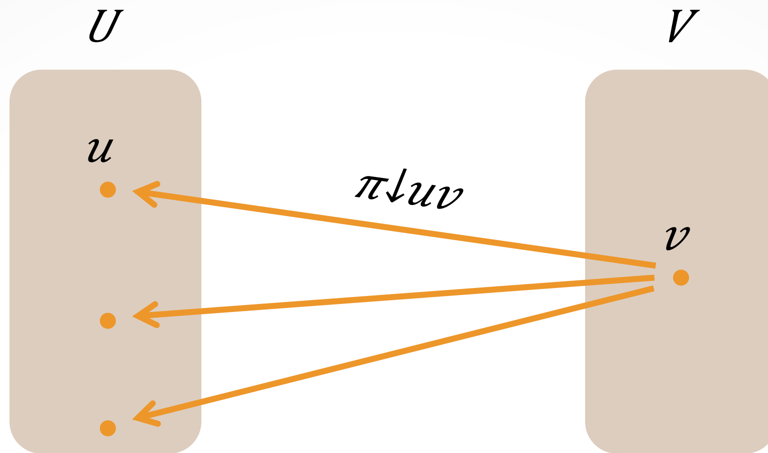
Parallel Repetition and Direct Products

Irit Dinur
Weizmann

Joint work with David Steurer and Thomas Vidick constraint graph



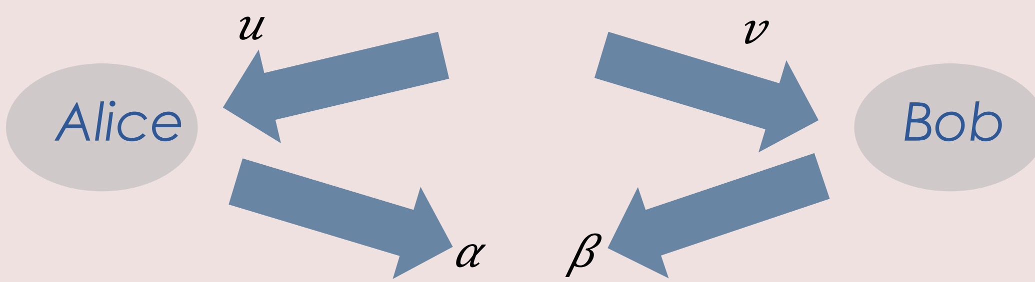
bipartite regular (for simplicity)

game G

random: uv

no communication between A & B

strategy $g{:}U\to\Sigma$



strategy $f{:}V\to\Sigma$

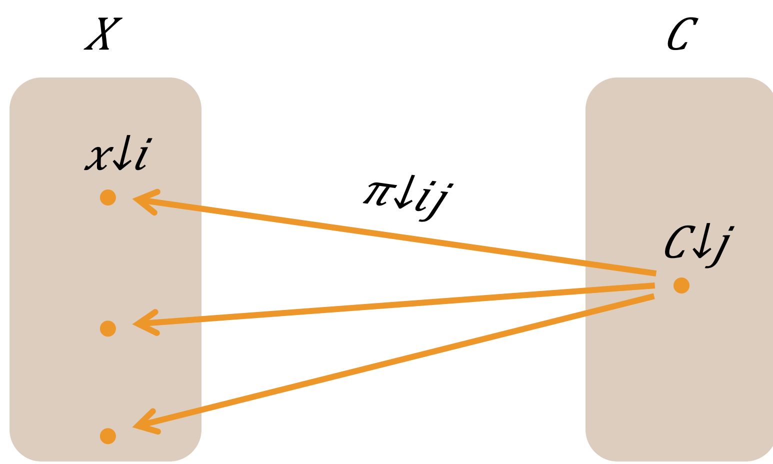
WIN: $\alpha\beta\in\pi_{uv}$

$$\text{value}(G) := \max_{f,g} \mathbb{P}_{uv}\{\, (g(u),f(v)) \in \pi_{uv} \,\}$$

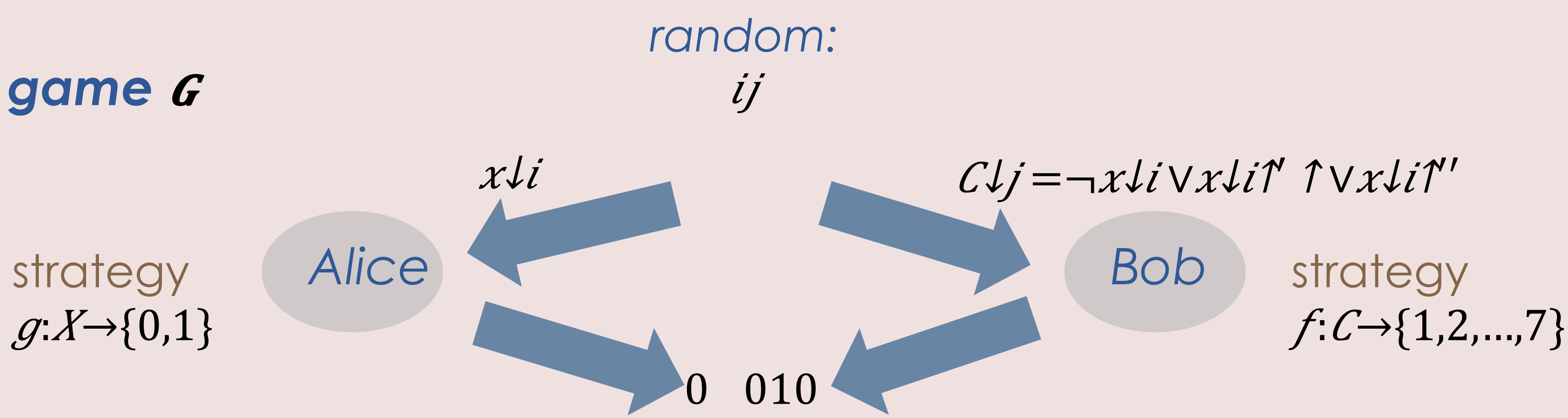
Example: the 3SAT game

Given a 3SAT formula with variables $X=\{x_1,\dots,x_n\}$ and clauses $C=\{C_1,\dots,C_m\}$

