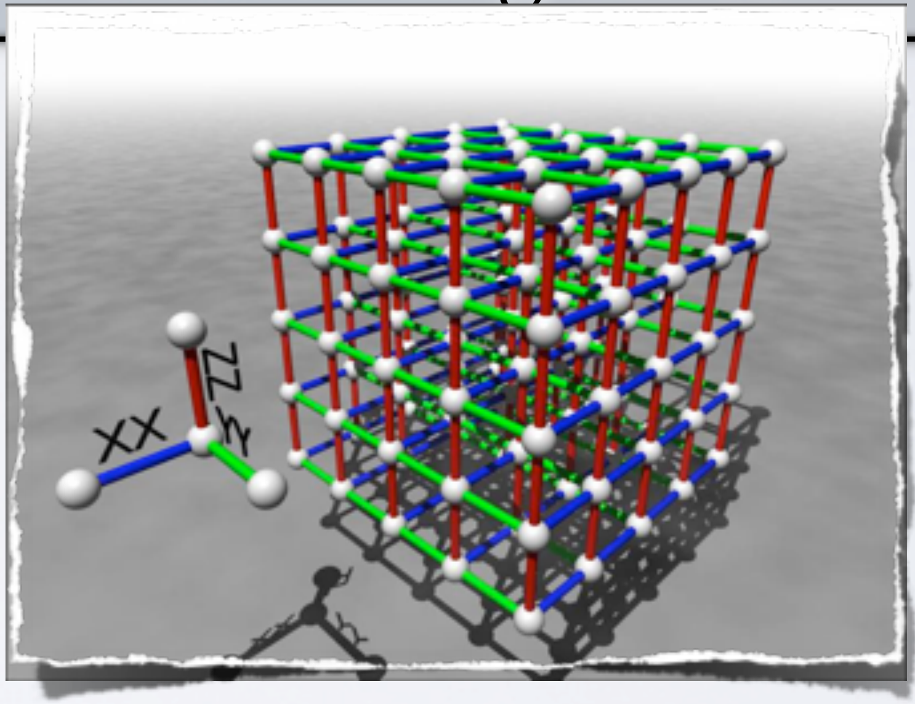
Kastoryano, Eisert, J Math Phys 54, 102201 (2013)

Mixing times and clustering of correlations

- **Lesson:** Rapidly mixing systems exhibit **exponentially clustering correlations**
- "Mixing in time related to mixing in space"
- **Liouvillian gap** (log-Sobolev constant) reminds of **Hamiltonian gap**

- Two **different regimes**, with quite different implications
- Quantum feature, difference **absent classically**

Quantum memories, topological order and mixing times



- Optimal **dissipative encoders for toric codes**

- **Interesting challenge:** Time to prepare topologically ordered states $O(L)$ for $L \times L$ lattice can be achieved, ...

Dengis, Koenig, Pastawski, arXiv:1310.1036

- ... stability relies on log-Sobolev-type clustering not allowing for topological order

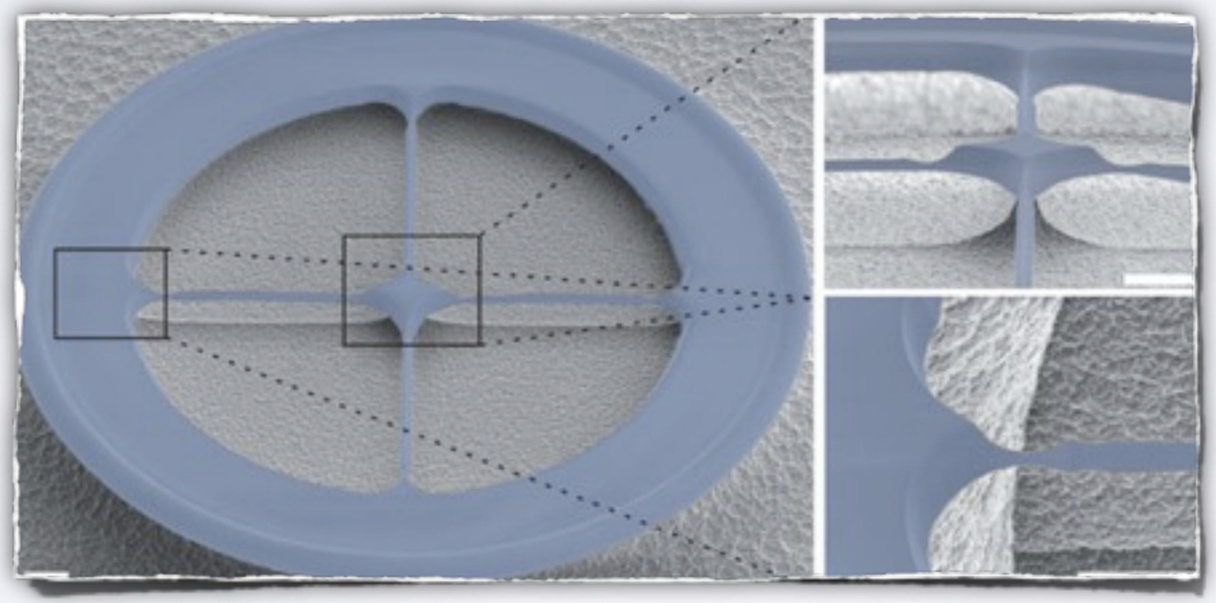
- How to reconcile that? **Dissipative stable passive quantum memories?**

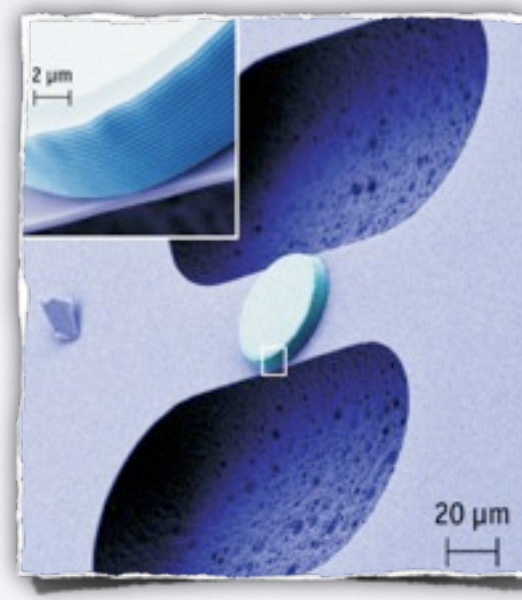
Kastoryano, Eisert, J Math Phys 54, 102201 (2013)
Cubitt, Lucia, Michalakis, Perez-Garcia, arXiv:1303.4744

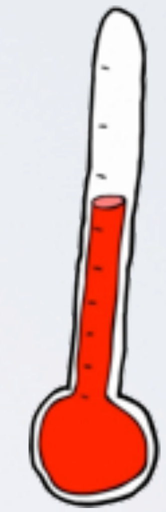
Clustering of correlations in thermal states

Locality of temperature?

- At what **length scales** is temperature well-defined?







Pothier, Gueron, Birge, Esteve, Devoret, Phys Rev Lett 79, 3490 (1997)
Peng, Su, Liu, Yu, Cheng, Bao, Nanoscale 5, 9532 (2013)
Hartmann, Mahler, Hess, Phys Rev Lett 93, 080402 (2004)
Ferraro, Garcia-Saez, Acin, Europhys Lett 98, 10009 (2012)

