AGSPs and an Area Law for Gapped 1D Systems

Itai Arad, Alexei Kitaev, Zeph Landau, Umesh Vazirani
The difficulty of understanding many-body physics

Each particle in a $d$-dimensional space—$C^d_n$ particles—becomes a tensor the individual spaces together creating $H = (C^d)^\otimes n$. System described by a state: a unit vector $|v\rangle \in H$.

The same property that leads to the power of quantum computation is the major barrier for understanding many-body physics: Exponential Dimensional Space. So even describing a state requires exponential amount of information.
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Exponential Dimensional Space

So even describing a state requires exponential amount of information.
A Basic Question

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- Do they have a special structure?
- Does that structure allow for meaningful short descriptions?
- Does that structure allow us to compute various properties of them?
Physically Relevant States: Ground States of Local Hamiltonians

- **Local term**: $H_i$ is a linear operator (self-adjoint) that acts "locally": non-trivial on only a few particles.

- **Local Hamiltonian**: $H = \sum_i H_i$, an operator formed from the sum of local terms.

- **Ground State**: The ground state $|\Gamma\rangle$ is the smallest eigenvector of $H$.

- **Gap**: Distance between the lowest two eigenvalues. Focus on unique ground state and constant gap.

Ground states model the state of the system at low temperatures.
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For (gapped) 1D systems: yes
For higher dimensions: ?

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Became known as an **Area Law**.

[ʼ01, Vidal, Latorre, Rico, Kitaev] Area Law formalized in terms of entanglement entropy.
Area Law in 1D systems

1D Area law proved [Hastings ’07].
- Established that 1D ground states (constant gap) satisfy an area law.
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Natural Questions:

- Does the result generalize to 2D?
- Does it suggest an algorithm for finding the ground state?
"If there is a problem you can’t solve, then there is an easier problem you can’t solve: find it." - George Polya
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A special case: frustration-free commuting case.

- Can assume $H_i$ are projections.
- $P = \prod_i (1 - H_i)$ projects onto the ground space.
- Apply to a tensor product state to immediately get an MPS representation.
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How to generalize this idea?
Approximate Ground State Projection (AGSP)

Properties:
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Properties:
- It "approximately" projects onto one vector you want (ground state).
Approximate Ground State Projection (AGSP)

Properties:

- It "approximately" projects onto one vector you want (ground state).
- It isn’t too complex.
Consequences of AGSPs

Two new results:

- ['11,’12, Arad, Kitaev, Landau, Vazirani] Exponential improvement in parameters of the 1D area law which → a sub-exponential time algorithm for finding solutions.

- [’13, Landau, Vazirani, Vidick] Polynomial time algorithm for finding solutions to constant gapped 1D systems.
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An area law in 2 steps

Area law proof:

1. Find a not very complex state that has constant overlap with the ground state.
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2. Repeatedly apply an AGSP to that state to rapidly get a good approximation to the ground state.
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Ground State

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Area law proof:

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Both steps use AGSPs– the first is much more delicate.
Measure of Complexity: Entanglement rank

A state on $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the form $\sum_1^C a_i \otimes b_i$ will be said to have entanglement rank $C$. 

Entanglement rank behavior

Multiplicative for operators applied to states or product of operators.

Additive for sums of states or operators.
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AGSP: almost projection with small entanglement rank

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- It approximately projects onto the ground state:

$$\Delta < \frac{1}{2}$$

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  - Eigenvalues of AGSP

  - Eigenvalues of $H$

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```
Ground state

1

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Δ

Ground state

Eigenvalues of H
```

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```
\cdot \cdot \cdot
\cdots
D
\cdots
\cdot \cdot \cdot
}
K
```

**Critical threshold** $D\Delta < 1$. 
Theorem (Area Law) [Arad, Landau, Vazirani] The existence of an AGSP $K$ for which $D\Delta < 1/2$ proves that the ground state has entropy $O(\log D)$.

Proof:
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Proof:
Role of AGSP in proving Area Law cont.

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**Proof:**

\[ |A\rangle |B\rangle \]

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![Diagram showing A and B regions cut into D pieces and ground state |A>|B> with AGSP K|A>|B> and Ground State direction]
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Proof:

\[
\begin{aligned}
\text{Ground State} &\quad \begin{array}{c}
|A'| \\
\cdots \\
|B'
\end{array} \\
&\quad K^1 \\
&\quad D^1 \\
&\quad l=O(\log n)
\end{aligned}
\]
AGSP construction

AGSP will be a well chosen polynomial in the local terms $H_i$.

Three key ingredients:
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- truncation away from the cut,
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Three key ingredients:

- **truncation** away from the cut,
- **Chebyshev polynomials**, 
- Analysis of the entanglement rank will involve **polynomial interpolation**.
Building Intuition

\( H \) has eigenvalues in \([0, n]\). So \( H/\|H\| \) has eigenvalues in \([0, 1]\).

\[ \Delta \epsilon \|H\| \]
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**How can we make $\Delta$ smaller without increasing $\ell$?**

- Smaller $\|H\|$ would be better but we don’t want to lose the 1D structure of $H \rightarrow$ **truncate** the ends to get $H' = (H_L + H_1 + H_2 + \cdots + H_s + H_R)$. 

\[
\begin{array}{c}
\text{1} \\
\vdots \\
\text{s+1}
\end{array}
\]
Building intuition: using Chebyshev polynomials

How can we make $\Delta$ smaller without increasing $\ell$?

- Truncate away from the cut.
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- Choose a better polynomial.
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Chebyshev polynomials: small in an interval:
Building intuition: using Chebyshev polynomials

**Candidate 2**: $C_\ell(H')$ = dilation and translation of Chebyshev applied to $H'$:

$$K = C_\ell(H')$$

with

$$\Delta = e^{\frac{-\ell \sqrt{\epsilon}}{\sqrt{||H'||}}}. $$
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This will be our AGSP. How complex is it?
AGSP complexity: Entanglement rank analysis

\[(H')^\ell = \sum \text{(product of } H_j)\].

For a single term:
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**Total**: \(d^{2\ell/s} + s\)
Problem: Too many \( (s^\ell) \) terms in naive expansion of \( (H')^\ell \).
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Need to group terms in a nice way (polynomial interpolation) but it all works out with total entanglement increase of the same order as the single term.
Putting things together: Area Law for $H'$

Chebyshev $C_\ell(H')$ has $\Delta \approx e^{-O(\ell/\sqrt{s})}$:

\[ f(x) \Delta \epsilon ||H|| \]

Entanglement analysis yields $D \approx O(d^{\ell/s+s})$.

Choosing $\ell = s^2$ yields $D \Delta \approx e^{-s^{3/2}+s \log d} < 1$ for appropriate choice of $s \approx \log^2 d$. 
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\begin{align*}
\Delta & \quad \quad f(x) \\
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Area Law of entanglement entropy $\log(D) = \tilde{O}\left(\frac{\log^3(d)}{\epsilon}\right)$
What about 2d? Any improvement in the entropy bound $\tilde{O}(\log^3 d \epsilon)$ would produce a sub-volume law for 2D systems.

Towards more local algorithms in 1D . . .

Of independent interest: entanglement rank has a "random walk" type behavior (added entanglement of $H_\ell$ is $dO(\sqrt{\ell})$).

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The Landscape

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