Clauses vs. variable constraint graph



game $\boldsymbol{G}$



Games and CSPs

Label Cover = the problem of finding the value of a given game
(this is a 2-local Constraint Satisfaction Problem - CSP)

Every CSP gives rise to a clause vs. variable game
(or to a multi-player game)

MIP = games

PCP = CSPs (fixed proof = assignment;
randomized verification = clauses)

(classically equivalent, but the quantum analogs are not)

COMPLEXITY OF LABEL-COVER

NP-hard to decide if $val(G)=1$ or $val(G)<1$ (Cook-Levin)

NP-hard to decide if $val(G)=1$ or $val(G)<1-\varepsilon$ (PCP)

NP-hard to decide if $val(G)=1$ or $val(G)<\varepsilon$ (strong PCP)

The PCP Theorem [AS, ALMSS '91]

strong PCP theorem: "gap-label-cover is NP-hard"
Proof: By reduction from tiny gap to constant gap,
aka **amplification**

Start with label-cover G and end up with G', s.t.

If $val(G) = 1$ then $val(G') = 1$ then $val(G'^{\otimes k})=1$

If $val(G) < 1$ then $val(G')<1-\varepsilon$ then $val(G'^{\otimes k})<\varepsilon$

How?

- by algebraic encoding [AS, ALMSS 1991]; or
- By "derandomized parallel repetition" (D. 2007)

Repetition

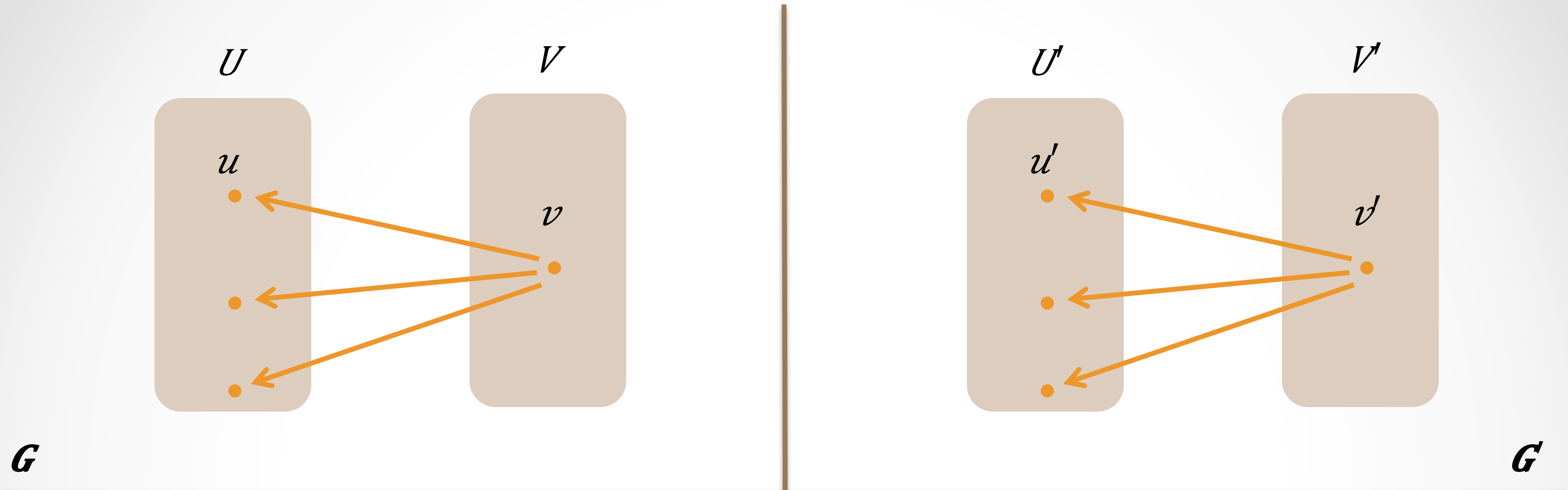
Sequential repetition: run the game t rounds
Probability of not catching an error = exp(-t)
CSP perspective: Number of bits read from proof = O(t)
Not a 2-local CSP (label-cover) anymore*. 2t-local.

Remark: derandomizing this is easy (e.g. via "expander walks")
So we can plug in t=log n and get the sequential analog of the sliding scale conjecture

Want: A 2-local CSP with the same properties
Why? Useful for inapprox, 2-local is simulateable by graph problems