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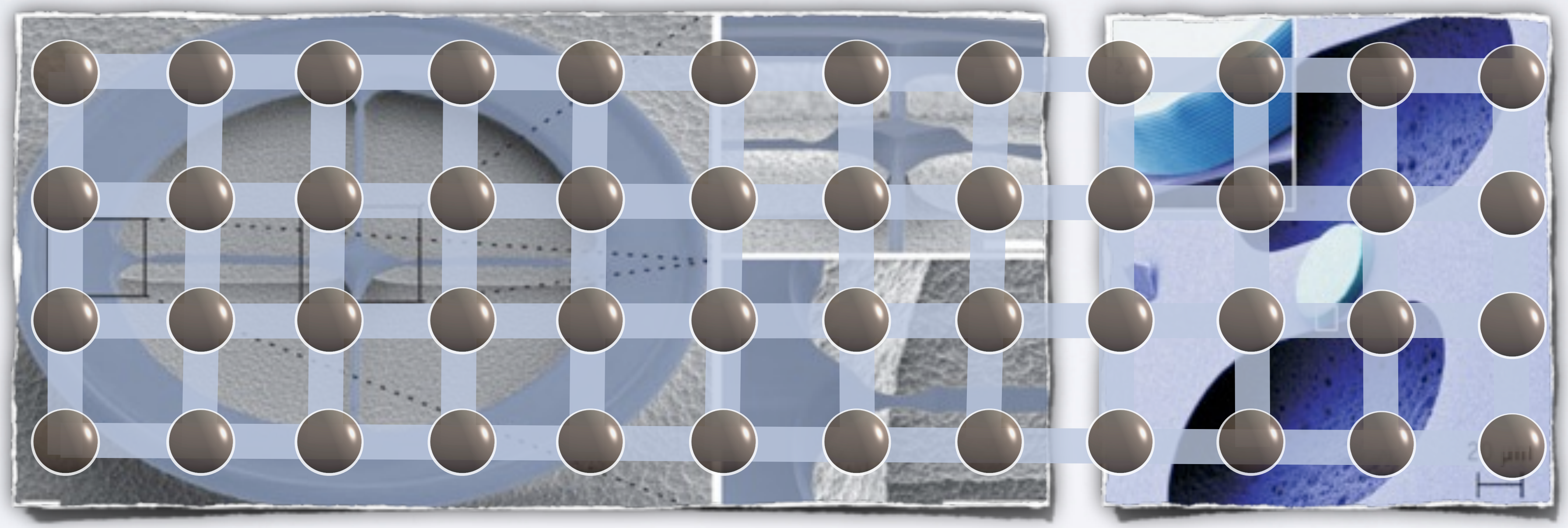

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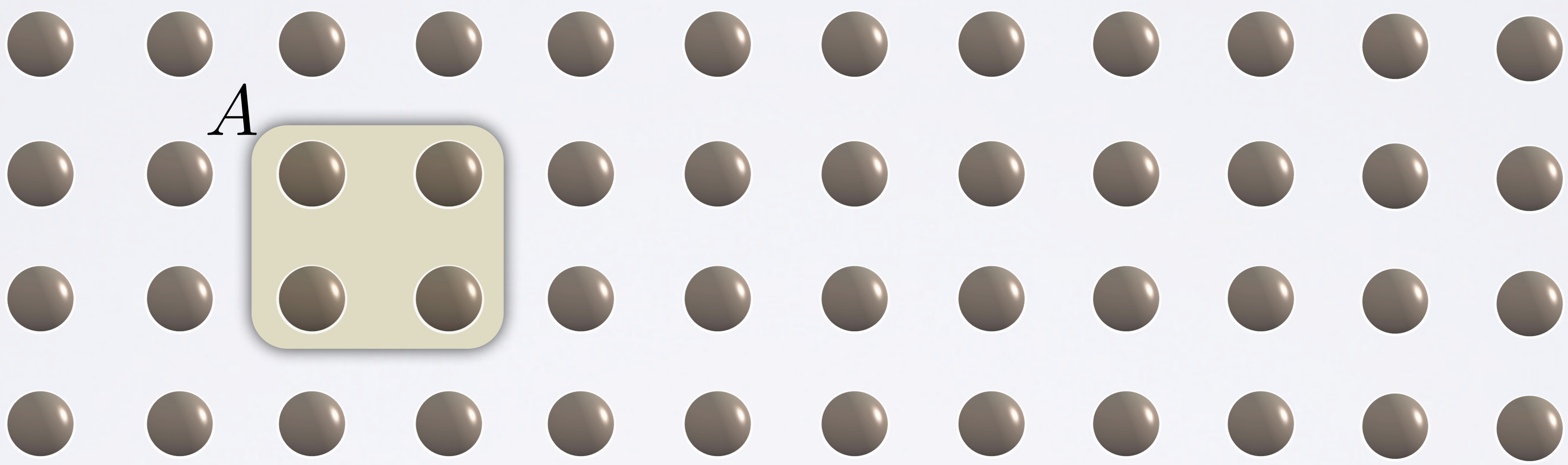

$g[H_A]$

- Gibbs states $g[H] = \dfrac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}$

Pothier, Gueron, Birge, Esteve, Devoret, Phys Rev Lett 79, 3490 (1997)
Peng, Su, Liu, Yu, Cheng, Bao, Nanoscale 5, 9532 (2013)
Hartmann, Mahler, Hess, Phys Rev Lett 93, 080402 (2004)
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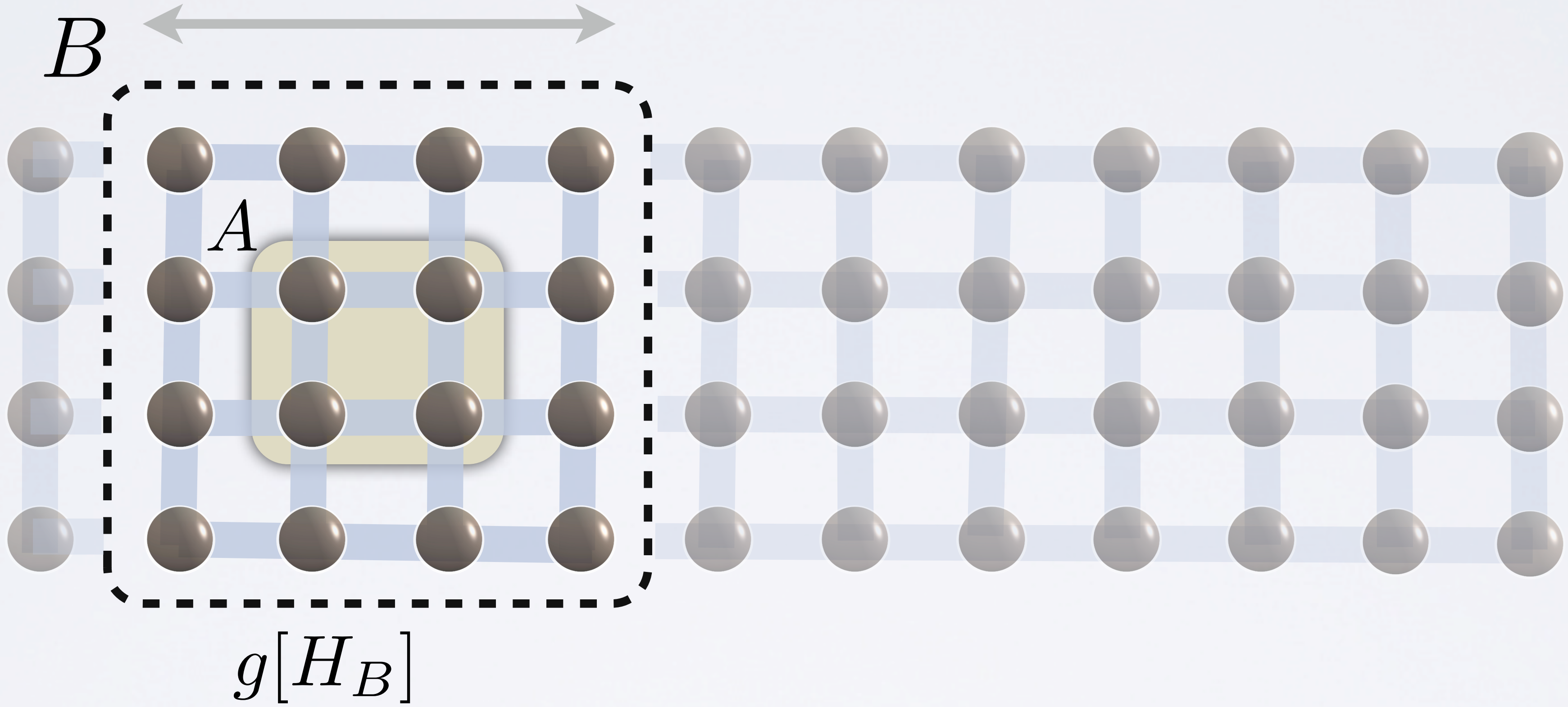

