the talk
1. strong promises and eigenvalue gaps

2. the history of the history state

3. running the clock: precise/faulty, qubit/qudit, sequential/parallel

4. on the qPCP road: questions & warnings
Hamiltonians and their eigenvalue gaps

\[ \sum_{m=1}^{M} H_m \]

\[ \sum_{m=1}^{M} G_m \]

\[ \sum_{m=1}^{M} F_m \]
1 Hamiltonians and their ground states

Is the ground state energy of a small?
1 Hamiltonians and their ground states

Is the ground state energy of a high?
The QMA protocol

YES?
Accept a good proof with $p > a$.

NO?
Probability of accepting $p < b$.

Is there an acceptable witness for this circuit?
Is some local Hamiltonian (nearly) frustration-free?
The QMA protocol

YES?
Accept a good proof with $p > a$.

NO?
Probability of accepting $p < b$.

- Is there an acceptable witness for this circuit?
- Does some local Hamiltonian have a low ground energy?
The promise gap for a problem

\[ \sum_{m=1}^{M} H_m \]

\[ \sum_{m=1}^{M} G_m \]

YES  little frustration

NO  lots of frustration
The promise gap for a simpler problem?

\[
\sum_{m=1}^{M} H_m
\]

\[
\sum_{m=1}^{M} G_m
\]

YES  little frustration

NO  lots of frustration

the promise gap
\[ \sum_{m=1}^{M} H_m \]

YES  little frustration  

NO  lots of frustration
\[ \sum_{m=1}^{M} H_m \quad \rightarrow \quad \sum_{m=1}^{M'} H'_m \]

YES    little frustration    NO    lots of frustration
Increase the promise gap?

**YES**  little frustration

**NO**  lots of frustration
Increase the promise gap?

Using the usual circuit encoding clock construction based ideas? NO
Small eigenvalue gaps ... small promise gaps

\[ H = A + B \]

- geometric lemma

\[ \lambda_0 \geq \sin^2 \frac{\theta}{2} \times \min(\Delta_A, \Delta_B) \]

YES
- little frustration
- a very low ground energy

NO
- without much frustration
- a pretty low ground energy
1 Small or large eigenvalue gaps?

- Anything close to the ground state?

  \[
  \Delta \geq const.
  \]

1D: area law, an algorithm
2D: area law?

---

a Heisenberg XXX spin-1 chain (AKLT)

\[
\sum_{j=1}^{N-1} X_j X_{j+1} + Y_j Y_{j+1} + Z_j Z_{j+1}
\]

---

a biased walk in 1D

\[
\sum_{j=1}^{N-1} (|j\rangle - B|j+1\rangle) (\langle j| - B\langle j + 1|)
\]
1 Small or large eigenvalue gaps?

- Anything close to the ground state?

  constant gap? \[ \Delta \geq \text{const.} \]
  1D: area law, an algorithm
  2D: area law?

  inverse-poly gap? \[ \Delta \propto N^{-c} \rightarrow 0 \]
  clock constructions
  NP, QCMA hard
  qubits? 1D?

transverse-field Ising
\[ \sum_{j=1}^{N-1} X_j - \sum_{j=1}^{N-1} Z_j Z_{j+1} \]

quantum walk on a line
\[ \sum_{j=1}^{N-1} |j\rangle\langle j+1| + |j+1\rangle\langle j| \]
Small or large eigenvalue gaps?

- Anything close to the ground state?

  - Constant gap? \( \Delta \geq \text{const.} \)
    - 1D: area law, an algorithm
    - 2D: area law?

  - Inverse-poly gap? \( \Delta \propto N^{-c} \to 0 \)
    - Clock constructions
    - NP, QCMA hard
    - Qubits? 1D?

  - Exponential-small gap? \( \Delta \propto 2^{-cN} \to 0 \)

- Constant degree LH: at most constant gap.

- Degeneracy: help or trouble?
Snapshots of a computation
Locally comparing strings.
Locally comparing products.

SWAP test
Locally comparing entangled states?

UGH!
Labeling the data

Hard to compare directly (locally).