Approach: Parallel Repetition

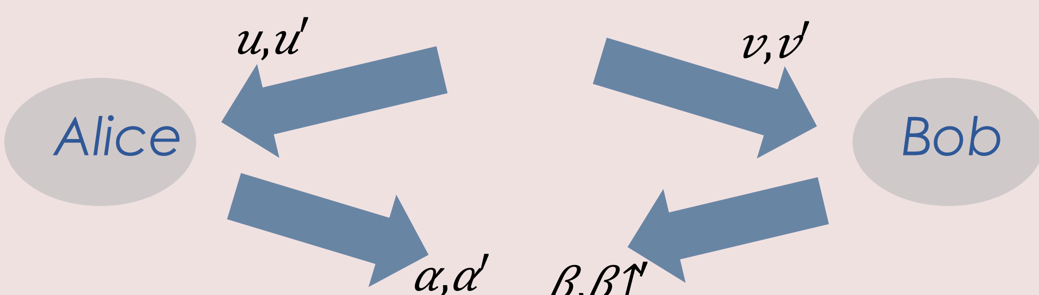
Parallel Repetition



Playing G, G' in parallel

random:
uv, $u'v'$

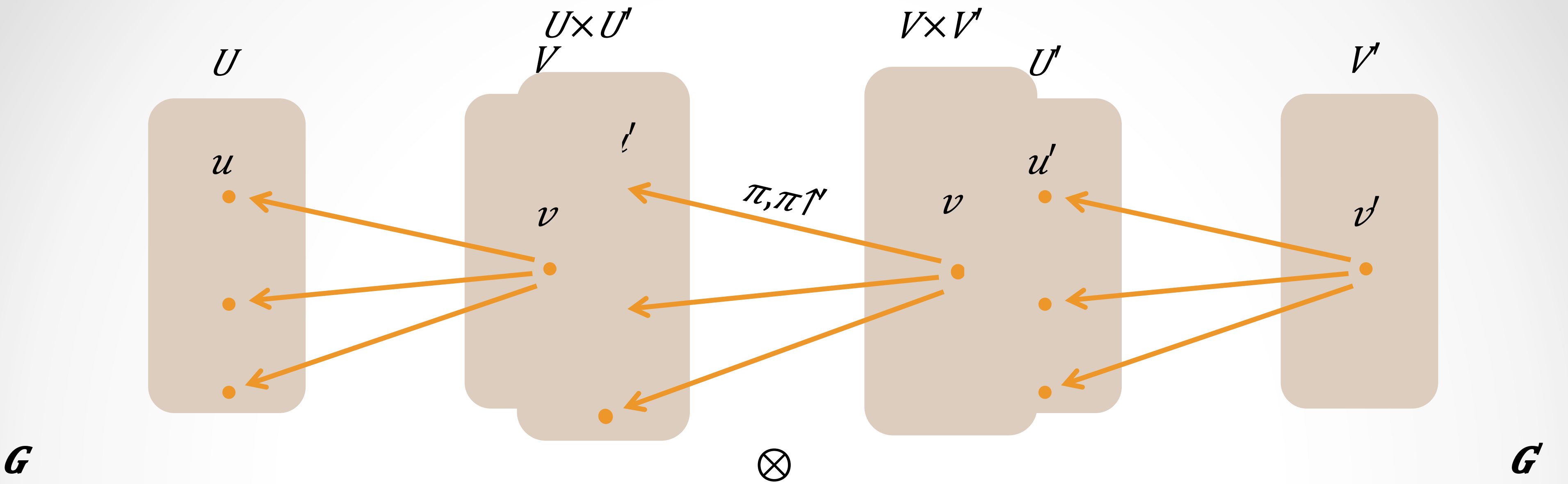
no communication between A & B



strategy
$g: U\times U' \to \Sigma\times\Sigma'$

strategy
$f: V\times V' \to \Sigma\times\Sigma'$

WIN: $\alpha\beta\in\pi_{uv}$ &
$\alpha'\beta'\in\pi_{u'v'}$



game $G\otimes G'$

random:
$uv,\ u'v'$

no communication between A & B

u,u' → Alice

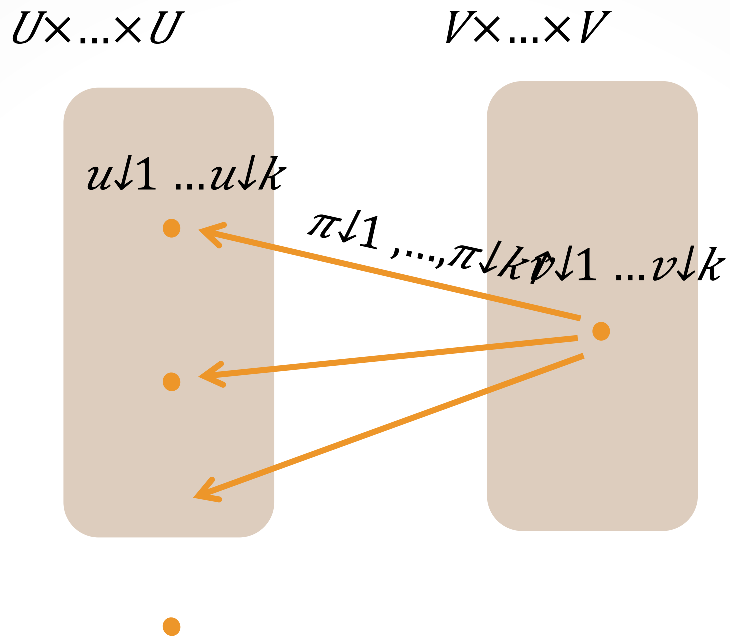
v,v' → Bob

strategy
g: $U\times U' \to \Sigma\times\Sigma'$

strategy
f: $V\times V' \to \Sigma\times\Sigma'$

α,α' β,β'

WIN: $\alpha\beta\in\pi_{uv}$ &
$\alpha'\beta'\in\pi_{u'v'}$



THIS IS A "PRODUCT" OPERATION ON GAMES

Product of Games: $\boldsymbol{G \otimes H}$

Parallel Repetition: $\boldsymbol{G{\uparrow}\otimes k}$

Q1: If $value(G1) = \omega 1$ and $value(G2) = \omega 2$

then what is $value(G1 \otimes G2)$?

Q2: If $value(G) = \alpha$, then what is $value(G \uparrow \otimes k)$ for $k > 1$?

One obvious candidate is the direct product strategy.