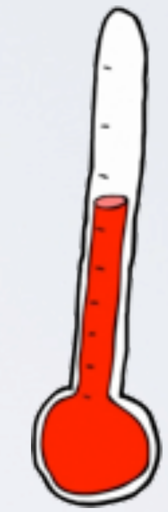

Pothier, Gueron, Birge, Esteve, Devoret, Phys Rev Lett 79, 3490 (1997)
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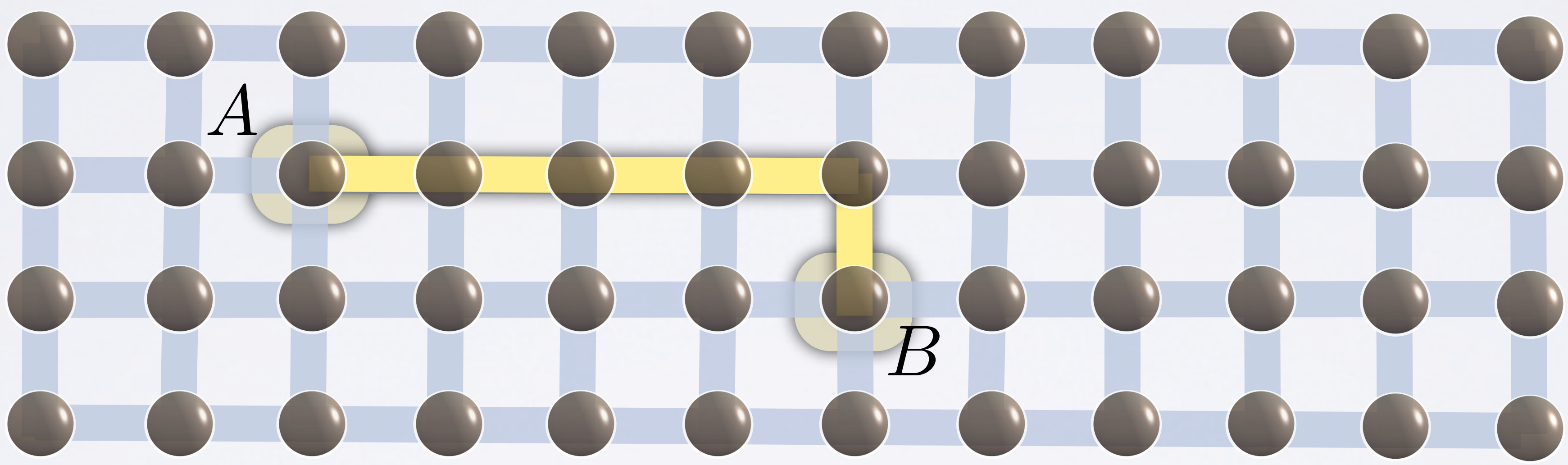
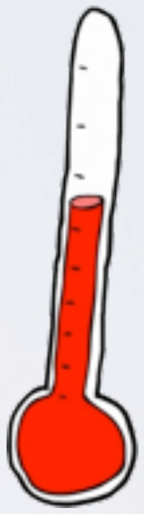
Thermal states of quantum many-body systems

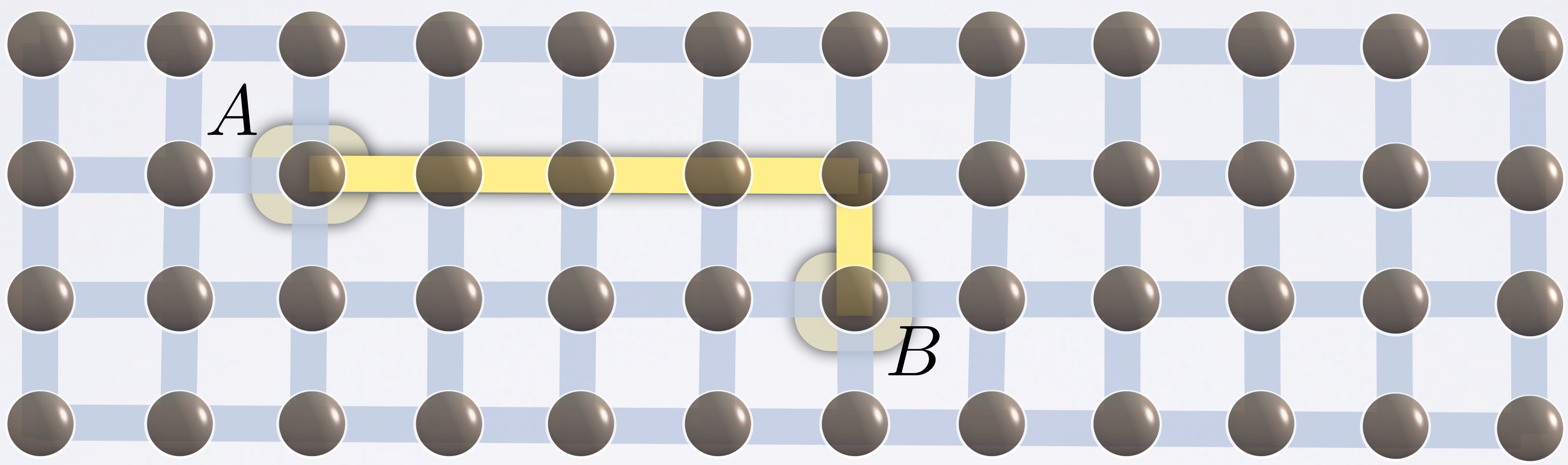


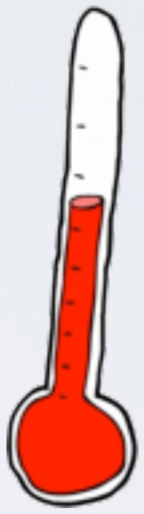


- Again, GS of gapped Hamiltonians have clustering correlations

- **Is there a thermal analogue?**

Thermal states of quantum many-body systems



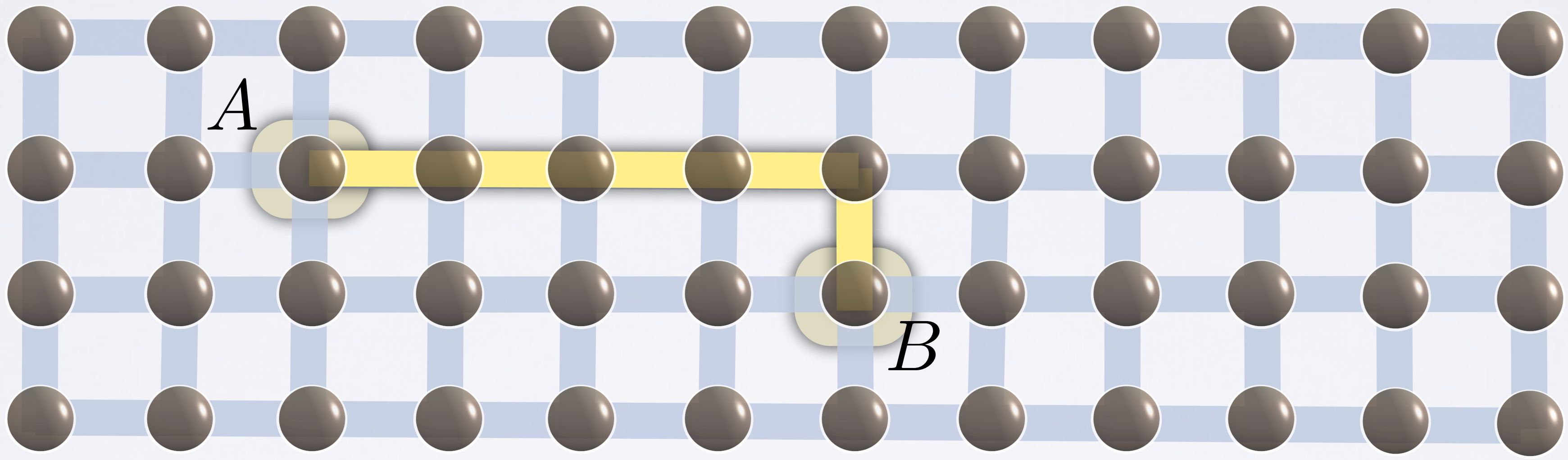


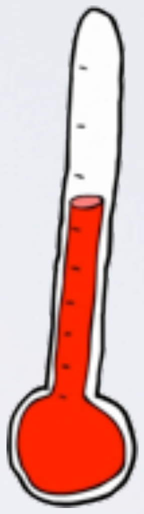
- **Critical temperature**, dependent only on crude properties of graph (+ coupling strength), above which correlations cluster?