$U^\dagger \downarrow U$

a clock
2 Labeling the data

a clock
2 The data & the clock

\[ U \otimes |1\rangle \langle 0| \]

\[ \begin{array}{ccc}
+ & \begin{array}{ccc}
\text{Yellow} & \text{Yellow} & \text{Yellow} \\
\text{Red} & \text{Red} & \text{Red} \\
\end{array} \\
\end{array} \]

\[ \begin{array}{ccc}
+ & \begin{array}{ccc}
\text{Yellow} & \text{Yellow} & \text{Red} \\
\text{Red} & \text{Red} & \text{Yellow} \\
\end{array} \\
\end{array} \]

\[ |0\rangle \]

\[ |1\rangle \]
The data & the clock

\[ U^\dagger \otimes |0\rangle\langle 1| \]

\[ |0\rangle \rightarrow |1\rangle \]
The data & the clock: locally comparing related states
The history state

\[ |\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle \]

\[ \underbrace{U_t \cdots U_1|\varphi_0\rangle} \]
The history state: a ground state

\[ \Pi_j = 0 \]

\[ |\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle \]

\[ U_t \cdots U_1 |\varphi_0\rangle \]
\[ \psi_{\text{hist}} = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\phi_t\rangle \otimes |t\rangle \]

\[ \langle L | \otimes \langle I | \cdots | \text{output} \]

\[ \cdots \]

\[ \langle 0 | \otimes \langle 0 | \]

k-local

conditions

clock encoding

state progression

initialization
a clock workshop
Making a local clock

- dynamic: a system that ticks
  local ticks (transitions)

- static: a unique ground state
  the uniform tick superposition

- identifiable states

- checking for bad states
A quantum walk on a line is a clock

- transitions

\[ T_{st} = |s\rangle \langle t| + |t\rangle \langle s| \]

Hamiltonian

\[ H_w = \sum_{\langle s,t\rangle} T_{st} \]
A line is a clock

transitions \[ T_{st} = |s\rangle \langle t| + |t\rangle \langle s| \]

projections \[ P_{st} = \frac{1}{2} (|s\rangle - |t\rangle) (\langle s| - \langle t|) \]

Hamiltonian \[ H = \sum_{\langle s,t\rangle} P_{st} \]

the ground state \[ |1\rangle + |2\rangle + |3\rangle + \cdots \]
3 A line is a clock

Transitions

$$T_{st} = |s\rangle\langle t| + |t\rangle\langle s|$$

Projections

$$P_{st} = \frac{1}{2} (|s\rangle - |t\rangle) (\langle s| - \langle t|)$$

Hamiltonian

$$H = \sum_{\langle s,t\rangle} P_{st}$$

The ground state

$$|1\rangle + |2\rangle + |3\rangle + \cdots$$

Other eigenstates

$$|\varphi_p\rangle \propto \sum_{s=1}^{N} \cos(ps) |s\rangle \quad E_p = 2 \cos p$$

The gap

$$\Delta = \Theta \left( \frac{1}{N^2} \right) \quad p = \frac{k\pi}{N}$$
3. A pulse clock

- **transitions**
  - 2-local
  
  \[ |10\rangle\langle01| + |01\rangle\langle10| \]

- **identification**
  
  \[ |1\rangle\langle1| \]

- **projections**
  - get a superposition for the ground state
  
  \[ |01 - 10\rangle\langle01 - 10| \]

- **invariant subspaces & tuning**
  - a given number of excitations
  
  tuning for a single excitation: prefer 1, hate 11

\[ \begin{align*}
1000 \\
+0100 \\
+0010 \\
+0001
\end{align*} \]
A domain wall (unary) clock

- clock checking  $|01\rangle\langle 01|$
- 2-local
- identification  $|10\rangle\langle 10|$
A domain wall (unary) clock

- **clock checking** \(|01\rangle\langle 01|\)
  - 2-local
- **identification** \(|10\rangle\langle 10|\)
- **projections** \(|100 - 110\rangle\langle 100 - 110|\)
  - 3-local
- **a single domain wall: fix the ends**
  - a unique ground state

\[ 10000 + 11000 + 11100 + 11110 \]
3 The DW clock in Kitaev’s 5-local Hamiltonian

- clock checking
  2-local
  \[ |01 \rangle \langle 01| \]

- identification
  \[ |10 \rangle \langle 10| \]

- projections
  3-local
  \[ |100 - 110 \rangle \langle 100 - 110| \]

- interacting with data
  5-local
  \[
  \frac{1}{2} \left( |t + 1 \rangle \langle t + 1| + |t \rangle \langle t| \right) - \frac{1}{2} \left( U_{t+1} \otimes |t + 1 \rangle \langle t| + U_{t+1}^\dagger \otimes |t \rangle \langle t + 1| \right)
  \]
Kitaev’s LH: the playground