But it is not, in general, the best strategy.

previous bounds

(long history, notoriety) [… Verbitsky'94, Feige-Kilian'94]

parallel repetition theorem [Raz'95,improved: Holenstein'07, Rao'08]
(for projection games)

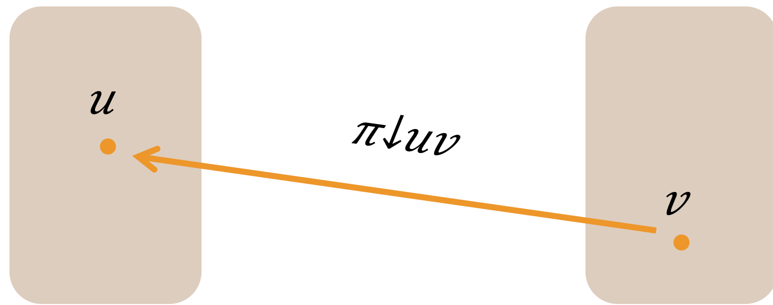
$$\mathrm{value}(G) \leq 1-\varepsilon \implies \mathrm{value}(G^{\otimes k}) \leq (1-\varepsilon^{32}/C)^{k}$$

(tight even for games with XOR constraints) [Raz'08]

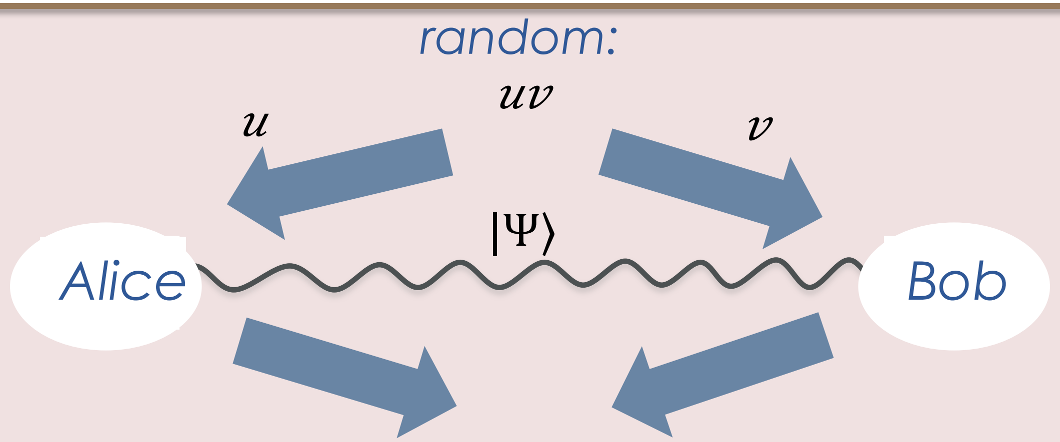
main application: hardness amplification for LABEL COVER
1 vs δ approximation is NP-hard (basis of inapproximability results)

Parallel repetition for entangled value

constraint graph



game $\boldsymbol{G}$



A & B also share an entangled state $|\Psi\rangle$

Alice's strategy: For each u, Alice has a set of measurements $\{A_u^a\}_a$ s.t. $\sum_a$

Bob's strategy: For each v, Bob has a set of measurements $\{B_v^b\}_b$ s.t. $\sum_b B_v$

Parallel repetition for entangled value

[Holenstein ‘06] – parallel repetition for non-signalling strategies

[Cleve, Slofstra, Unger, Upadhya ‘08] – perfect parallel repetition for entangled value of XOR games

[Kempe Regev Toner ‘08] - unique games

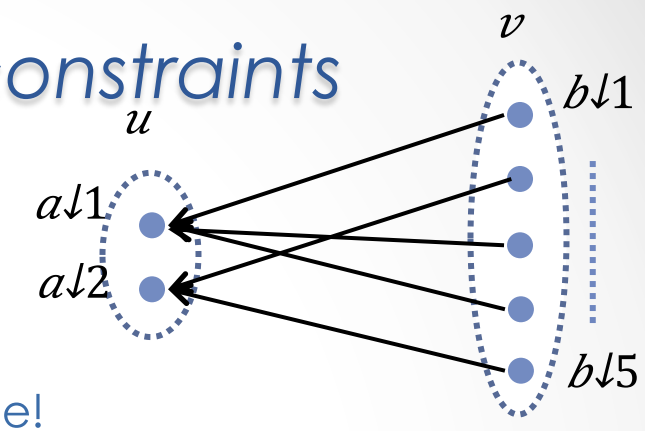
[Kempe-Vidick ‘11] : polynomially decaying bounds for parallel repetition of entangled value
Extending the Feige-Kilian proof to the quantum setting
(adding consistency queries to force provers to play a direct product strategy)

[Chailloux, Scarpa ‘14] – parallel repetition for general games, bound depends on size of game

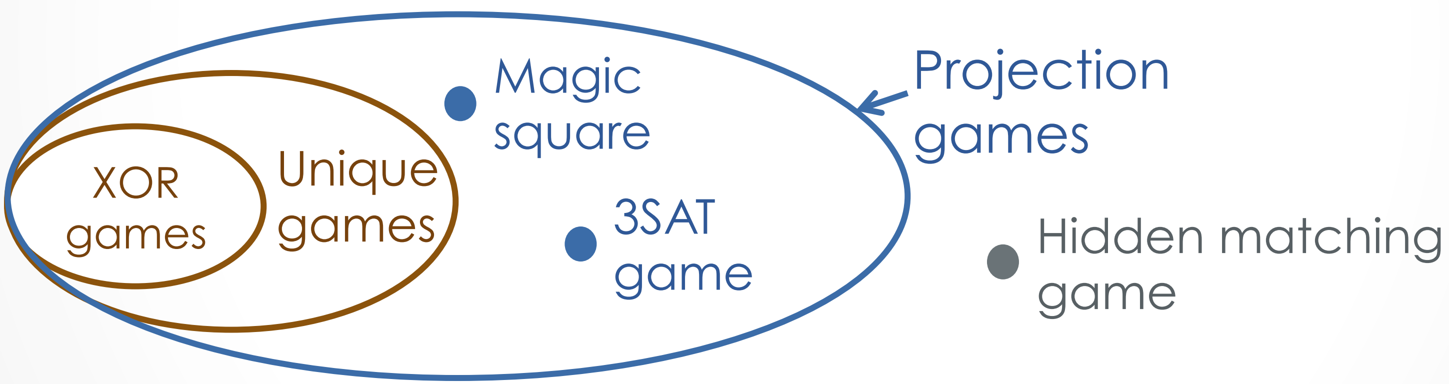
[Jain, Pereszlényi, Yao ‘14] – product-distribution games

[D.-Steurer-Vidick, to appear in CCC‘14]:
Let G be a projection-constraint game, then
If $val^*(G) < 1-\epsilon$ then $val^*(G^{\otimes k}) < (1-poly(\epsilon))^k$

Games with projection constraints



- Projection game: for every pair of questions, any answer from B determines unique valid answer from A
- Your favorite two-player game is a projection game!