- Long-standing **open question**, results known for **classical and continuum models**, some (few) insights into quantum lattice models

Araki, Commun Math Phys 38,1 (1974)
Ruelle, Rev Mod Phys 36, 580 (1964)
Ginibre, J Math Phys 6, 252 (1965)
Greenberg, Commun Math Phys 13, 335 (1969)
Brattelli, Robinson, Operator algebras in quantum statistical mechanics (Springer, 1981)

Thermal states of quantum many-body systems



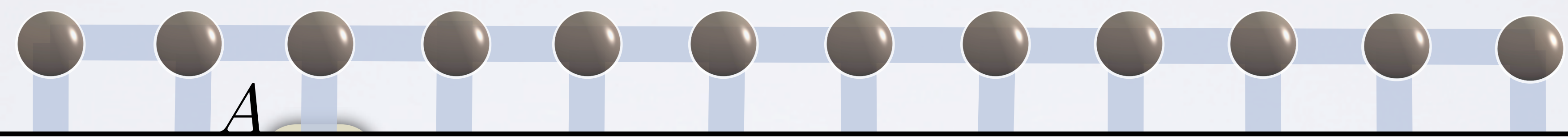


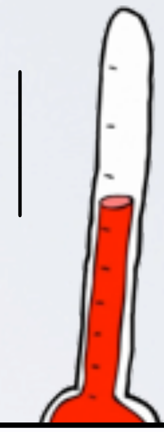
- **Yes :)**

Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816

General clustering of correlations at high temperatures

$$\xi(\beta) = \left|1/\ln(\alpha e^{2|\beta|J}(e^{2|\beta|J}-1))\right|$$





- **Clustering of correlations in thermal states:** Consider local Hamiltonian on arbitrary regular lattice, $J := \max \|h_k\|$ coupling strength, then exists critical inverse temperature

$$\beta^* := \log((1+\sqrt{1+4/\alpha}/2)/(2J)$$

such that for all $\beta < \beta^*$ and $d(A,B) \geq L_0$

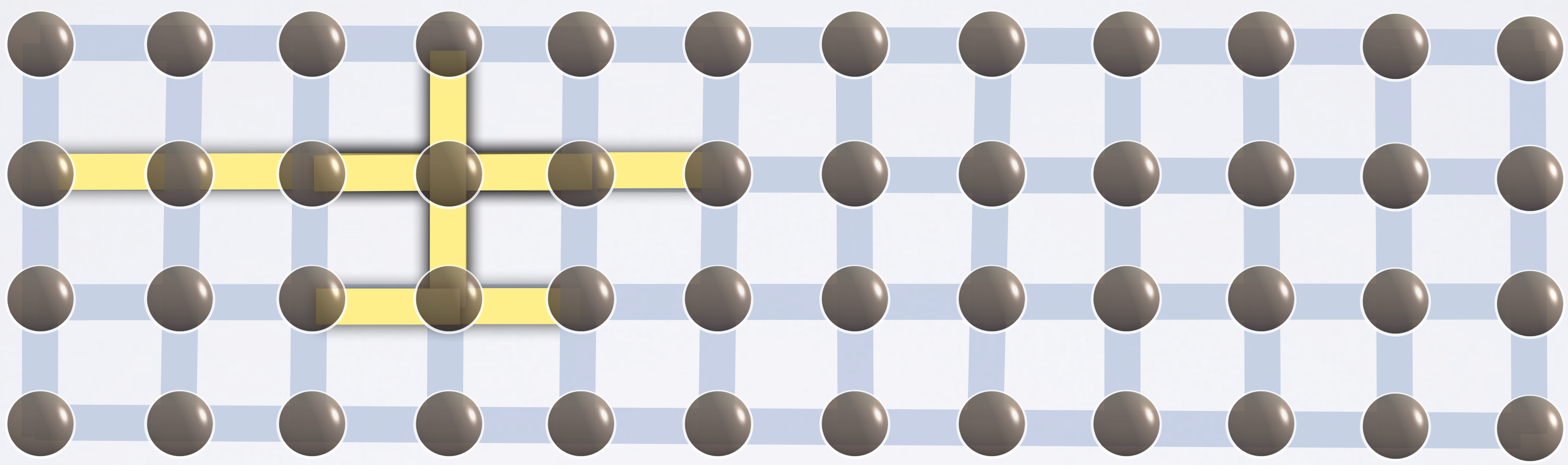
$$C_{g[H]}(A,B) \leq \frac{4\min\{|\partial A|, |\partial B|\}}{\log(3)} \frac{\|f\|\,\|g\|}{1-e^{-1/\xi(\beta)}} e^{-d(A,B)/\xi(\beta)}$$

$$\xi(\beta) := \left|1/\ln(\alpha e^{2|\beta|J}(e^{2|\beta|J}-1))\right|$$

- α **lattice animal constant**
- General statement for **arbitrary lattices** and **covariances**

Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816

Lattice animal constants



- Connected set of edges: **Lattice animal**
- Number a_m of lattice animals F of size $m = |F|$
- Lattice animal constant: Smallest α such that $a_m \leq \alpha^m$

Hastings, Phys Rev B 73, 085115 (2006)
Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816

Flavour of (involved) proof

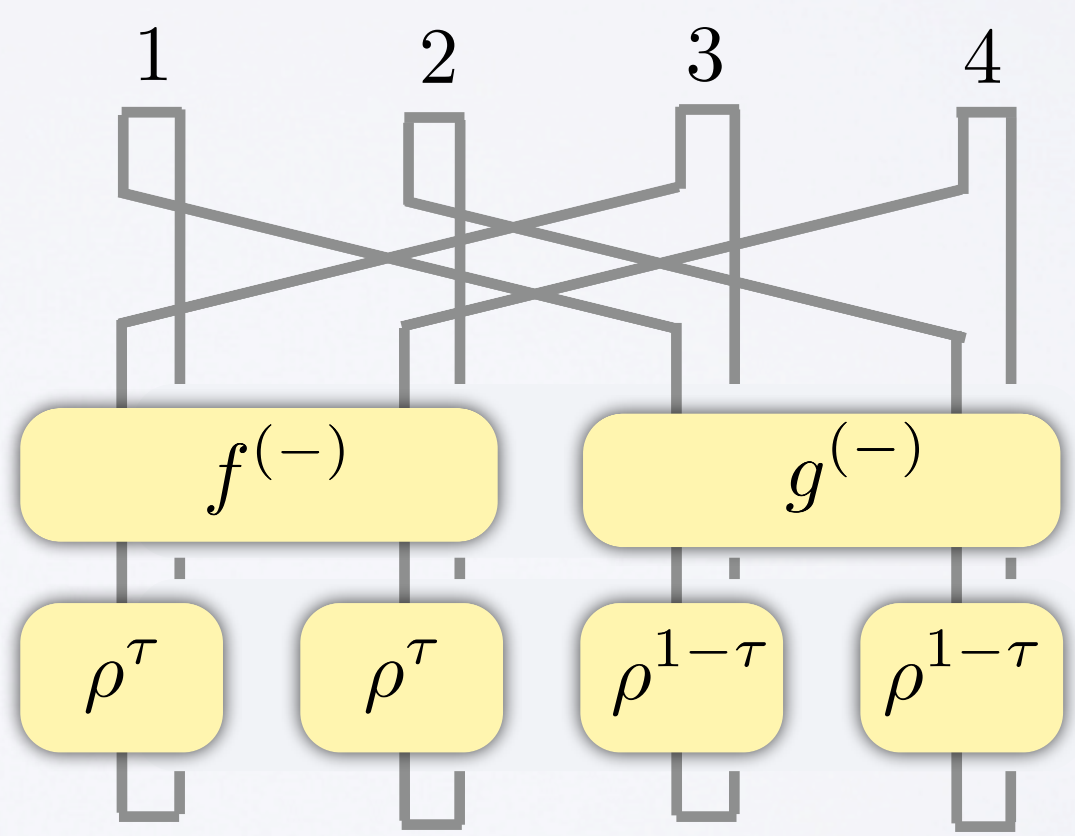
Define **generalized covariance** $\mathrm{Cov}_\rho^\tau(f,g) = \mathrm{tr}(\rho^\tau f \rho^{1-\tau} g) - \mathrm{tr}(\rho f)\mathrm{tr}(\rho g), \;\; \tau \in [0,1]$