Invarient subspaces with bad clock states labeled by the initial state

\[
|\varphi_1\rangle \otimes |0\rangle_c \\
U_1 |\varphi_1\rangle \otimes |1\rangle_c \\
U_2 U_1 |\varphi_1\rangle \otimes |2\rangle_c \\
U_3 U_2 U_1 |\varphi_1\rangle \otimes |3\rangle_c \\
U_4 U_3 U_2 U_1 |\varphi_1\rangle \otimes |4\rangle_c \\
U_1 |\varphi'_1\rangle \otimes |1\rangle_c \\
U_2 U_1 |\varphi'_1\rangle \otimes |2\rangle_c \\
U_3 U_2 U_1 |\varphi'_1\rangle \otimes |3\rangle_c \\
U_4 U_3 U_2 U_1 |\varphi'_1\rangle \otimes |4\rangle_c
\]}
The history state: a line of states

- a projector Hamiltonian kernel: the uniform superposition

\[ |t + 1\rangle\langle t + 1| + |t\rangle\langle t| \]

\[ - U_{t+1} \otimes |t + 1\rangle\langle t| \]

\[ - U_{t+1}^\dagger \otimes |t\rangle\langle t + 1| \]

\[ |\varphi_t\rangle \otimes |t\rangle \]

\[ |\varphi_{t+1}\rangle \otimes |t + 1\rangle \]

\[ |\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\varphi_t\rangle \otimes |t\rangle \]

- endpoints: ancilla initialization/final acceptance
ground state

YES
ground state

• NO
lower bound on the ground state energy
history states

non-uniform superpositions
history states

a polynomially small gap

\[ \Delta = O(L^{-2}) \]
history states
initialized history states
$H_A + H_B$

$$\lambda_0 \geq \sin^2 \frac{\psi}{2} \times \min(\Delta_A, \Delta_B)$$

accepted states
3 - LH and QMA verification

\[ H_{\text{clock}} + H_{\text{init}} + H_{\text{prop}} + H_{\text{out}} \]

\[ V \]

\[ |\psi\rangle \]
\[ |0\rangle \]
\[ 0/1 \]

\[ \text{NO} \]
\[ V \text{ is unlikely to accept anything (} \epsilon \text{)} \]

\[ \text{promise gap } L^{-2} \]

\[ \text{YES} \]
\[ \text{some proof is likely (} 1-\epsilon \text{) accepted} \]

\[ \text{lowest eigenvalue} \]
\[ \geq \frac{c (1 - \sqrt{\epsilon})}{L^2} \]

\[ \text{(needs } \epsilon = L^{-1}) \]

\[ \text{energy of the history} \]
\[ \leq \frac{\epsilon}{L + 1} \]

[N, Mozes 07]
projections & gadgets
3 Lower locality for the price of bad transitions

- the domain wall

- projections
  now just 1-local

- clock checks
  2-local, STRONG

- the ground state is close to

\[ |01\rangle \langle 01| \]
Lemma 1 Let $H = H_1 + H_2$ be the sum of two Hamiltonians operating on some Hilbert space $\mathcal{H} = S + S^\perp$. The Hamiltonian $H_2$ is such that $S$ is a zero eigenspace and the eigenvectors in $S^\perp$ have eigenvalue at least $J > 2\|H_1\|$. Then,

$$\lambda(H_1|S) - \frac{\|H_1\|^2}{J - 2\|H_1\|} \leq \lambda(H) \leq \lambda(H_1|S)$$
The projection lemma to estimate eigenvalues

\[ H^+ H_1 |S \rangle \]

\[ S \quad S' \quad H_2 \]

\[ |01 \rangle \langle 01| \]
The projection lemma to estimate eigenvalues

$H_1 |_S$

$|1110000\rangle$
$|1111000\rangle$
$|1111100\rangle$

$|1 - 0\rangle \langle 1 - 0| + |01\rangle \langle 01|$
Lower locality (3-LH) for the price of bad transitions

- the domain wall

- non-projector 2-local terms

- clock checks

- from 5- to 3-local Hamiltonian [Kempe, Regev]

restricted to good clocks: qw on a line
Further decreasing locality: a “3 from 2” gadget