- Exists universal transformation $G \to G'$ such that G' projection game and $\omega(G) \approx \omega(G')$
-
- …but could have $\omega^*(G') \ll \omega^*(G)$

Recent but earlier work with David Steurer, (to appear in STOC '14) :

analytical framework to analyze parallel repetition for projection constraints

(contrast to previous information-theoretic approach)

new bounds

low value: $\text{value}(G) \le \rho \implies \text{value}(G^{\otimes k}) \le (2\rho)^{k/2}$

(for projection constraints)

few repetitions: $\text{value}(G) \le 1-\varepsilon \implies \text{value}(G^{\otimes k}) \le (1-\varepsilon)^{\sqrt{k}}$

(for projection constraints, $k \ll 1/\varepsilon^2$)

optimal NP-hardness for SET COVER (and better NP-hardness for LABEL COVER)

$(1-\varepsilon)\ln n$ -approximation, via [Feige, Moshkovitz–Raz, Moshkovitz]

Raz's parallel-repetition counterexample tight even for small k

some G have $\text{value} \le 1-\varepsilon$ but $\text{value}(G^{\otimes k}) \ge 1-\varepsilon\sqrt{k}$

(answers question of O'Donnell)

Analytical framework of DS extends nicely to entangled value setup

Plan:

Describe the “analytical / linear-algebraic” proof of [DS], show how to generalize to the entangled value setup

[DS] proof overview

I. Analytical Setup View a projection game as a linear operator acting on (Bob)-assignments

II. Prelim Step: move to "collision value" $\|\cdot\|$ instead of value(G)

$$\text{value}(G) \le \|G\| \le \text{value}(G)^{1/2}$$ for all G (easy)

III. Main Step: further relax $\|\cdot\|$ to val_+ and prove

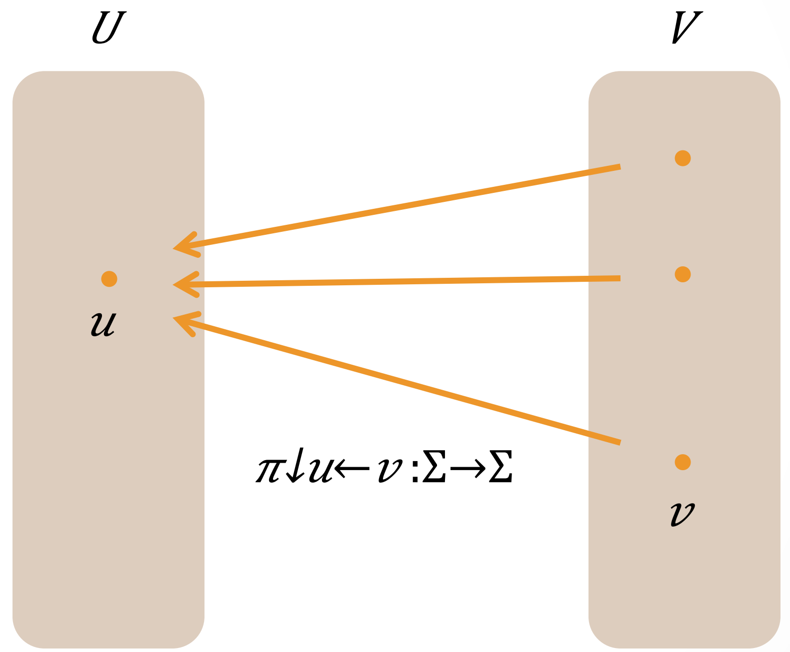
1. $val_+(G \otimes H) = val_+(G) \cdot val_+(H)$ for all G, H *(multiplicativity)*
2. $val_+(G) \approx \text{value}(G)$ for all G *(approximation)*

proof of parallel-repetition bound

$$\text{value}(G^{\otimes k}) \overset{\text{II}}{\le} \|G^{\otimes k}\| \overset{\text{III}}{\le} val_+(G^{\otimes k}) \overset{1.}{=} val_+(G)^k \overset{2.}{\approx} \text{value}(G)^k$$