Multiple "swap-trick"

Write $\mathrm{Cov}_\rho^\tau(f,g) = \frac{1}{2}\mathrm{tr}\left(\mathcal{S}^{(1,3)}\mathcal{S}^{(2,4)}(f^{(-)} \otimes g^{(-)})(\rho^\tau \otimes \rho^\tau \otimes \rho^{1-\tau} \otimes \rho^{1-\tau})\right)$

on four copies, where $f^{(-)} = f \otimes 1 - 1 \otimes f$

$2\mathrm{Cov}_\rho^\tau(f,g) =$



Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816

Flavour of (involved) proof

Define **generalized covariance** $\mathrm{Cov}_\rho^\tau(f,g) = \mathrm{tr}(\rho^\tau f \rho^{1-\tau} g) - \mathrm{tr}(\rho f)\mathrm{tr}(\rho g), \;\; \tau \in [0,1]$

Multiple "swap-trick"

Write $\mathrm{Cov}_\rho^\tau(f,g) = \frac{1}{2}\mathrm{tr}\left(\mathcal{S}^{(1,3)}\mathcal{S}^{(2,4)}(f^{(-)} \otimes g^{(-)})(\rho^\tau \otimes \rho^\tau \otimes \rho^{1-\tau} \otimes \rho^{1-\tau})\right)$

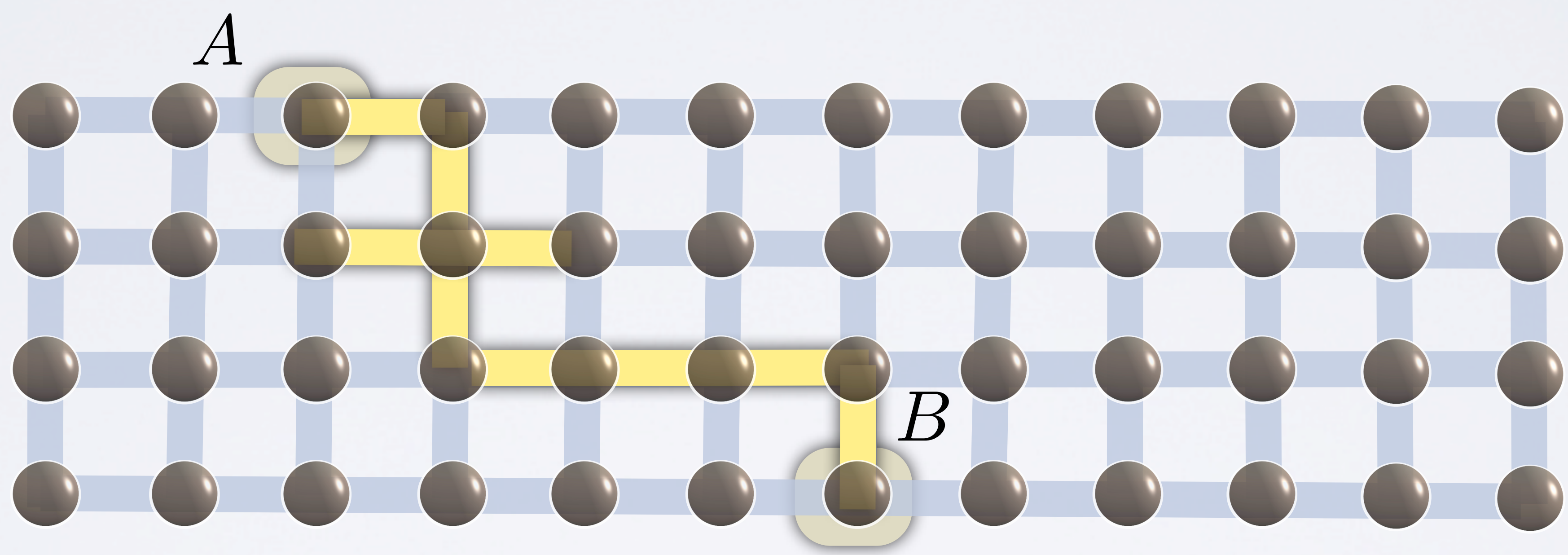
on four copies, where $f^{(-)} = f \otimes 1 - 1 \otimes f$

Cluster expansion of new Hamiltonian $\tilde{H}$

$$\frac{e^{-\beta \tilde{H}}}{\mathrm{tr}(e^{-\beta \tilde{H}})} = \rho^\tau \otimes \rho^\tau \otimes \rho^{1-\tau} \otimes \rho^{1-\tau}$$

Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816

Flavour of (involved) proof



Combinatorics

Symmetry: Only clusters connecting A and B contribute

Cluster expansion of new Hamiltonian $\tilde{H}$

$$\frac{e^{-\beta \tilde{H}}}{\mathrm{tr}(e^{-\beta \tilde{H}})} = \rho^{\tau} \otimes \rho^{\tau} \otimes \rho^{1-\tau} \otimes \rho^{1-\tau}$$

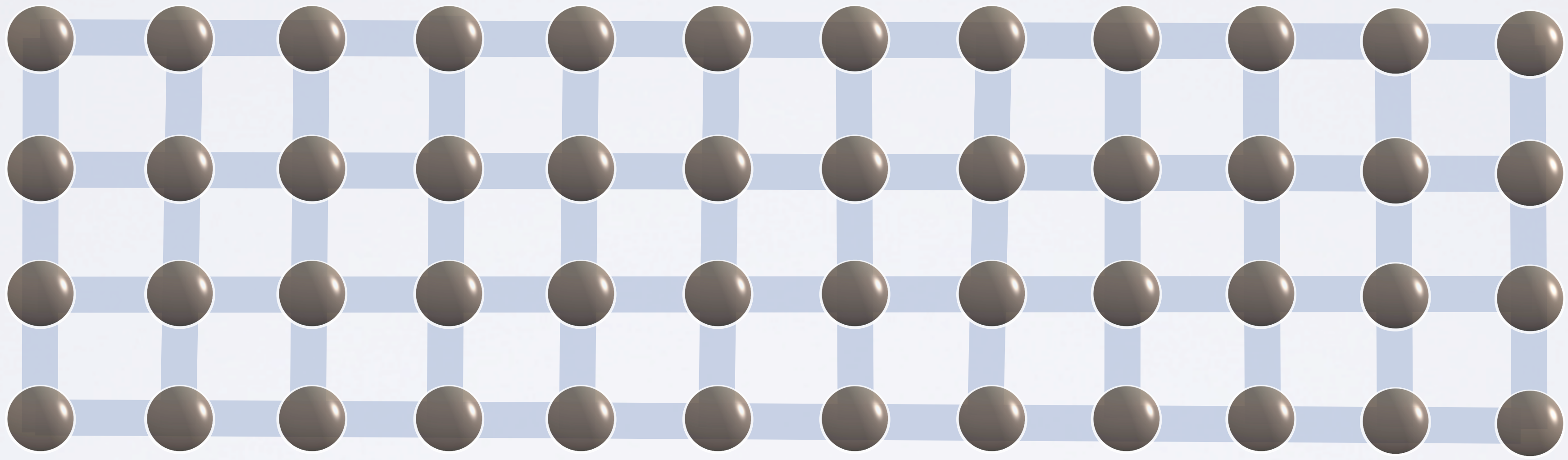
Truncated cluster expansion

$$\left\| \sum_{w \in C_{\geq L}(F)} \frac{(-\beta)^{|w|}}{|w!|} h(w) \right\|_1 \leq Z(\beta) \left(e^{|F| \frac{b(\beta)^L}{1-b(\beta)}} - 1 \right)$$

Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816
Hastings, Phys Rev B 73, 085115 (2006)

Physical implications: Bounds to Curie temperatures

1. "Critical temperature" is **universal upper bound** to phase transition points