- strongly coupled ancillas (a new energy scale)
- perturbation theory

\[
G'(z) = (zI - H')^{-1}
\]

\[
H' = H + V
\]

\[
||H|| \gg ||V||
\]

\[
S' = \text{span} \{ |000\rangle, |111\rangle \}
\]

[Kempe, Kitaev, Regev '03]
Further decreasing locality: a “3 from 2” gadget

- strongly coupled ancillas (a new energy scale)
- perturbation theory gives us an effective Hamiltonian

\[
V \big|_S \quad V^2 \big|_S \quad V^3 \big|_S
\]

projection lemma \hspace{1cm} \text{unwanted (subtract)} \hspace{1cm} \text{the effective 3-local term}

\[
H' = H + V
\]

\[
\|H\| \gg \|V\|
\]

\[
S' = \text{span} \left\{ |000\rangle, |111\rangle \right\}
\]

[Kempe, Kitaev, Regev '03]
3 STRONG local fields, OK interactions

- strongly bound a single ancilla
- no superstrong interactions

- perturbation theory gives us an effective Hamiltonian

\[ S = \{ |0\rangle \} \]

\[ H' = H + V \]
\[ ||H|| \gg ||V|| \]

\[ V \mid S \quad V^2 \mid S \quad V^3 \mid S \]

projection lemma unwanted (subtract) the effective 3-local term

special cases (Z-basis) exact gadgets!

[Cao et al., 1311.2555]

[Jacob Biamonte 0801.3800]
“Strengthening”, intermediary gadgets?

- classically easy: copy

- quantumly?

[N., Yudong Cao]
locality & dimensionality
clock/work registers

constant degree geometric locality

a geometric clock

[Mizel] [Janzing] [AvDKLLR] [BT13]

[AGIK07] moving data in 1D
Moving a special site: the qutrit surfer

- clock checking
  - 2-local + ends
    - $|10\rangle\langle 10|$, $|02\rangle\langle 02|$, $|22\rangle\langle 22|$
    - $|01\rangle\langle 01|$, $|21\rangle\langle 21|$
- identification
  - $|2\rangle\langle 2|$
- projections
  - 2-local
    - $|20-12\rangle\langle 20-12|$
3. Constructing local, geometric clocks: moving the data

- telling “time” by where the data is

- carrying/moving data? larger qudits (local dim.) larger locality

internal states ... dual-rail

[Childs Gosset Webb 13]
Making a good local clock

- identifiable states
  - domain-wall structure
- local transitions
- easily checkable states

- different geometry?
  - simpler terms?
  - locality/qudits?
  - beyond linear?
  - larger (promise) gaps?
Hamiltonian Quantum Cellular Automata in 1D

- moving the program instead of the data
  [N., Wocjan 07]

program particles diffuse above data
special states stand in their way
moving the program instead of the data

[N., Wocjan 07]

program particles diffuse above data
special states stand in their way
Hamiltonian Quantum Cellular Automata in 1D

- moving the program instead of the data [N., Wocjan 07]

program particles diffuse above data
special states stand in their way

- BQP in 1D with a trans. invariant, time independent LH computational basis programmable

- a nonlinear clock, polynomial expected runtime
A local, sequential geometric clock in 2D

2D “sequential” evaluation

[Aharonov van Dam Kempe Landau Lloyd Regev 04]

universality of adiabatic QC

2-local interactions, d=6 qudits
A local, sequential geometric clock in 2D

- 2D “sequential” evaluation
  [Aharonov van Dam Kempe Landau Lloyd Regev 04]

Universality of adiabatic QC
2-local interactions, d=6 qudits
A local, sequential geometric clock in 2D

- 2D “sequential” evaluation
  [Aharonov van Dam Kempe Landau Lloyd Regev 04]

universality of adiabatic QC
2-local interactions, d=6 qudits
Another geometric clock in 2D: a string on a torus

- 2D “parallelized” evaluation

[Mizel Lidar 06] [Janzing 07]
[Breuckmann Terhal 13]
Another geometric clock in 2D: a string on a torus

- 2D “parallelized” evaluation
  - [Mizel Lidar 06]
  - [Janzing 07]
  - [Breuckmann Terhal 13]

QMA-complete
4-local operations
b, b† fermions, spin
or d=4 (spin 3/2)

particle # tuning
(motivation: AQC)