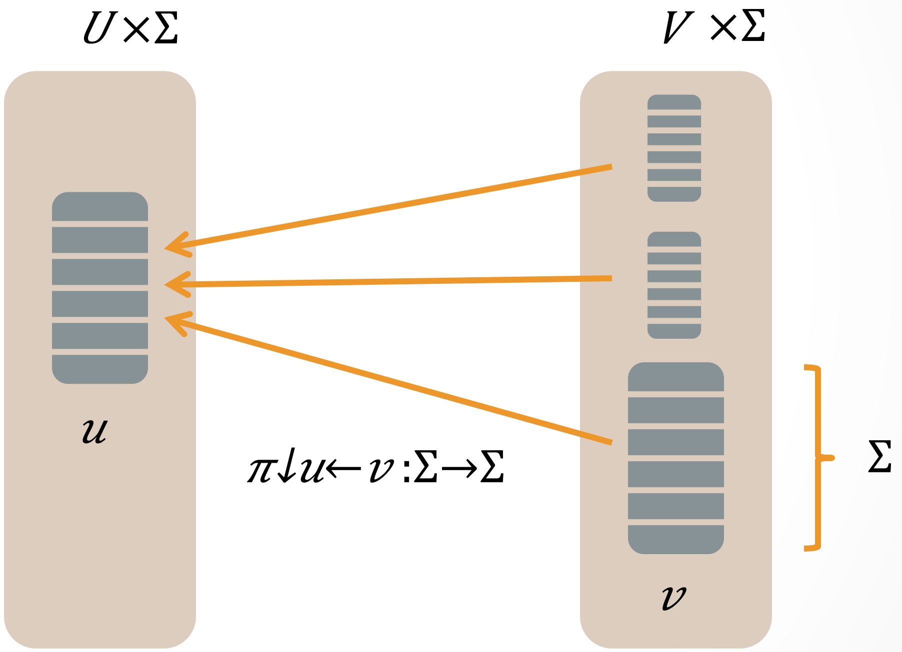
analytical setup

constraint graph



analytical setup

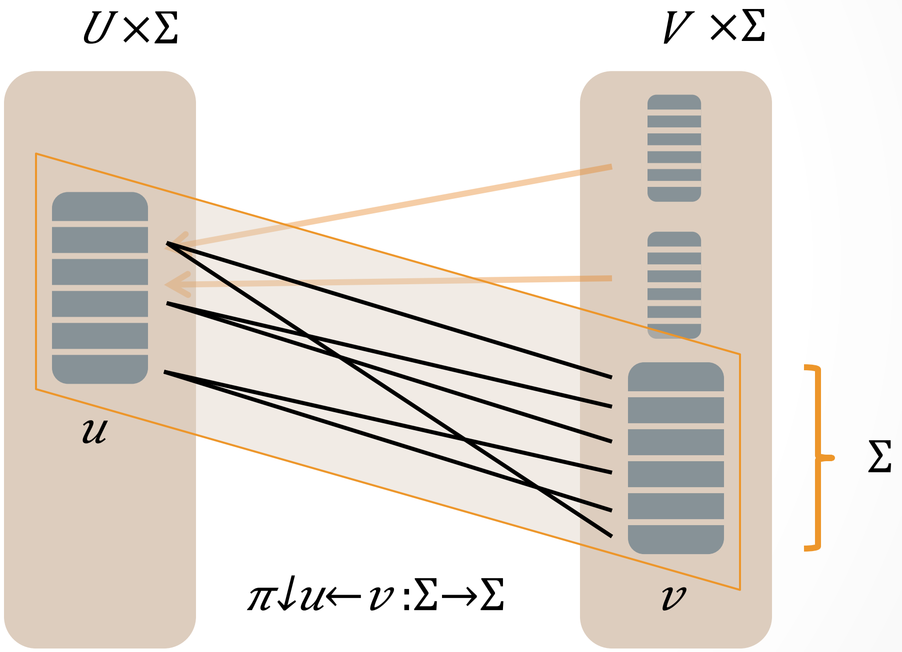
constraint graph



analytical setup

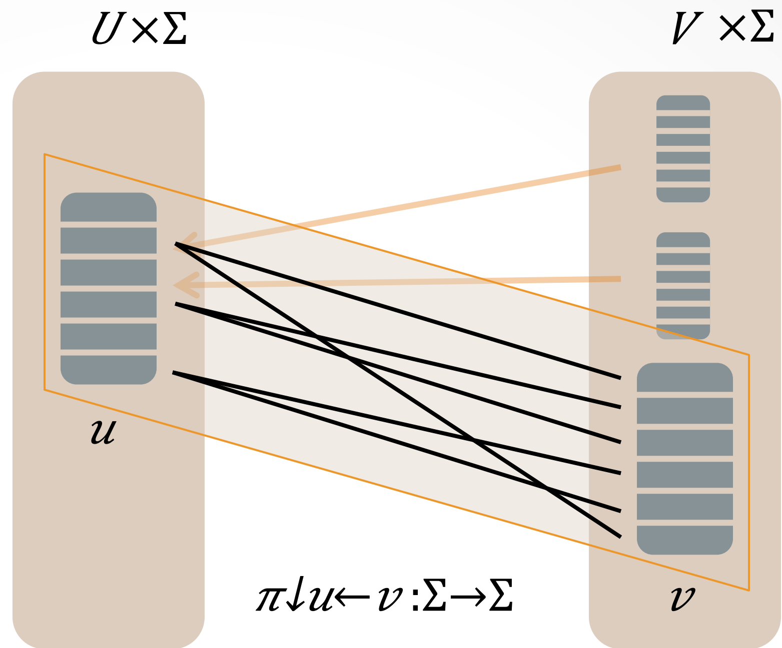
label-extended graph

~~constraint graph~~



analytical setup

label-extended graph G



$f: V\times\Sigma\to\mathbb{R}$

$g: U\times\Sigma\to\mathbb{R}$

$f: V\times\Sigma\to\mathbb{R}$ is an *assignment* if $f\geq 0$ and $\sum_{\beta} f(v,\beta)=1$ for all $v\in V$