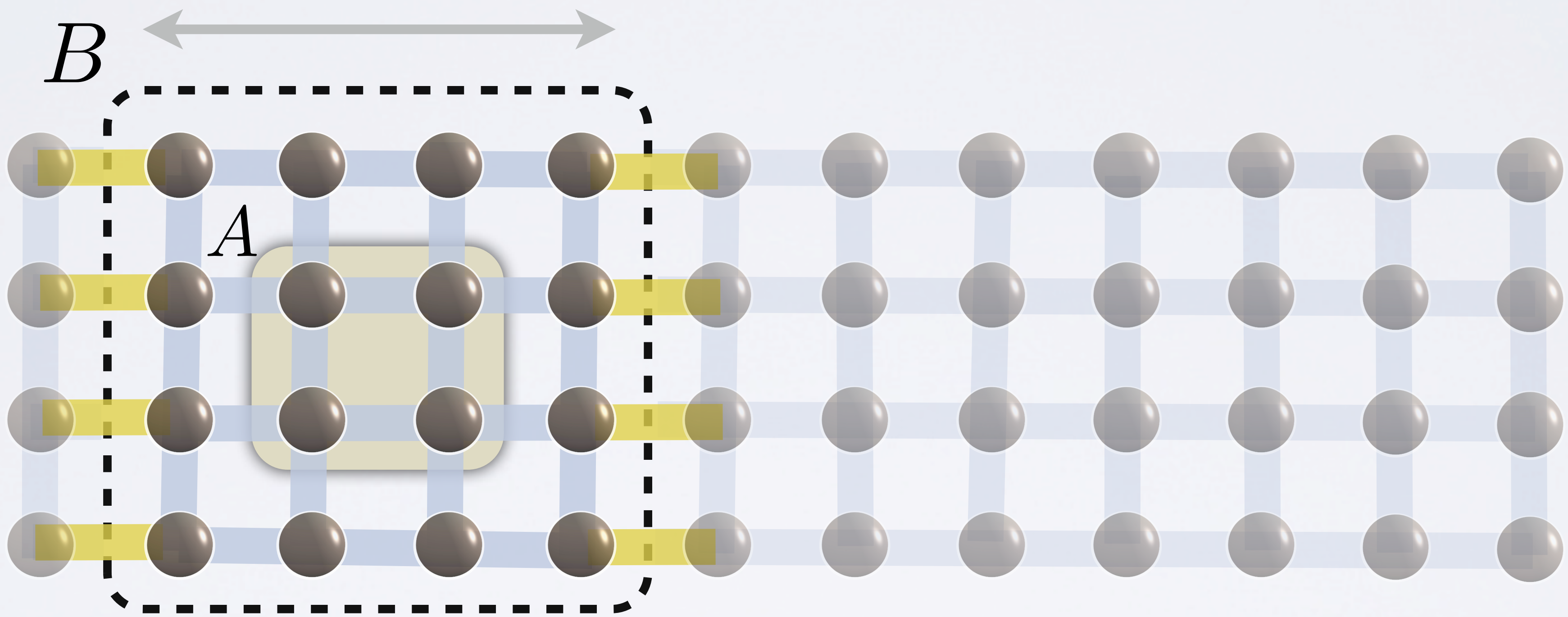


- E.g., ferromagnetic **2d isotropic Ising model** without external field, $1/(J\beta^*) = 24.58$, while phase transition known to happen at 2.27

Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816

Locality of temperature

2. Length scale of temperature



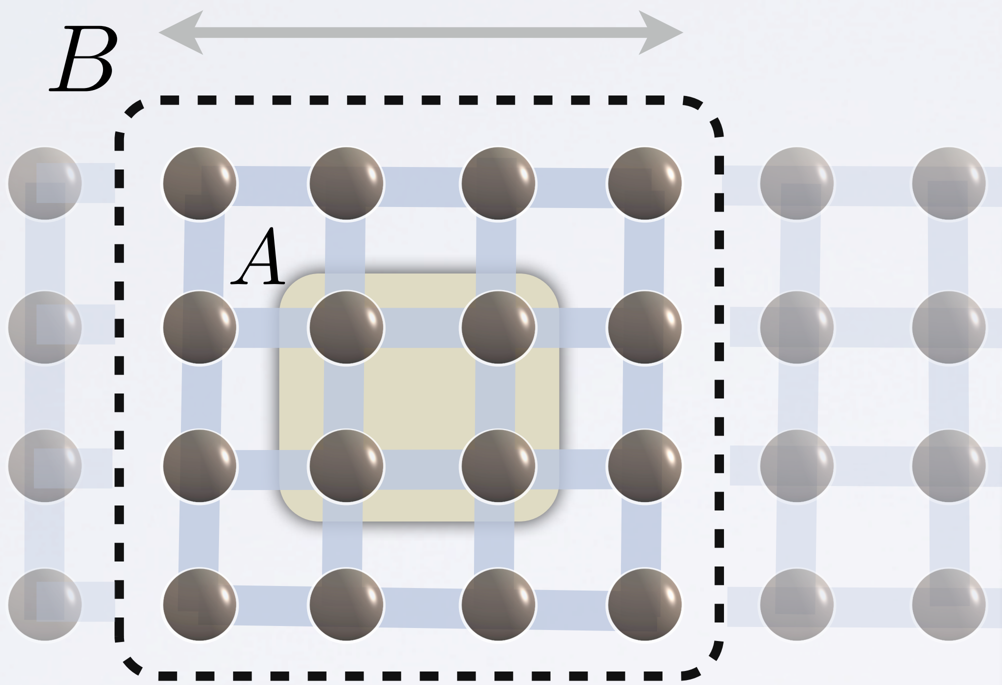
$$\mathrm{tr}(Ag[H(0)]) - \mathrm{tr}(Ag[H]) = \beta \int_0^1 d\tau \int_0^1 ds \mathrm{Cov}^{\tau}_{g[H(s)]}(A, H_I)$$