- promise gap: proven $N^{-3}D^{-3}$, conjectured $N^{-2}D^{-2}=L^{-2}$
Constructing local, geometric clocks in 1D

- moving the data with 2-local interactions

\[
\begin{align*}
\langle|s\rangle - |t\rangle\rangle (\langle s | - \langle t |)
\end{align*}
\]

\[
|XY\rangle \langle XY|_{j, j+1} + |ZW\rangle \langle ZW|_{k, k+1}
\]

\[
- |PQ\rangle \langle NO|_{i, i+1} - |NO\rangle \langle PQ|_{i, i+1}
\]

- higher local dimension: qudits
  carry the data
  mark transitions
  detect bad states
we’ve been here  the data  undiscovered territory

moves

the power of quantum systems on a line

[Aharonov, Gottesman, Irani, Kempe]
$U_{ab}$

$U_{bc}$
1. LH in 1D (2-local) with qudits

- **unique state progression**
  every legal state goes to exactly 2 states

- **clairvoyance**
  allowed but illegal states evolve to forbidden ones

- **the promise gap:** $L^{-3}$

- **an entangled ground state**
  special case: NP-hard [Schuch]

QMA-complete

$d = 13$

[AGIK '06]
1 LH in 1D: more space = smaller qudits

- unique state progression
  every legal state goes
to exactly 2 states

\[ d = 11 \]

[N. 08]

- bad but detectable transitions

\[ d = 8 \]

[Hallgren, N., Narayanaswami '13]
2-local Hamiltonian is QMA-complete

[Oliveira, Terhal '05]

a global minimum

[Hallgren, N, Narayanaswami '13]
QMA$_1$-complete problems

11 - 11 - 11 - 11 - 11

[N. ‘08]

2 - 2 - 2 - 2

[Gosset, N. ‘13]

5 - 3 - 3 - 3 - 5

[Eldar, Regev ‘08]

unfrustrated

qSAT

[Bravyi]
linear

clock progression
com-pos-ite
clock progression
composite

clock progression
single - path
clock progression
double = path

clock progression
double = path
clock progression
double = path

clock progression
double = path
clock progression
2 clocks: 2D clock progression
Applying 2-qubit gates 3-locally

- the railroad switch
Applying 2-qubit gates 3-locally

- the railroad switch
Applying 2-qubit gates 3-locally

- the railroad switch
Applying 2-qubit gates 3-locally

- the railroad switch
Applying 2-qubit gates 3-locally

- the railroad switch

CNOT: 3-local needs initialization
2D clocks (with two registers)

- two clocks
2D clocks (with two registers)

- two clocks
2D clocks (with two registers)

- add non-commuting (data) operations
2D clocks (with two registers)

- like a railroad switch... with a single active site ensured
3  Can a clock be shorter than unary?

- a qutrit surfer on a cycle: 2C states
A smaller clock using two coupled cogs
A smaller clock using two coupled cogs
3. A smaller clock using two coupled cogs

- 2 cogs of length $C$ give us $(2C)^2$ clock states
- transitions: 4-local, gates: 6-local (can be improved)
- the promise gap for a circuit with $L$ gates: still $L^{-2}$
3 Questions about the qPCP conjecture

- equivalence of the two formulations? [AAV13]

LH with a fractional promise gap

- look at a few qubits of a proof

- locally checking the (expected) very entangled states? [D.Aharonov, L.Eldar]

translating the random small verification to a LH?
3 Questions about the qPCP conjecture

- clock constructions have a 1/poly promise gap consistent, effective interaction strengthening? error-correction/detection based quantum gadgets?

- direct Hamiltonian methods? beyond the history state? [Itai Arad]

\[
M := \mathbb{I} - \frac{H}{m}
\]

\[
\text{Tr}(M^\ell) \quad \Gamma = \frac{1}{\text{poly}(n)}
\]

\[
\ell = \Omega(mn/\Gamma)
\]

an interactive protocol to check the trace:

\[
\begin{align*}
\rho_1 \quad & \quad \rho_2, |\psi_1\rangle \\
|\psi_1\rangle \quad & \quad |\psi_1\rangle |\psi_2\rangle \\
\rho_3 \quad & \quad \rho_3, |\psi_1\rangle |\psi_2\rangle
\end{align*}
\]
1. strong promises and eigenvalue gaps

2. the history of the history state

3. running the clock precise/faulty, qubit/qudit, sequential/parallel

4. on the qPCP road questions & warnings