linear operator

= adjacency matrix of label-extended graph

$G:\mathbb{R}^{V\times\Sigma}\to\mathbb{R}^{U\times\Sigma}$

For assignment f, $Gf(u,\alpha)$ = prob. that random u-neighbor "demands" α

$$Gf(u,\alpha) := \mathbb{E}_{v:u\leftarrow v}\sum_{\beta\to\alpha} f(v,\beta)$$

success probability for assignments f,g

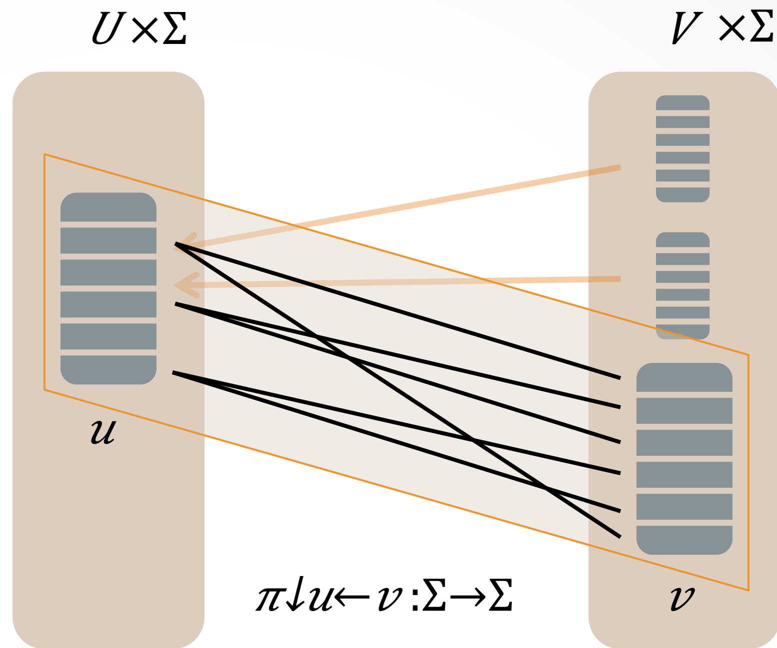
bilinear form of G

$$\langle Gf,g\rangle := \mathbb{E}_{u}\sum_{\alpha} Gf(u,\alpha)\cdot g(u,\alpha)$$

$\mathrm{value}(G)=\max\langle Gf,g\rangle$ over assignments f,g

analytical setup

label-extended graph G



$f:V\times\Sigma\to\mathbb{R}$

$g:U\times\Sigma\to\mathbb{R}$

$G:\mathbb{R}^{V\times\Sigma}\to\mathbb{R}^{U\times\Sigma}$ $\quad$ $H:\mathbb{R}^{V'\times\Sigma'}\to\mathbb{R}^{U'\times\Sigma'}$

tensor product

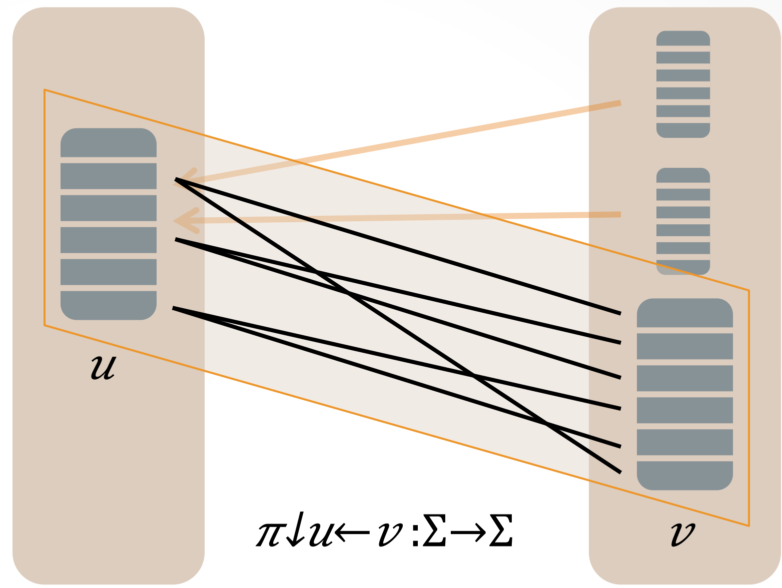
$G\otimes H:\mathbb{R}^{V\times V'\times\Sigma\times\Sigma'}\to\mathbb{R}^{U\times U'\times\Sigma\times\Sigma'}$

= parallel repetition

$$(G\otimes H)f(u,u',\alpha,\alpha') := \mathbb{E}_{v\leftarrow u,\, v'\leftarrow u'} \sum_{\beta\to\alpha,\, \beta'\to\alpha'} f(v,v',\beta,\beta')$$

analytical setup*

label-extended graph G



~~$f:V\times\Sigma\to\mathbb{R}$~~

$f:V\times\Sigma\to\mathcal{L}(\mathbb{C}^d)$

$\sum_b f(v,b)=Id$

~~$g:U\times\Sigma\to\mathbb{R}$~~

$g:U\times\Sigma\to\mathcal{L}(\mathbb{C}^d)$

$$G:\mathcal{L}(\mathbb{C}^d)^{V\times\Sigma}\to\mathcal{L}(\mathbb{C}^d)^{U\times\Sigma}$$

$$Gf(u,\alpha):=\mathbb{E}_{v:u\leftarrow v}\sum_{\beta\to\alpha} f(v,\beta)$$

$$val^*(G)=\sup_{g,G,f,\Psi}\langle\Psi|\ldots|\Psi\rangle := \mathbb{E}_u\sum_\alpha\langle\Psi|g(u,\alpha)\otimes Gf(u,\alpha)|\Psi\rangle$$

$$:=\mathbb{E}_{u\sim v}\sum_{\alpha\sim\beta}\langle\Psi|g(u,\alpha)\otimes f(v,\beta)|\Psi\rangle$$

Collision value

II. Prelim Step: move to "collision value" instead of $value(G)$

$$\text{value}(\mathrm{G}) = \sup_{f,g} \langle Gf, g\rangle \approx \sup_{f} \langle Gf, Gf\rangle^{} =: \; col{-}val(G)$$

Simple Cauchy-Schwarz

$$\text{value}^{*}(\mathrm{G})^{} = \sup_{f,g,\Psi} \langle Gf, g\rangle_{\Psi} \approx \sup_{f,\Psi} \langle Gf, Gf\rangle_{\Psi}^{} =: \; col{-}val^{*}(G$$

Generalized Cauchy-Schwarz due to Haagerup

Vector relaxation – $val_+(G)$

III. Define $val_+(G)$ by replacing $f(v,b)$ by a vector $f(v,b,\omega)$

$$val_+(G) = \sup_{f \geq 0} \langle Gf, Gf \rangle^{1/2} = \sup_f \mathbb{E}_u \sum_a \mathbb{E}_\omega Gf(u,a,\omega)^2$$

Where the sup is over vector strategies f, with proper normalization

$$\forall v, ||\sum_b f(v,b)||_2 \leq 1$$

Why vector strategies ?