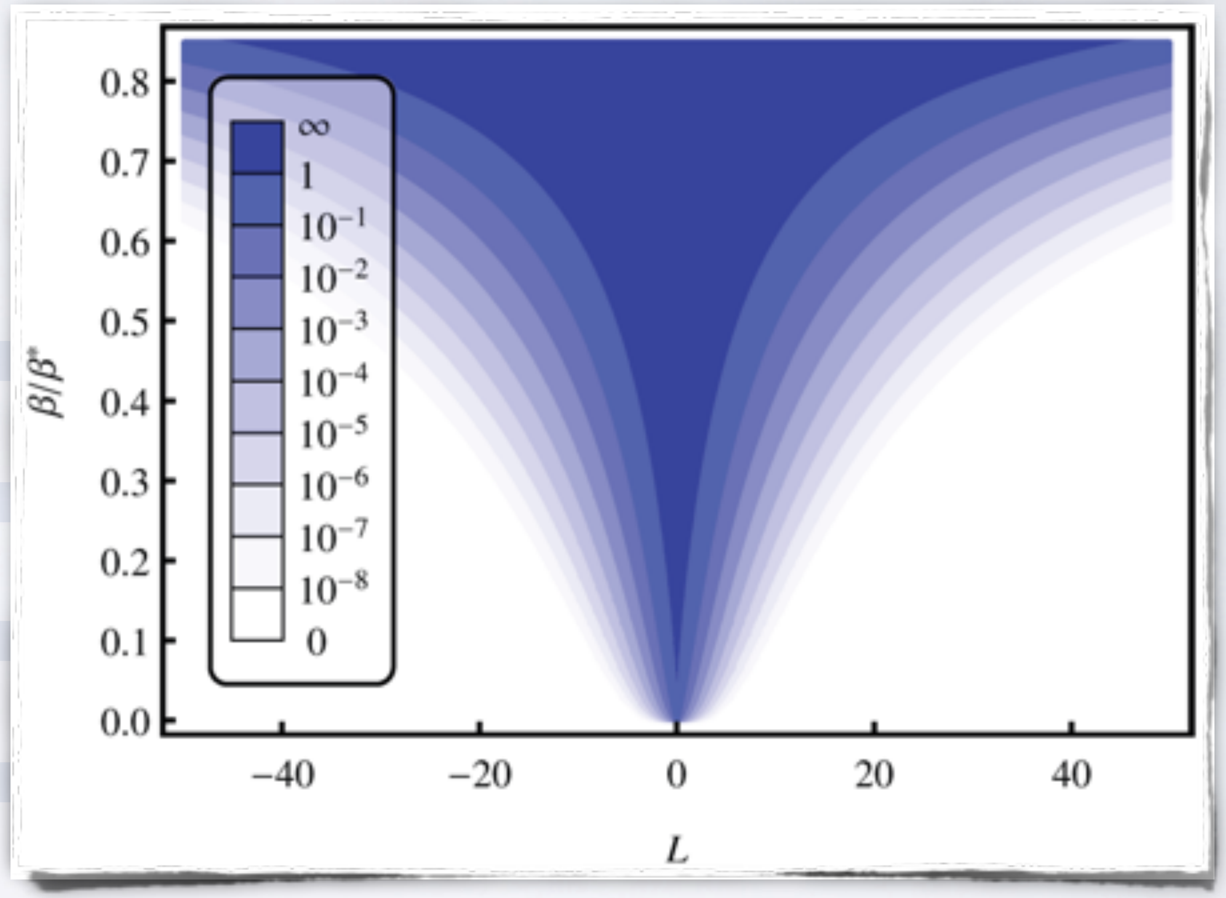
$$H(s) = H - (1-s)H_I$$

Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816

Locality of temperature

2. Length scale of temperature





$$\|g_A[H] - g_A[H(0)]\|_1 \leq \frac{v|\beta|J}{1 - e^{-1/\xi(\beta)}} e^{-d(A,\partial A)/\xi(\beta)}$$

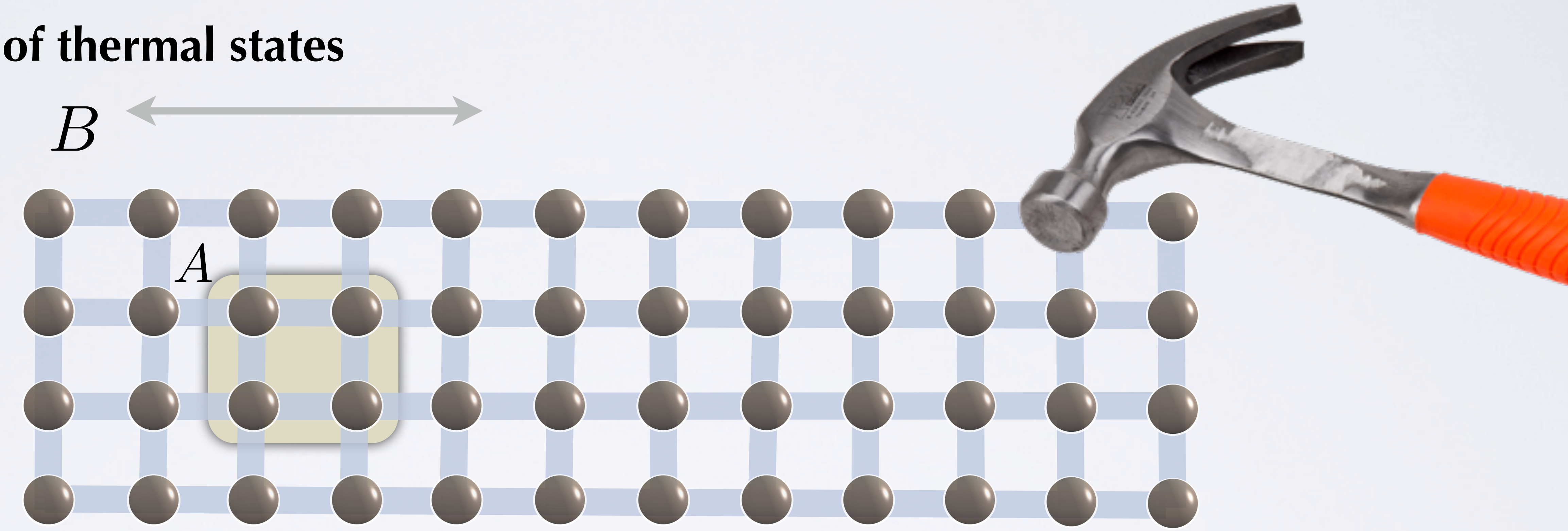
Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816

- Tool in rigorous approaches in **quantum thermodynamics** and **canonical typicality**

Popescu, Short, Winter, Nature Phys 2, 754 (2006)
Goldstein, Lebowitz, Tumulka, Zanghi, Phys Rev Lett 96, 050403 (2006)
Reimann, Phys Rev Lett 101, 190403 (2008)
Linden, Popescu, Short, Winter, Phys Rev E 79, 061103 (2009)
Riera, Gogolin, Eisert, Phys Rev Lett 108, 080402 (2012)