A strategy for $G \otimes H$ is automatically a vector strategy,
if viewed as a strategy for G alone.
The normalization "eliminates" the effects of the game H

This facilitates proving $val_+(G \otimes H) = val_+(G) \cdot val_+(H)$

Vector relaxation – $val_+(G)$

III. Define $val_+^*(G)$ by replacing $f(v,b)$ by a vector $f(v,b,\omega)$

$$val_+^*(G) = \sup_{f,\Psi} \langle Gf, Gf \rangle_\Psi = \sup_{f,\Psi} \mathbb{E}_u \sum_a \mathbb{E}_\omega \langle \Psi | Gf(u,a,\omega) \otimes Gf(u,a,\omega)$$

Where the sup is over vector strategies f, with proper normalization

$$\sup_\Psi \sum_{b,b'} \mathbb{E}_\omega \langle \Psi | f(v,b,\omega) \otimes f(v,b',\omega) | \Psi \rangle \le 1$$

Why vector strategies ?

A strategy for $G \otimes H$ is automatically a vector strategy,
if viewed as a strategy for G alone.
The normalization "eliminates" the effects of the game H

This facilitates proving $val_+(G \otimes H) = val_+(G) \cdot val_+(H)$

Vector relaxation – $val_{+}(G)$–approximation

III. Remaining goal: prove that $val_{+}(G) \approx val(G)$

Given a vector strategy, derive a standard strategy that has similar value

Naïvely: focus on one coordinate ω and hope for the best

Generally: combine different coordinates through correlated sampling

[DSV] proof overview

I. Analytical Setup — View a projection game as a linear operator acting on (Bob)-assignments

II. Prelim Step: move to "collision value" $\|\cdot\|^*$ instead of value(G)

$\mathrm{value}^*(G) \le \|G\|^* \le \mathrm{value}(G)^{*\,1/2}$ for all G (easy)

III. Main Step: further relax $\|\cdot\|^*$ to val_+^* and prove

1. $val_+^*(G \otimes H) = val_+^*(G) \cdot val_+^*(H)$ for all G, H *(multiplica*
2. $val_+^*(G) \approx \mathrm{value}^*(G)$ for all G *(approxim*

proof of parallel-repetition bound

$$\mathrm{value}^*(G^{\otimes k}) \overset{\text{II}}{\le} \|G^{\otimes k}\|^* \overset{\text{III}}{\le} val_+^*(G^{\otimes k}) \overset{1.}{=} val_+^*(G)^k \overset{2.}{\approx} \mathrm{value}^*(G)^k$$

Main source of trouble: non-product strategies that are too good

Why are they too good?

How good can they be ?

How does this depend on the game ?

Back to the basics – design a simple game in which this can be studied

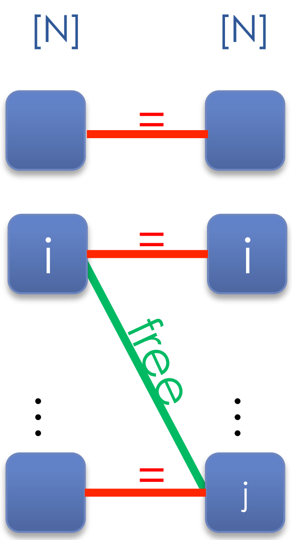
THE CONFUSE & COMPARE GAME

*[Feige & Kilian]

The Confuse & Compare game, with parameters p, N

Alice

$A : [N] \to \{0,1\}$



Bob

$B : [N] \to \{0,1\}$

p fraction of edges are equality edges (i,i) "compare"

(1-p) fraction of edges are free edges (i,j) "confuse"

If A = B then win with probability 1

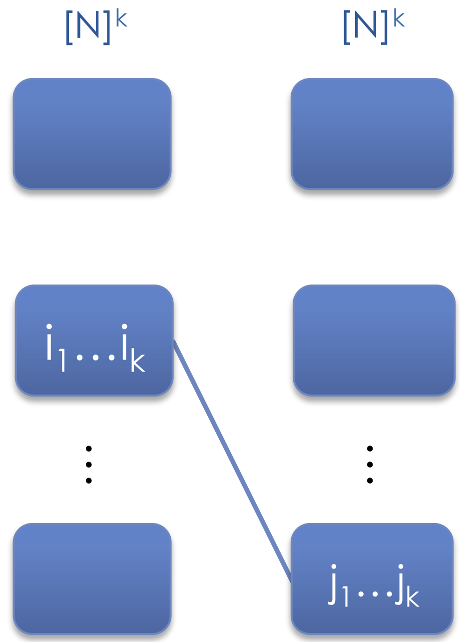
k-fold parallel repetition: choose k independent edges $i_1 j_1 \ldots i_k j_k$

Send $i_1 \ldots i_k$ to Alice and $j_1 \ldots j_k$ to Bob

The Confuse & Compare* game, k-fold direct product

Alice

$A : [N]^k \to \{0,1\}^k$



Bob

$B : [N]^k \to \{0,1\}^k$

Clearly, a product strategy A^k obtains value 1

Question : can something be said about A if val > 0.99 ? val > 0.001 ?

Direct Product Testing

The symmetrized version of this game is known as "direct product testing". Both players use a strategy

$$A:[N]^k \rightarrow [M]^k$$

and the goal is to prove

THM: If Confuse-Compare-value (A) > a,
then A is close to being direct-product

For some values of p, N, M this question is solved,
[Goldreich-Safra, D.-Reingold, D.-Goldenberg, Impagliazzo-Kabanets-Wigderson, D.-Steurer]

but much is open

This is a clean question, can also be formulated in the entangled setting, but little is known there…

Summary

- Parallel repetition is good for amplification
- Analytical approach useful for analyzing parallel repeated projection games
- Direct product testing