Stability of high temperature thermal states

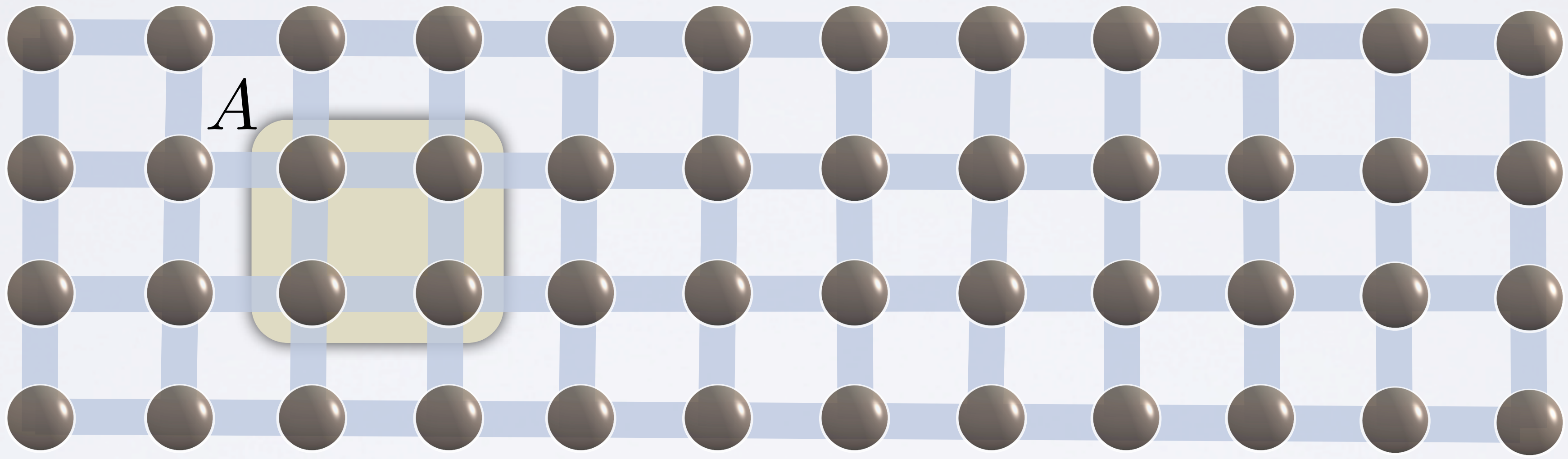
3. Stability of thermal states



Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816

Efficient computation of expectation values

4. Computing of **local expectation values is in P**

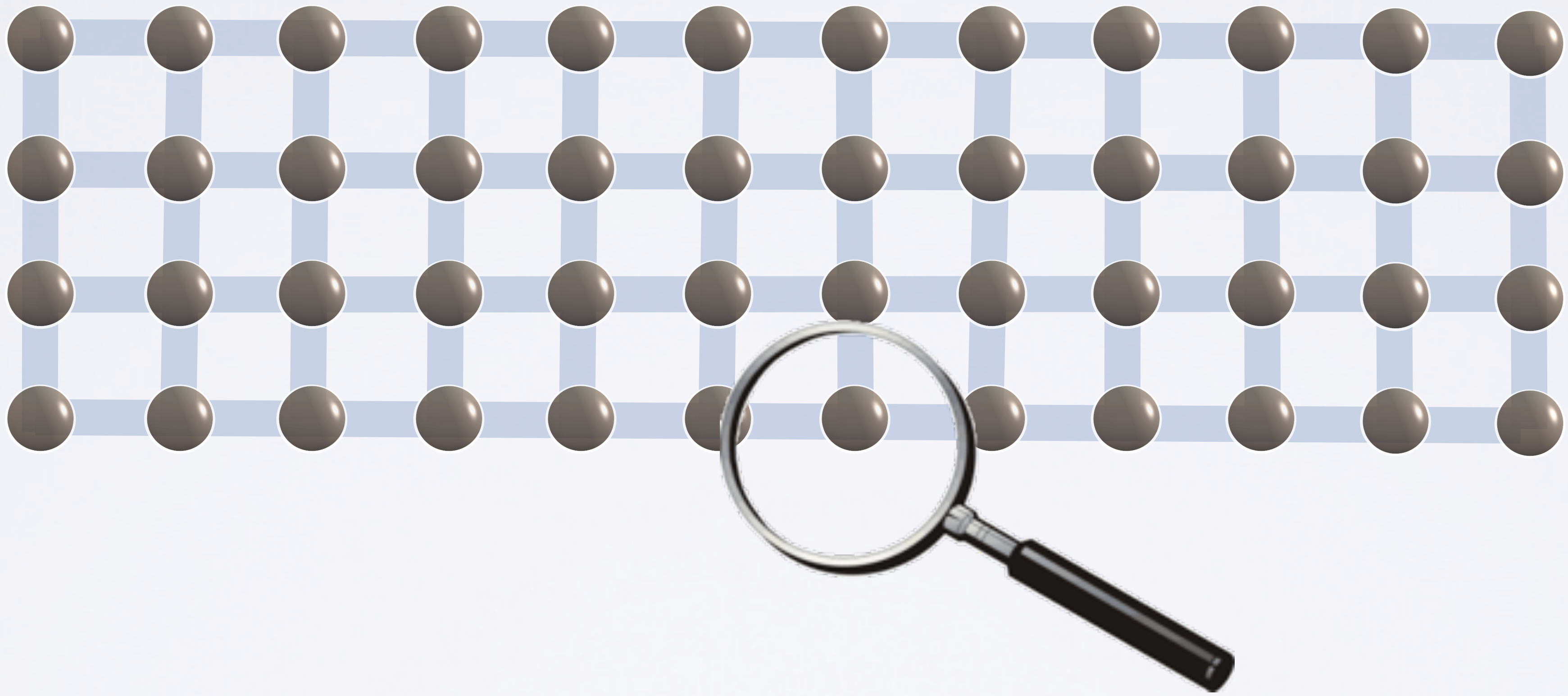


- Guideline for quantum Monte Carlo etc

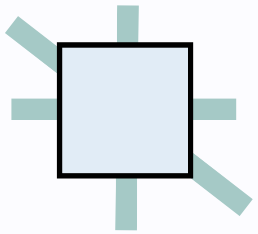
Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816

Matrix-product operators

5. Matrix-product operator approximation



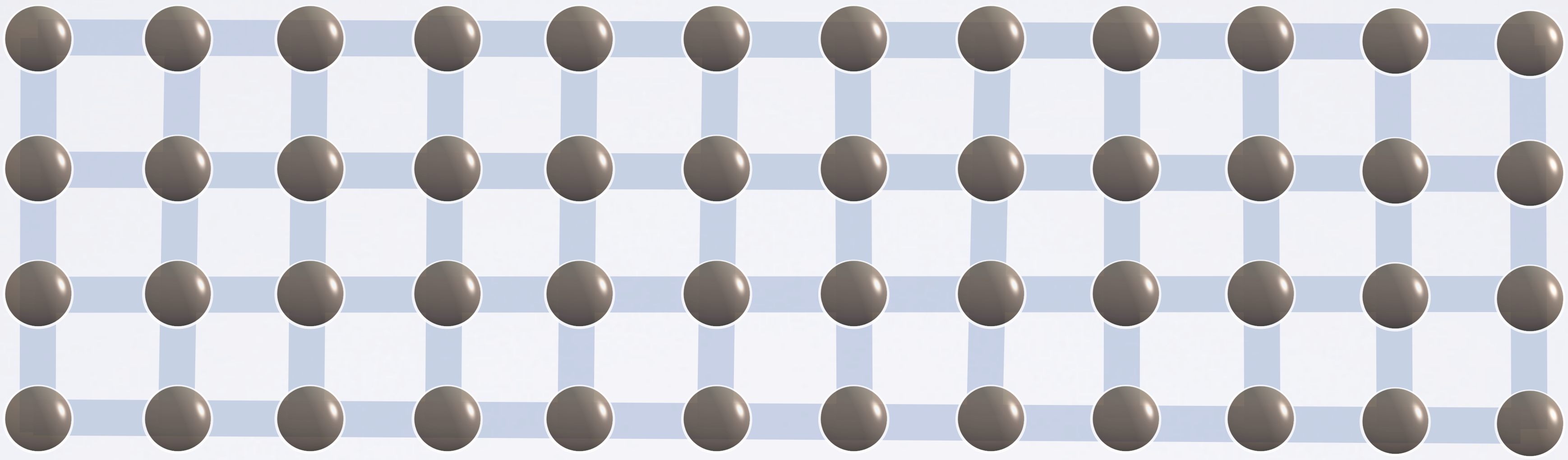
MPO approximation of sub-exponential dimension, efficient in 1D



Kliesch, Gogolin, Kastoryano, Riera, Eisert, arXiv:1309.0816
Hastings, Phys Rev B 73, 085115 (2006)

Interacting fermions

6. All also true **for interacting fermions**



$$\{f_j, f_k^\dagger\} = \delta_{j,k}$$

- Generalizing earlier results on fermionic covariance matrices

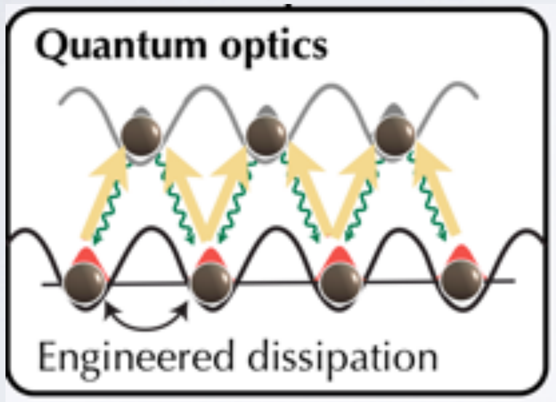
Hastings, Phys Rev Lett 93, 126402 (2004)

Thermal many-body systems

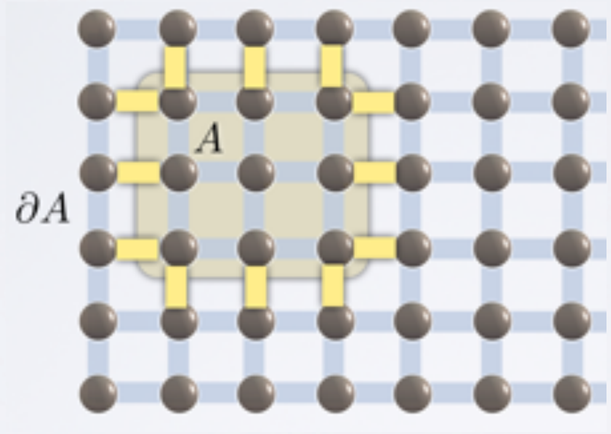
- **Lessons**:

- **Length scale** at which one can speak of temperature!
- High temperature thermal states have **clustering correlations**
- Can **efficiently compute local expectation values**

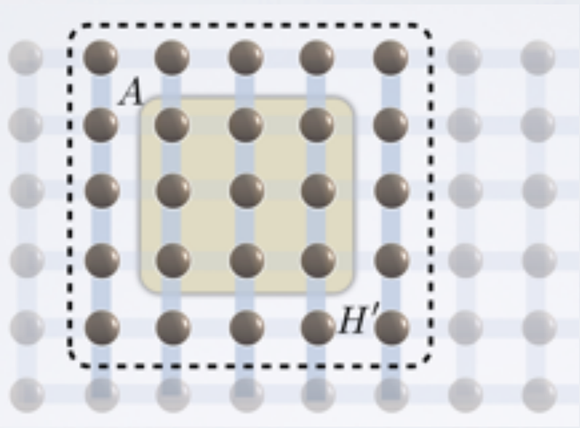
Summary: Liouvillians and thermal states



"Liouvillian complexity": Interesting arena to study many-body problems



Clustering of correlations, area laws, topological order



Intensivity of temperature and correlations in thermal many-body systems

- Does the exponential complexity of general quantum systems persist at high temperature?

No

Thanks for your attention!